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ZAHID AHMAD

MUHAMMAD SHOAIB KHAN



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# Higher-dimensional dust collapse in *f(R)* gravity

**Zahid AHMAD***, **Muhammad SHOAIB KHAN**
Mathematics Department, COMSATS Institute of Information Technology, Abbottabad, Pakistan

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**Abstract:** The $n+2$-dimensional gravitational collapse of pressureless fluid is investigated in metric $f(R)$ gravity. Matching conditions are derived by taking the $n+2$-dimensional Friedmann–Robertson–Walker (FRW) metric as interior spacetime and the $n+2$-dimensional Schwarzschild metric as exterior spacetime. In the analysis of the solution of field equations, the scalar curvature is assumed to be a constant. It is observed that the scalar curvature constant term $f(R_0)$ slows the collapse.

**Key words:** Dust collapse, *f(R)* gravity, higher dimensional

## 1. Introduction

Recently, a number of research papers have been written to investigate gravitational collapse in extended theories of gravity [1–12]. It is shown that the further one goes from general relativity, there is a greater chance of naked singularity [4]. Sharif and Kausar [10] studied the spherically symmetric perfect fluid gravitational collapse in the *f(R)* theory of gravity. Farasat et al. [11] investigated collapse with dust in the *f(R)* theory of gravity. They concluded that $f(R_0)$ slows the collapse. For more references on this topic see [1–12].

Recent advancements in string theory and other field theories indicate that gravity is a higher-dimensional interaction. It would be important to study gravitational collapse and singularity formation in higher dimensions. Dadhich et al. [13] studied the gravitational collapse in pure Lovelock gravity in higher dimensions. Ghosh and Beesham [14] investigated the higher-dimensional inhomogeneous dust collapse and cosmic censorship. Patil et al. [15] investigated the naked singularities and structure of geodesics in higher-dimensional dust collapse. Sharif and Ahmad [16] studied higher-dimensional perfect fluid collapse with a cosmological constant. In *f(R)* gravity, higher-dimensional null dust collapse was studied by Ghosh and Maharaj [12] and they investigated a condition for the formation of a naked singularity. In this paper, the work done by Farasat et al. [11] is extended for $n+2$-dimensional spacetime.

The scheme of the paper is as follows. In section two, the field equations are given. Section three is devoted to junction conditions. The results are discussed in section four. In section five, the apparent horizons are discussed. Section six contains the summarized results.

*Correspondence: zahidahmad@ciit.net.pk

## 2. Equations in *f(R)* gravity

Here the $n+2$-dimensional spacetime is divided into interior and exterior space-times by $n+1$-dimensional hypersurface $\Sigma$. For the interior region, we take the $n+2$-dimensional FRW metric given by

$$ds_-^2 = dt^2 - a^2(t)(dr^2 + b^2(r)d\Omega^2), \tag{1}$$

where a(t) represents the cosmic scale factor and

$$b(r) = \begin{cases} \sin r, & k=1, \\ r, & k=0, \\ \sinh r & k=-1. \end{cases}$$

$$\begin{aligned} d\Omega^2 &= \sum_{k=1}^{n}[\prod_{l=1}^{k-1} sin^2\theta_l]d\theta_k^2 = d\theta_1^2 + sin^2\theta_1 d\theta_2^2 + sin^2\theta_1 sin^2\theta_2 d\theta_3^2 \\ &+ \quad ... + sin^2\theta_1 sin^2\theta_2 sin^2\theta_3 ... sin^2\theta_{n-1} d\theta_n^2. \end{aligned} \tag{2}$$

The Einstein field equations in $f(R)$ gravity are given as [17]

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\nabla^\sigma\nabla_\sigma F(R) = \kappa T_{\mu\nu}^m. \tag{3}$$

