Turkish Journal of Mathematics

Volume 44 | Number 5 Article 5

1-1-2020

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Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2020) 44: 1578 – 1594
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doi:10.3906/mat-1911-35

Linear mappings satisfying some recursive sequences

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Received: 10.11.2019 • **Accepted/Published Online:** 07.06.2020 • **Final Version:** 21.09.2020

Abstract: Let $\mathcal{A}$ be a unital, complex normed $*$-algebra with the identity element $\mathbf{e}$ such that the set of all algebraic elements of $\mathcal{A}$ is norm dense in the set of all self-adjoint elements of $\mathcal{A}$ and let $\{D_n\}_{n=0}^{\infty}$ and $\{\Delta_n\}_{n=0}^{\infty}$ be sequences of continuous linear mappings on $\mathcal{A}$ satisfying

$$\begin{cases} D_{n+1}(p) = \sum_{k=0}^{n} D_{n-k}(p)D_k(p), \\ \Delta_{n+1}(p) = \sum_{k=0}^{n} \Delta_{n-k}(p)D_k(p), \end{cases}$$

for all projections p of $\mathcal{A}$ and all nonnegative integers n. Moreover, suppose that $D_0(p) = D_0(p)^2$ holds for all projections p of $\mathcal{A}$. Then

$$\Delta_n = \frac{C_n}{2}\left(R_{D_0(\mathbf{e})}\Delta_0 + L_{\Delta_0(\mathbf{e})}D_0\right)$$

for all $n \in \mathbb{N}$, where C_n denotes the nth Catalan number and $R_{D_0(\mathbf{e})}(a) = aD_0(\mathbf{e})$ and $L_{\Delta_0(\mathbf{e})}(a) = \Delta_0(\mathbf{e})a$ for all $a \in \mathcal{A}$. Using this result, we present a characterization of left τ-centralizers satisfying a certain recursive relation. In addition, a characterization of generalized higher derivations is presented. Moreover, we show that higher derivations, prime higher derivations, left higher derivations, and σ-derivations are zero under certain conditions.

Key words: Recursive sequence, $*$-algebra, derivation, left τ-centralizer, higher derivation, prime higher derivation

1. Introduction and preliminaries

Today, the use of recursive sequences is seen in most applied fields, such as the number theory, combinatorial theory, and computer sciences. In combinatorial theory, the Catalan numbers form a sequence of natural numbers that occur in various counting problems, often involving recursively-defined objects; for more details, see [2, 12, 13] and references therein. Thus, solving these sequences in Banach algebras and C^*-algebras seems to be useful and necessary. The main purpose of this paper is to solve some recursive sequences in some $*$-algebras.

We first introduce some basic notations that play a fundamental role in what follows. Throughout the paper, let $\mathcal{A}$ be a unital complex normed $*$-algebra with the identity element $\mathbf{e}$. By $\mathcal{A}_{sa}$, we denote the set of all self-adjoint elements of $\mathcal{A}$ (i.e. $\mathcal{A}_{sa} = \{a \in \mathcal{A} \mid a^* = a\}$), and the set of all projections of $\mathcal{A}$ is denoted by $P(\mathcal{A})$ (i.e., $P(\mathcal{A}) = \{p \in \mathcal{A} \mid p^2 = p,\ p^* = p\}$). An element of $\mathcal{A}$ is usually called an algebraic element if it

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2010 *AMS Mathematics Subject Classification:* 47C10, 47C15, 46L57, 03D20

can be written as a finite real-linear combination of mutually orthogonal projections of $\mathcal{A}$. We denote the set of all algebraic elements of $\mathcal{A}$ by $\mathfrak{O}_{\mathcal{A}}$. Thus, we have $P_{\mathcal{A}} \subseteq \mathfrak{O}_{\mathcal{A}} \subseteq \mathcal{A}_{sa}$. If $\mathcal{A}$ is a von Neumann algebra, then $\mathfrak{O}_{\mathcal{A}}$ is norm dense in $\mathcal{A}_{sa}$, that is, $\overline{\mathfrak{O}_{\mathcal{A}}} = \mathcal{A}_{sa}$. More generally, the same is true for AW^*-algebras. For more material about C^*-algebras and W^*-algebras, see, for example, [8, 11, 14] and the references therein.