Here $F(R) \equiv f'(R)$, $\nabla_\mu$ is the covariant derivative, $T_{\mu\nu}^m$ is the energy momentum tensor, and $\kappa$ is the coupling constant. Contracting with metric tensor $g^{\mu\nu}$ one can find the following trace of the field equations (3)

$$F(R)R - 2f(R) + 3\nabla^\sigma\nabla_\sigma F(R) = \kappa T^m. \tag{4}$$

We consider dust for which the energy momentum tensor is

$$T_{\mu\nu}^m = \rho u_\mu u_\nu, \tag{5}$$

where $\rho$ denotes the energy density and $u_\mu = \delta_\mu^0$ is the $n+2$-dimensional velocity. For metric (1), the independent field equations are

$$-(n+1)\frac{\ddot{a}}{a} = \frac{1}{F}[\kappa\rho + \frac{f}{2} - (n+1)\frac{\dot{a}}{a}\dot{F}], \tag{6}$$

$$\frac{\ddot{a}}{a} + n(\frac{\dot{a}}{a})^2 - n\frac{b''}{a^2 b} = \frac{1}{F}[-\frac{f}{2} + n\frac{\dot{a}}{a}\dot{F} + \ddot{F}], \tag{7}$$

$$\frac{\ddot{a}}{a} + n(\frac{\dot{a}}{a})^2 - \frac{b''}{a^2 b} - (n-1)(\frac{b'}{ab})^2 + \frac{n-1}{a^2 b^2} = \frac{1}{F}[-\frac{f}{2} + n\frac{\dot{a}}{a}\dot{F} + \ddot{F}]. \tag{8}$$

Here dot means differentiation with respect to $t$ and prime represents differentiation with respect to $r$.

### 3. Matching conditions

The matching conditions are defined as

(a) The continuity of interior and exterior spacetimes on $\Sigma$ gives

$$(ds^2_+)_\Sigma = (ds^2_-)_\Sigma = (ds^2)_\Sigma. \tag{9}$$

(b) The continuity of the extrinsic curvature $K_{cd}$ over $\Sigma$ gives

$$[K_{cd}] = K^+_{cd} - K^-_{cd} = 0, \qquad\qquad (c, d = 0, 2, 3, ...n + 1). \tag{10}$$

These equations give necessary and sufficient conditions for smooth matching of interior and exterior spacetimes. The condition (9) implies that the first fundamental form for interior and exterior spacetimes is the same on hypersurface $\Sigma$. The second condition (10) shows that the second fundamental form is the same on hypersurface $\Sigma$. In (10),

$$K^\pm_{cd} = -n^\pm_\omega \left( \frac{\partial^2 x^\omega_\pm}{\partial\xi^c \partial\xi^d} + \Gamma^\omega_{\gamma\delta} \frac{\partial x^\gamma_\pm \partial x^\delta_\pm}{\partial\xi^c \partial\xi^d} \right), \qquad (\omega, \gamma, \delta = 0, 1, 2, 3, ...n + 1). \tag{11}$$

Here $n^\pm_\omega$ , $x^\omega_\pm$, and $\xi^c$ denote the unit outward normals, coordinates on $V^\pm$ and $\Sigma$, respectively.

The $n + 2$-dimensional Schwarzschild metric is considered as exterior spacetime

$$ds^2_+ = (1 - \frac{2M}{\tilde{R}})dT^2 - \frac{1}{1 - \frac{2M}{\tilde{R}}} d\tilde{R}^2 - \tilde{R}^2 d\Omega^2, \tag{12}$$

where $M$ is constant, and $T$ and $\tilde{R}$ are the time and radial coordinates for exterior spacetime, respectively. The equations of hypersurfaces are given by

$$h_-(r, t) = r - r_\Sigma = 0, \tag{13}$$

$$h_+(\tilde{R}, T) = \tilde{R} - \tilde{R}_\Sigma(T) = 0, \tag{14}$$

where $r_\Sigma$ is a constant. These equations imply that the radial coordinate of interior spacetime is constant and the radial coordinate of exterior spacetime depends on its time coordinate, i.e. $\tilde{R}_\Sigma(T)$ on $\Sigma$, respectively. Using Eq. (13) in Eq. (1), the interior metric on the hypersurface $\Sigma$ takes the following form:

$$ds^2_- = dt^2 - a(t)^2 b(r)^2 d\Omega^2. \tag{15}$$

Moreover, substituting Eq. (14) in Eq. (12), we get

$$ds^2_+ = (1 - \frac{2M}{\tilde{R}_\Sigma} - \frac{1}{1 - \frac{2M}{\tilde{R}_\Sigma}} (\frac{d\tilde{R}_\Sigma}{dT})^2) dT^2 - \tilde{R}^2_\Sigma d\Omega^2. \tag{16}$$

For $T$, a timelike coordinate, we assume

$$(1 - \frac{2M}{\tilde{R}_\Sigma} - \frac{1}{1 - \frac{2M}{\tilde{R}_\Sigma}} (\frac{d\tilde{R}_\Sigma}{dT})^2) > 0. \tag{17}$$

Now from matching condition (9), we get

$$\tilde{R}_\Sigma = a(t)b(r_\Sigma), \tag{18}$$

$$[1 - \frac{2M}{\tilde{R}_\Sigma} - \frac{1}{1-\frac{2M}{\tilde{R}_\Sigma}}(\frac{d\tilde{R}_\Sigma}{dT})^2]^{\frac{1}{2}} dT = dt. \tag{19}$$

The unit normals on $\Sigma$ can be found from Eqs. (13) and (14) as

$$n^-_\omega = (0, a(t), 0, 0, 0, ...0), \tag{20}$$

$$n^+_\omega = (-\dot{\tilde{R}}_\Sigma, \dot{T}, 0, 0, 0, ...0). \tag{21}$$

The components of $K^\pm_{cd}$ are

$$K^-_{00} = 0, \tag{22}$$

$$\begin{aligned} K^-_{22} &= \csc^2\theta_1 K^-_{33} = \csc^2\theta_1 \csc^2\theta_2 K^-_{44} = \\ ... &= \csc^2\theta_1 \csc^2\theta_2 ... \csc^2\theta_n K^-_{n+1n+1} = (abb')_\Sigma, \end{aligned} \tag{23}$$

$$K^+_{00} = [\dot{\tilde{R}}_\Sigma \ddot{T} - \dot{T}\ddot{\tilde{R}}_\Sigma + \frac{3M\dot{\tilde{R}}^2_\Sigma \dot{T}}{\tilde{R}_\Sigma(\tilde{R}_\Sigma - 2M)} - \frac{M(\tilde{R}_\Sigma - 2M)\dot{T}^3}{\tilde{R}^3_\Sigma}]_\Sigma, \tag{24}$$

$$\begin{aligned} K^+_{22} &= \csc^2\theta_1 K^+_{33} = \csc^2\theta_1 \csc^2\theta_2 K^+_{44} = \\ ... &= \csc^2\theta_1 \csc^2\theta_2 ... \csc^2\theta_n K^+_{n+1n+1} = (\dot{T}(\tilde{R}_\Sigma - 2M))_\Sigma. \end{aligned} \tag{25}$$

Now from continuity of extrinsic curvature (10), it follows that

$$K^+_{00} = 0, \qquad K^+_{22} = K^-_{22}. \tag{26}$$

From Eqs. (22)–(26) and (18) and (19), the matching conditions take the form

$$(\dot{b}')_\Sigma = 0, \tag{27}$$

$$M = [\frac{n-1}{2}(ab)^{n-1} - \frac{n-1}{2}a^{n-1}\dot{a}^2 b^{n+1} - \frac{n-1}{2}(ab)^{n-1}b'^2]_\Sigma. \tag{28}$$

Eqs. (18), (19), (27), and (28) are necessary and sufficient matching conditions.