Let a be an arbitrary element of an algebra $\mathcal{A}$. We define the linear mappings $L_a, R_a : \mathcal{A} \to \mathcal{A}$ by $L_a(b) = ab$ and $R_a(b) = ba$ $(b \in \mathcal{A})$. A straightforward verification shows that $R_b L_a = L_a R_b$ for all $a, b \in \mathcal{A}$. One of the main goals of this research is to show that left τ-centralizers are the solutions of certain recursive sequences. Let $\mathcal{A}$ be an algebra and let τ be an endomorphism of $\mathcal{A}$. A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a left (resp. right) τ-centralizer if $f(ab) = f(a)\tau(b)$ (resp. $f(ab) = \tau(a)f(b)$) for all $a, b \in \mathcal{A}$. A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a Jordan left (resp. right) τ-centralizer of $\mathcal{A}$ if $f(a^2) = f(a)\tau(a)$ (resp. $f(a^2) = \tau(a)f(a)$) for all $a \in \mathcal{A}$. For more details about τ-centralizers and Jordan τ-centralizers, see, e.g. [1, 3]. Let $\mathbb{N}_k = \{k, k+1, k+2, ...\}$ for all $k \in \{0, 1, 2, ...\}$. Our main result regarding τ-centralizers says that let $\mathcal{A}$ be a unital, complex normed $*$-algebra with the identity element $\mathbf{e}$ such that $\mathfrak{O}_{\mathcal{A}}$ is norm dense in $\mathcal{A}_{sa}$ and let the sequences $\{D_n\}_{n=0}^{\infty}$ and $\{\Delta_n\}_{n=0}^{\infty}$ satisfy

$$\begin{cases} D_{n+1}(p) = \sum_{k=0}^{n} D_{n-k}(p) D_k(p), \\ \Delta_{n+1}(p) = \sum_{k=0}^{n} \Delta_{n-k}(p) D_k(p), \end{cases}$$

for all $p \in P(\mathcal{A})$ and all $n \in \mathbb{N}_0$. Let $\Delta_0(p) = \Delta_0(p) D_0(p)$ for all $p \in P(\mathcal{A})$. Suppose $\mathcal{A}$ is semiprime and D_0 is a surjective endomorphism such that $D_0(Z(\mathcal{A})) = Z(\mathcal{A})$, where $Z(\mathcal{A})$ denotes the center of $\mathcal{A}$. Then Δ_n is a left D_0-centralizer on $\mathcal{A}$ for all $n \in \mathbb{N}_0$.

In this article, we also present a characterization of generalized higher derivations via generating functions. As another purpose of this study, we present some conditions under which higher derivations, prime higher derivations, and left higher derivations are identically zero. For instance, our result concerning higher derivations is as follows. Let $\mathcal{A}$ be a unital, prime complex normed $*$-algebra such that $\mathfrak{O}_{\mathcal{A}}$ is norm dense in $\mathcal{A}_{sa}$. If $\{d_n\}_{n=0}^{\infty}$ is a bounded higher derivation such that $d_n(p) \in Z(\mathcal{A})$ for all $n \in \mathbb{N}$ and all $p \in P(\mathcal{A})$, then $d_n = 0$ for all $n \in \mathbb{N}$. Some other related results are also discussed.

2. Results and proofs

Let $\mathfrak{O}_{\mathcal{A}}$, $\mathcal{A}_{sa}$, $P(\mathcal{A})$, and $\mathbb{N}_k$ be the symbols which have been introduced in the Introduction. Throughout this section, without further mention, $\mathbf{e}$ stands for the identity element of any unital algebra and any unital ring. We use C_n to denote the nth Catalan number and we know that

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

for all $n \in \mathbb{N}_0$.