## 4. Solution

For obtaining a solution, we assume constant curvature scalar $R = R_0$. Using this assumption, Eq. (4) implies that trace of energy momentum tensor $T^m$ is constant and it is possible for constant energy density, i.e. $\rho = \rho_0$. Considering these assumptions, Eqs. (6)–(8) become

$$-(n+1)\frac{\ddot{a}}{a} = \frac{1}{F(R_0)}[\kappa\rho_0 + \frac{f(R_0)}{2}], \tag{29}$$

$$\frac{\ddot{a}}{a} + n(\frac{\dot{a}}{a})^2 - n\frac{b''}{a^2 b} = -\frac{f(R_0)}{2F(R_0)}, \tag{30}$$

$$\frac{\ddot{a}}{a} + n(\frac{\dot{a}}{a})^2 - \frac{b''}{a^2 b} - (n-1)(\frac{b'}{ab})^2 + \frac{n-1}{a^2 b^2} = -\frac{f(R_0)}{2F(R_0)}. \tag{31}$$

Now from Eqs. (29)–(31), we can obtain

$$2\frac{\ddot{a}}{a}+(n-1)(\frac{\dot{a}}{a})^2+(n-1)(\frac{1-b'^2}{a^2b^2})=-\frac{1}{F(R_0)}(\frac{\kappa\rho_0}{n}+\frac{f(R_0)}{2}). \tag{32}$$

Now from Eq. (27), it follows that

$$b'=X, \tag{33}$$

where $X(r)$ is an arbitrary function. Using Eq. (33), Eq. (32) becomes

$$2\frac{\ddot{a}}{a}+(n-1)(\frac{\dot{a}}{a})^2+(n-1)(\frac{1-X^2}{a^2b^2})=-\frac{1}{F(R_0)}(\frac{\kappa\rho_0}{n}+\frac{f(R_0)}{2}). \tag{34}$$

Integrating the above equation, we get

$$(\dot{a})^2=\frac{X^2-1}{b^2}+2\frac{m(r)}{a^{n-1}b^{n+1}}-\frac{a^2}{(n+1)F(R_0)}(\frac{\kappa\rho_0}{n}+\frac{f(R_0)}{2}), \tag{35}$$

where arbitrary function $m(r)$ represents the mass and is given by

$$m(r)=\frac{\kappa\rho_0 a^{n+1}b^{n+1}}{n(n+1)F(R_0)}. \tag{36}$$

The function $m(r)$ must be positive. Using the second matching condition from Eqs. (28) and (35), we get

$$M=(n-1)m-\frac{(n-1)a^{n+1}b^{n+1}}{2(n+1)F(R_0)}(\frac{\kappa\rho_0}{n}+\frac{f(R_0)}{2}). \tag{37}$$

Now using the mass function [18], the total energy $\tilde{M}(r,t)$ for the interior spacetime is defined as

$$\tilde{M}(r,t)=(n-1)\frac{(ab)^{n-1}}{2}[1+g^{\mu\nu}(ab)_{,\mu}(ab)_{,\nu}]. \tag{38}$$

Using Eq. (35), the mass function becomes

$$\tilde{M}(r,t)=(n-1)m(r)-\frac{(n-1)a^{n+1}b^{n+1}}{2(n+1)F(R_0)}(\frac{\kappa\rho_0}{n}+\frac{f(R_0)}{2}). \tag{39}$$

Now we consider $X(r)=1$ to find the solution of Eq. (35). Eq. (35) implies that

$$\dot{a}^2b^2=\frac{4n(n+1)mF(R_0)-a^{n+1}b^{n+1}(2\kappa\rho_0+nf(R_0))}{2n(n+1)F(R_0)a^{n-1}b^{n-1}}, \tag{40}$$

and the above equation gives

$$\dot{a}b=\pm\sqrt{\frac{4n(n+1)mF(R_0)-a^{n+1}b^{n+1}(2\kappa\rho_0+nf(R_0))}{2n(n+1)F(R_0)a^{n-1}b^{n-1}}}, \tag{41}$$

for the collapsing process, taking a negative sign only, and after some simplification Eq. (41) can be written as follows:

$$\frac{\frac{n+1}{2}(ab)^{\frac{n-1}{2}}d(ab)}{\sqrt{(\sqrt{\frac{-4n(n+1)mF(R_0)}{2\kappa\rho_0+nf(R_0)}})^2+((ab)^{\frac{n+1}{2}})^2}} = -\sqrt{\frac{-(n+1)(2\kappa\rho_0+nf(R_0))}{8nF(R_0)}}dt. \tag{42}$$

Integrating the last equation with assumption $2\kappa\rho_0 + nf(R_0) < 0$, we get

$$ab = (\frac{-4n(n+1)m(r)F(R_0)}{2\kappa\rho_0+nf(R_0)})^{\frac{1}{n+1}}\sinh^{\frac{2}{n+1}}\alpha(r,t), \tag{43}$$

where

$$\alpha(r,t) = \sqrt{-\frac{(n+1)[2\kappa\rho_0+nf(R_0)]}{8nF(R_0)}}[t_s(r)-t]. \tag{44}$$

Here $t_s(r)$ is considered as an arbitrary function. For $f(R_0) \to -\frac{2\kappa\rho_0}{n}$, the solution reduces to the solution obtained by Tolman–Bondi [19]

$$ab = [\frac{(n+1)^2m(r)}{2}(t_s-t)^2]^{\frac{1}{n+1}}. \tag{45}$$

## 5. Apparent horizons

The apparent horizon is obtained when the boundary of trapped $n$-sphere with outward null normals is formed. For spacetime (1), this boundary is given by

$$g^{\mu\nu}(ab)_{,\mu}(ab)_{,\nu} = (\dot{a})^2b^2 - b'^2. \tag{46}$$

Using Eqs. (33) and (35) with assumption $X(r) = 1$ , above equation yields

$$\frac{1}{F(R_0)}[\frac{\kappa\rho_0}{n}+\frac{f(R_0)}{2}]a^{n+1}b^{n+1}+(n+1)a^{n-1}b^{n-1}-2(n+1)m = 0. \tag{47}$$

The values of $ab$ give the apparent horizons. For $f(R_0) = \frac{-2\kappa\rho_0}{n}$, we have $ab = (2m)^{\frac{1}{n-1}}$, i.e. the Schwarzschild horizon. It gives the de-Sitter horizon when $m = 0$, i.e.

$$ab = \sqrt{\frac{-(n+1)F(R_0)}{\frac{\kappa\rho_0}{n}+\frac{f(R_0)}{2}}}. \tag{48}$$

The approximate solutions of Eq. (47) up to first order in $m$ and $\frac{1}{F(R_0)}[\frac{\kappa\rho_0}{n}+\frac{f(R_0)}{2}]$ can be obtained by perturbation method. For a solution up to first order in $m$, let

$$ab = (ab)_0 + m(ab)_1 + ... \tag{49}$$

be the solution of Eq. (47). Putting Eq. (49) in Eq. (47) and comparing coefficients of power of $m$, we get

$$(ab)_0 = \Big(\frac{-2n(n+1)F(R_0)}{2\kappa\rho_0+nf(R_0)}\Big)^{\frac{1}{2}}, \tag{50}$$

and

$$(ab)_1 = 2\Big((\frac{2nF(R_0)}{2\kappa\rho_0 + nf(R_0)})\Big(\frac{-2\kappa\rho_0 + nf(R_0)}{2nF(R_0)}\Big)^{\frac{n}{2}} + \Big(\frac{-2\kappa\rho_0 + nf(R_0)}{2n(n-1)(n+1)F(R_0)}\Big)^{\frac{n-2}{2}}\Big). \tag{51}$$