Our first theorem, which has been motivated by [4] is as follows.

Theorem 2.1 *Let $\mathcal{A}$ be a unital, complex normed $*$-algebra such that $\mathfrak{O}_{\mathcal{A}}$ is norm dense in $\mathcal{A}_{sa}$ and let*

$\{D_n\}_{n=0}^{\infty}$ and $\{\Delta_n\}_{n=0}^{\infty}$ be sequences of continuous linear mappings on $\mathcal{A}$ satisfying

$$\begin{cases} D_{n+1}(p) = \sum_{k=0}^{n} D_{n-k}(p)D_k(p), \\ \Delta_{n+1}(p) = \sum_{k=0}^{n} \Delta_{n-k}(p)D_k(p), \end{cases}$$

for all $p \in P(\mathcal{A})$ and all $n \in \mathbb{N}_0$. Moreover, suppose that $D_0(p) = D_0(p)^2$ for all $p \in P(\mathcal{A})$. Then

$$\Delta_n = \frac{C_n}{2}\left(R_{D_0(e)}\Delta_0 + L_{\Delta_0(e)}D_0\right)$$

for all $n \in \mathbb{N}$.

Proof By induction, we obtain that

$$\begin{cases} D_n(p) = C_n D_0(p), \\ \Delta_n(p) = C_n\Delta_0(p)D_0(p) \end{cases}$$

for all $n \in \mathbb{N}$. Let p and q be orthogonal projections of $\mathcal{A}$. Clearly, $p+q$ is a projection and we have

$$\Delta_n(p+q) = \Delta_n(p) + \Delta_n(q). \tag{2.1}$$

On the other hand, we have

$$\begin{aligned} \Delta_n(p+q) &= \Delta_n((p+q)^2) = C_n\Delta_0(p+q)D_0(p+q) \\ &= C_n\Delta_0(p)D_0(p) + C_n\Big(\Delta_0(p)D_0(q) + \Delta_0(q)D_0(p)\Big) + C_n\Delta_0(q)D_0(q) \\ &= \Delta_n(p) + C_n\Big(\Delta_0(p)D_0(q) + \Delta_0(q)D_0(p)\Big) + \Delta_n(q), \end{aligned}$$

which means that

$$\Delta_n(p+q) = \Delta_n(p) + C_n\Big(\Delta_0(p)D_0(q) + \Delta_0(q)D_0(p)\Big) + \Delta_n(q). \tag{2.2}$$

Comparing (2.1) and (2.2), we get that

$$\Delta_0(p)D_0(q) + \Delta_0(q)D_0(p) = 0.$$

Therefore, if $p_1, \ldots, p_m$ are projections of $\mathcal{A}$ such that $p_ip_j = p_jp_i = 0$ $(i \neq j)$, then

$$\Delta_0(p_i)D_0(p_j) + \Delta_0(p_j)D_0(p_i) = 0 \tag{2.3}$$

for all $i, j \in \{1, \ldots, m\}$ with $i \neq j$. Assume that a is an arbitrary algebraic element of $\mathcal{A}$. Therefore, $a = \sum_{i=1}^{m} \lambda_i p_i$ for some mutually orthogonal projections $p_1, \ldots, p_m$ of $\mathcal{A}$. We know that $\Delta_n(p) = C_n\Delta_0(p)D_0(p)$ for all $p \in P(\mathcal{A})$ and all $n \in \mathbb{N}$. Therefore, we have

$$\Delta_n(a^2) = \Delta_n\Big(\sum_{i=1}^{m} \lambda_i^2 p_i\Big) = \sum_{i=1}^{m} \lambda_i^2 \Delta_n(p_i) = C_n \sum_{i=1}^{m} \lambda_i^2 \Delta_0(p_i)D_0(p_i). \tag{2.4}$$

On the other hand, using (2.3) and (2.4), we have

$$\begin{aligned}
C_n\Delta_0(a)D_0(a) &= C_n\Delta_0\Big(\sum_{i=1}^{m}\lambda_i p_i\Big)D_0\Big(\sum_{i=1}^{m}\lambda_i p_i\Big)\\
&= C_n\sum_{i=1}^{m}\lambda_i^2\Delta_0(p_i)D_0(p_i)\\
&\quad + C_n\sum_{j=1}^{m-1}\sum_{i=j+1}^{m}\lambda_i\lambda_j\Bigg(\Delta_0(p_i)D_0(p_j)+\Delta_0(p_j)D_0(p_i)\Bigg)\\
&= C_n\sum_{i=1}^{m}\lambda_i^2\Delta_0(p_i)D_0(p_i)\\
&= \Delta_n(a^2).
\end{aligned}$$

Hence, we see that $\Delta_n(a^2) = C_n\Delta_0(a)D_0(a)$ for all $a \in \mathfrak{O}_{\mathcal{A}}$ and all $n \in \mathbb{N}$. Since every self-adjoint element of $\mathcal{A}$ can be approximated in norm by algebraic elements and Δ_n and D_n are continuous (equivalently, bounded) linear mappings for all $n \in \mathbb{N}_0$, we obtain that

$$\Delta_n(a^2) = C_n\Delta_0(a)D_0(a) \quad (a \in \mathcal{A}_{sa}, n \in \mathbb{N}). \tag{2.5}$$

Let a_1 and a_2 be two arbitrary self-adjoint elements of $\mathcal{A}$. Replacing a by $a_1 + a_2$ in (2.5) and then using that, we arrive at

$$\Delta_n(a_1a_2 + a_2a_1) = C_n\Delta_0(a_1)D_0(a_2) + C_n\Delta_0(a_2)D_0(a_1)$$

for all $a_1, a_2 \in \mathcal{A}_{sa}$. Let a be an arbitrary element of $\mathcal{A}$. Then there exist two self-adjoint elements a_1, a_2 of $\mathcal{A}$ such that $a = a_1 + ia_2$. Hence,

$$\begin{aligned}
\Delta_n(a^2) &= \Delta_n\Big(a_1^2 - a_2^2 + i(a_1a_2 + a_2a_1)\Big)\\
&= C_n\Delta_0(a_1)D_0(a_1) - C_n\Delta_0(a_2)D_0(a_2)\\
&\quad + iC_n\Big(\Delta_0(a_1)D_0(a_2) + \Delta_0(a_2)D_0(a_1)\Big)\\
&= C_n\Delta_0(a_1 + ia_2)D_0(a_1 + ia_2)\\
&= C_n\Delta_0(a)D_0(a),
\end{aligned}$$

which means that

$$\Delta_n(a^2) = C_n\Delta_0(a)D_0(a) \quad (a \in \mathcal{A},\ n \in \mathbb{N}). \tag{2.6}$$

Replacing a by $a + b$ in (2.6) and using that, we have

$$\Delta_n(ab + ba) = C_n\Big(\Delta_0(a)D_0(b) + \Delta_0(b)D_0(a)\Big) \quad (a, b \in \mathcal{A},\ n \in \mathbb{N}).$$

Considering $b = \mathbf{e}$ in the previous equation, we have

$$\begin{aligned}\Delta_n(a) &= \frac{C_n}{2}\Big(\Delta_0(a)D_0(\mathbf{e}) + \Delta_0(\mathbf{e})D_0(a)\Big)\\ &= \frac{C_n}{2}\Big(R_{D_0(\mathbf{e})}\Delta_0(a) + L_{\Delta_0(\mathbf{e})}D_0(a)\Big)\\ &= \frac{C_n}{2}\Big(R_{D_0(\mathbf{e})}\Delta_0 + L_{\Delta_0(\mathbf{e})}D_0\Big)(a),\end{aligned}$$

which means that

$$\Delta_n = \frac{C_n}{2}\Big(R_{D_0(\mathbf{e})}\Delta_0 + L_{\Delta_0(\mathbf{e})}D_0\Big) \qquad (n \in \mathbb{N}). \tag{2.7}$$