Substituting the value of $(ab)_0$ and $(ab)_1$ in Eq. (49), we get

$$(ab)_c = (\frac{-2n(n+1)F(R_0)}{2\kappa\rho_0 + nf(R_0)})^{\frac{1}{2}} + 2((\frac{2nF(R_0)}{2\kappa\rho_0 + nf(R_0)})(\frac{-2\kappa\rho_0 + nf(R_0)}{2nF(R_0)})^{\frac{n}{2}} + (\frac{-2\kappa\rho_0 + nf(R_0)}{2n(n-1)(n+1)F(R_0)})^{\frac{n-2}{2}})m + ..., \tag{52}$$

and for a solution up to first order in $\frac{1}{F(R_0)}[\frac{\kappa\rho_0}{n} + \frac{f(R_0)}{2}]$, let

$$ab = (ab)_0 + \frac{1}{F(R_0)}[\frac{\kappa\rho_0}{n} + \frac{f(R_0)}{2}](ab)_1 + ... \tag{53}$$

be the solution of Eq. (47). Putting Eq.(53) in Eq. (47) and comparing coefficients of power of $\frac{1}{F(R_0)}[\frac{\kappa\rho_0}{n} + \frac{f(R_0)}{2}]$, we get

$$(ab)_0 = (2m)^{\frac{1}{n-1}}, \tag{54}$$

and

$$(ab)_1 = -\frac{(2m)^{\frac{3}{n-1}}}{(n+1)(n-1)}. \tag{55}$$

Thus, putting the value of $(ab)_0$ and $(ab)_1$ in Eq.(53), it follows that

$$(ab)_{bh} = (2m)^{\frac{1}{n-1}} - (\frac{(2m)^{\frac{3}{n-1}}}{(n-1)(n+1)})\frac{1}{F(R_0)}(\frac{\kappa\rho_0}{n} + \frac{f(R_0)}{2}) + .... \tag{56}$$

$(ab)_c$ and $(ab)_{bh}$ are called cosmological and black hole respectively. Now from Eqs. (44) and (47), the time for the formation of apparent horizon is given by

$$t_k = t_s - \sqrt{\frac{8nF(R_0)}{(n+1)(2\kappa\rho + nf(R_0))}}\sinh^{-1}[\frac{(ab_k)^{n-1}}{2m(r)} - 1]^{\frac{1}{2}}, \quad k = 1, 2. \tag{57}$$

For $f(R_0) \to -\frac{2\kappa\rho_0}{n}$, the result corresponds to the Tolman–Bondi case

$$t_k = t_s - \frac{(2^n m)^{\frac{1}{n-1}}}{n+1}. \tag{58}$$

Eq. (57) shows that both the horizons (cosmological and black hole) form before the singularity $t = t_s$ formation. Eq. (58) gives the time for the formation of apparent horizons in higher-dimensional Tolman–Bondi spacetime.

## 6. Conclusion

In this paper, we have investigated gravitational collapse in $f(R)$ gravity by considering the metric approach . We have considered the $n+2$-dimensional FRW and $n+2$-dimensional Schwarzschild spacetimes for interior and exterior regions, respectively. Matching interior and exterior regions, the matching conditions have been derived. For the solution, constant curvature scalar is assumed.

Two horizons, which are black horizon and cosmological horizon, are formed. The black hole and cosmological horizon are formed before the singularity formation. This favors the cosmic censorship conjecture. The collapsing rate is calculated from (35) as

$$\ddot{a}b = -\frac{(n-1)m}{a^n b^n} - \frac{ab}{2n(n+1)F(R_0)}[2\kappa\rho_0 + nf(R_0)]. \quad (59)$$

For the collapsing process, the acceleration should be negative, which is possible when

$$ab < [-\frac{2n(n-1)(n+1)mF(R_0)}{2\kappa\rho_0 + nf(R_0)}]^{\frac{1}{n+1}}. \quad (60)$$

It follows from Eq. (59) that the $f(R_0)$ slows the collapse when $2\kappa\rho_0 + nf(R_0) < 0$. Further, due to $f(R_0)$ there exist two physical horizons, namely block hole horizon and cosmological horizon. It is mentioned here that the effects of $f(R_0)$ are the same as the cosmological constant in general relativity [16]. For $n = 2$, our results reduced to the four-dimensional case given by Farasat et al. [11]. Therefore, this work is a generalization of Farasat et al. [11].

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