Now, we show that $\Delta_n = C_n\Delta_1$ for all $n \in \mathbb{N}$. It follows from (2.6) that $\Delta_1(a^2) = \Delta_0(a)D_0(a)$ for all $a \in \mathcal{A}$. Thus, we have

$$\Delta_n(a^2) = C_n\Delta_1(a^2) \qquad (a \in \mathcal{A},\ n \in \mathbb{N}). \tag{2.8}$$

Putting $a + b$ instead of a in (2.8), we get that

$$\Delta_n(ab + ba) = C_n\Delta_1(ab + ba) \quad (n \in \mathbb{N}).$$

Letting $b = \mathbf{e}$ in the previous equation, we find that $\Delta_n(a) = C_n\Delta_1(a)$ for all $a \in \mathcal{A}$ and all $n \in \mathbb{N}$. This completes the proof. □

We denote the center of an algebra $\mathcal{A}$ by $Z(\mathcal{A})$. Recall that $Z(\mathcal{A}) = \{c \in \mathcal{A} \mid ac = ca \ for\ all\ a \in \mathcal{A}\}$.

The following is the immediate consequence of Theorem 2.1.

Corollary 2.2 *Let $\mathcal{A}$ be a unital, complex normed $*$-algebra such that $\mathfrak{D}_{\mathcal{A}}$ is norm dense in $\mathcal{A}_{sa}$ and let the sequences $\{D_n\}_{n=0}^{\infty}$ and $\{\Delta_n\}_{n=0}^{\infty}$ satisfy all the conditions of the above theorem. Let $\Delta_0(p) = \Delta_0(p)D_0(p)$ for all projections $p \in \mathcal{A}$. Suppose $\mathcal{A}$ is semiprime and D_0 is a surjective endomorphism such that $D_0(Z(\mathcal{A})) = Z(\mathcal{A})$. Then Δ_n is a left D_0-centralizer on $\mathcal{A}$ for all $n \in \mathbb{N}_0$.*

Proof Using an argument similar to the proof of Theorem 2.1, we can show that Δ_0 is a Jordan left D_0-centralizer on $\mathcal{A}$, that is, $\Delta_0(a^2) = \Delta_0(a)D_0(a)$ for all $a \in \mathcal{A}$. By [3, Theorem 1], Δ_0 is a left D_0-centralizer, that is, $\Delta_0(ab) = \Delta_0(a)D_0(b)$ for all $a, b \in \mathcal{A}$. Now, we define $\Phi_n = \Delta_n - \frac{C_n}{2}R_{D_0(\mathbf{e})}\Delta_0$ for all $n \in \mathbb{N}$. It follows from (2.7) that $\Phi_n = \frac{C_n}{2}L_{\Delta_0(\mathbf{e})}D_0$ for all $n \in \mathbb{N}$. Then we have the following equalities:

$$\Phi_n(ab) = \frac{C_n}{2}L_{\Delta_0(\mathbf{e})}D_0(ab) = \frac{C_n}{2}L_{\Delta_0(\mathbf{e})}D_0(a)D_0(b) = \Phi_n(a)D_0(b),$$

which means that Φ_n is a left D_0-centralizer for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}\Big(\Delta_n(a) - \frac{C_n}{2}R_{D_0(\mathbf{e})}\Delta_0(a)\Big)D_0(b) &= \Phi_n(a)D_0(b) = \Phi_n(ab)\\ &= \Delta_n(ab) - \frac{C_n}{2}R_{D_0(\mathbf{e})}\Delta_0(ab)\\ &= \Delta_n(ab) - \frac{C_n}{2}R_{D_0(\mathbf{e})}\Delta_0(a)D_0(b).\end{aligned}$$

Therefore, we get that $\Delta_n(ab) = \Delta_n(a)D_0(b)$ for all $a, b \in \mathcal{A}$ and all $n \in \mathbb{N}_0$. It means that Δ_n is a left D_0-centralizer for all $n \in \mathbb{N}_0$. □

Let $\mathcal{A}$ be an algebra. Recall that a sequence $\{d_n\}$ of linear mappings from $\mathcal{A}$ into itself is a higher derivation if $d_n(ab) = \sum_{k=0}^{n} d_{n-k}(a)d_k(b)$ for all $a, b \in \mathcal{A}$ and for each $n \in \mathbb{N}_0$. Moreover, a sequence $\{f_n\}$ of linear mappings from $\mathcal{A}$ into itself is called a generalized higher derivation if there exists a higher derivation $\{d_n\}$ such that $f_n(ab) = \sum_{k=0}^{n} f_{n-k}(a)d_k(b)$ for all $a, b \in \mathcal{A}$ and for each $n \in \mathbb{N}_0$. In the next theorem, we present a characterization of generalized higher derivations on the complex normed $*$-algebras under certain conditions. In the proof of the next theorem, generating functions play an essential rule. An interesting connection between higher derivations and generating functions was given by Miller [9].

Theorem 2.3 *Let $\mathcal{A}$ be a semiprime Banach $*$-algebra such that $\mathfrak{O}_{\mathcal{A}}$ is norm dense in $\mathcal{A}_{sa}$ and let $\{d_n\}_{n=0}^{\infty}$ and $\{f_n\}_{n=0}^{\infty}$ be two uniformly bounded sequences of linear mappings with $d_0 = f_0 = I$, where I is the identity mapping on $\mathcal{A}$. If*

$$\begin{cases} f_n(p) = \sum_{k=0}^{n} f_{n-k}(p)d_k(p), \\ d_n(p) = \sum_{k=0}^{n} d_{n-k}(p)d_k(p), \end{cases}$$

for all $p \in P(\mathcal{A})$ and all $n \in \mathbb{N}_0$, then $\{f_n\}$ is a generalized higher derivation associated with the higher derivation $\{d_n\}$ on $\mathcal{A}$.

Proof Since $\{f_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=0}^{\infty}$ are uniformly bounded sequences of linear mappings, there exist two positive real numbers M_1 and M_2 such that $\|f_n\| \leq M_1$ and $\|d_n\| \leq M_2$ for all $n \in \mathbb{N}_0$. For any $t \in (-1, 1)$ and any $a \in \mathcal{A}$, we have

$$\Big\| \sum_{n=0}^{\infty} f_n(a)t^n \Big\| \leq \sum_{n=0}^{\infty} \|f_n\|\|a\||t|^n \leq M_1\|a\| \sum_{n=0}^{\infty} |t|^n = \frac{M_1\|a\|}{1-|t|} < \infty.$$

Similarly, we have $\Big\| \sum_{n=0}^{\infty} d_n(a)t^n \Big\| < \infty$ for all $a \in \mathcal{A}$ and for each $t \in (-1, 1)$. Thus, we can define the functions $\alpha, \beta : (-1, 1) \to B(\mathcal{A})$ (the set of all bounded linear mappings from $\mathcal{A}$ into itself) by $\alpha_t = \alpha(t) = \sum_{n=0}^{\infty} f_n t^n$ and $\beta_t = \beta(t) = \sum_{n=0}^{\infty} d_n t^n$, respectively. Note that $\alpha_t(a) = \sum_{n=0}^{\infty} f_n(a)t^n$ and $\beta_t(a) = \sum_{n=0}^{\infty} d_n(a)t^n$ for all $a \in \mathcal{A}$. We continue the proof similar to the proof of Theorem 2.1. Let p be an arbitrary projection of $\mathcal{A}$. Then

$$\begin{aligned} \alpha_t(p)\beta_t(p) &= \left(\sum_{n=0}^{\infty} f_n(p)t^n \right) \left(\sum_{n=0}^{\infty} d_n(p)t^n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} f_{n-k}(p)d_k(p) \right) t^n \\ &= \sum_{n=0}^{\infty} f_n(p)t^n = \alpha_t(p), \end{aligned}$$

which means that

$$\alpha_t(p) = \alpha_t(p)\beta_t(p) \tag{2.9}$$

for all $p \in P(\mathcal{A})$ and for each $t \in (-1, 1)$. Let p and q be orthogonal projections of $\mathcal{A}$; then $p + q$ is a projection and we have

$$\alpha_t(p + q) = \alpha_t(p) + \alpha_t(q). \tag{2.10}$$

On the other hand, we have

$$\begin{aligned}\alpha_t(p+q) &= \alpha_t((p+q)^2) = \alpha_t(p+q)\beta_t(p+q)\\ &= \alpha_t(p)\beta_t(p) + \alpha_t(p)\beta_t(q) + \alpha_t(q)\beta_t(p) + \alpha_t(q)\beta_t(q)\\ &= \alpha_t(p) + \alpha_t(p)\beta_t(q) + \alpha_t(q)\beta_t(p) + \alpha_t(q),\end{aligned}$$

which means that

$$\alpha_t(p+q) = \alpha_t(p) + \alpha_t(p)\beta_t(q) + \alpha_t(q)\beta_t(p) + \alpha_t(q) \tag{2.11}$$

for each $t \in (-1,1)$. Comparing (2.10) and (2.11), we obtain that

$$\alpha_t(p)\beta_t(q) + \alpha_t(q)\beta_t(p) = 0.$$

Let a be an arbitrary algebraic element of $\mathcal{A}$. Therefore, $a = \sum_{i=1}^m \lambda_i p_i$ for some mutually orthogonal projections $p_1, \ldots, p_m$ of $\mathcal{A}$. Therefore, we have

$$\alpha_t(p_i)\beta_t(p_j) + \alpha_t(p_j)\beta_t(p_i) = 0, \tag{2.12}$$

for all $i, j \in \{1, 2, \ldots, m\}$ with $i \neq j$ and for each $t \in (-1,1)$. Note that $a^2 = \sum_{i=1}^m \lambda_i^2 p_i$. Thus, we have

$$\alpha_t(a^2) = \alpha_t\Big(\sum_{i=1}^m \lambda_i^2 p_i\Big) = \sum_{i=1}^m \lambda_i^2 \alpha_t(p_i) = \sum_{i=1}^m \lambda_i^2 \alpha_t(p_i)\beta_t(p_i). \tag{2.13}$$

On the other hand, using Equations (2.12) and (2.13), we have

$$\begin{aligned}\alpha_t(a)\beta_t(a) &= \alpha_t\Big(\sum_{i=1}^m \lambda_i p_i\Big)\beta_t\Big(\sum_{i=1}^m \lambda_i p_i\Big)\\ &= \sum_{i=1}^m \lambda_i^2 \alpha_t(p_i)\beta_t(p_i) + \sum_{j=1}^{m-1}\sum_{i=j+1}^m \lambda_i\lambda_j\Bigg(\alpha_t(p_i)\beta_t(p_j) + \alpha_t(p_j)\beta_t(p_i)\Bigg)\\ &= \sum_{i=1}^m \lambda_i^2 \alpha_t(p_i)\beta_t(p_i)\\ &= \alpha_t(a^2).\end{aligned}$$

Hence, we see that $\alpha_t(a^2) = \alpha_t(a)\beta_t(a)$ for all $a \in \mathfrak{D}_{\mathcal{A}}$. Since every self-adjoint element of $\mathcal{A}$ can be approximated in norm by algebraic elements and α_t and β_t are continuous linear mappings for any $t \in (-1,1)$, we obtain that $\alpha_t(a^2) = \alpha_t(a)\beta_t(a)$ for all $a \in \mathcal{A}_{sa}$. Replacing a by $a_1 + a_2$ in the previous equation, we get that $\alpha_t(a_1a_2 + a_2a_1) = \alpha_t(a_1)\beta_t(a_2) + \alpha_t(a_2)\beta_t(a_1)$ for all $a_1, a_2 \in \mathcal{A}_{sa}$. Let a be an arbitrary element of $\mathcal{A}$. Then there exist two self-adjoint elements a_1 and a_2 of $\mathcal{A}$ such that $a = a_1 + ia_2$. Hence,

$$\begin{aligned}\alpha_t(a^2) &= \alpha_t\Big(a_1^2 - a_2^2 + i(a_1a_2 + a_2a_1)\Big)\\ &= \alpha_t(a_1)\beta_t(a_1) - \alpha_t(a_2)\beta_t(a_2) + i\Big(\alpha_t(a_1)\beta_t(a_2) + \alpha_t(a_2)\beta_t(a_1)\Big)\\ &= \alpha_t(a_1 + ia_2)\beta_t(a_1 + ia_2)\\ &= \alpha_t(a)\beta_t(a),\end{aligned}$$

which means that $\alpha_t(a^2) = \alpha_t(a)\beta_t(a)$ for all $a \in \mathcal{A}$ and for each $t \in (-1,1)$. Thus, we have

$$\sum_{n=0}^{\infty} f_n(a^2)t^n = \alpha_t(a^2) = \alpha_t(a)\beta_t(a) = \left(\sum_{n=0}^{\infty} f_n(a)t^n\right)\left(\sum_{n=0}^{\infty} d_n(a)t^n\right)$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} f_{n-k}(a)d_k(a)t^n.$$

We simply conclude that

$$f_n(a^2) = \sum_{k=0}^{n} f_{n-k}(a)d_k(a) \tag{2.14}$$

for all $a \in \mathcal{A}$. Reasoning like above, we can show that $\beta_t(a^2) = \beta_t(a)^2$ for all $a \in \mathcal{A}$ and for each $t \in (-1,1)$. Hence, we can obtain that

$$d_n(a^2) = \sum_{k=0}^{n} d_{n-k}(a)d_k(a) \tag{2.15}$$

for all $a \in \mathcal{A}$. It follows from (2.14) and (2.15) that $\{f_n\}$ is a generalized Jordan higher derivation associated with the Jordan higher derivation $\{d_n\}$. Theorems 2.15 and 2.16 of [15] together show that $\{f_n\}$ is a generalized higher derivation associated with the higher derivation $\{d_n\}$. Thereby, we get the required result. □

In the following, we are going to prove that every higher derivation, prime higher derivation, and left higher derivation of continuous linear mappings on a unital, prime complex normed $*$-algebra is identically zero under mid conditions.

Given an integer $n \geq 2$, a ring $\mathcal{R}$ is said to be n-torsion free, if for $x \in \mathcal{R}$, $nx = 0$ implies $x = 0$. Recall that a ring $\mathcal{R}$ is prime if for $a, b \in \mathcal{R}$, $a\mathcal{R}b = \{0\}$ implies $a = 0$ or $b = 0$. To show that every higher derivation is zero under certain conditions, we first establish the auxiliary result below.

Lemma 2.4 *Let $\mathcal{R}$ be a unital 2-torsion free prime ring containing the element $\frac{\mathbf{e}}{2}$ and let $\{a_n\}_{n=0}^{\infty}$ be a sequence of $\mathcal{R}$ satisfying $a_n = \sum_{k=0}^{n} a_{n-k}a_k$ for all $n \in \mathbb{N}$. If $a_n \in Z(\mathcal{R})$ $(n \in \mathbb{N})$, then $a_n = 0$ for all $n \in \mathbb{N}$.*

Proof Since $a_n \in Z(\mathcal{R})$ for all $n \in \mathbb{N}$, we have $a_1 = a_1a_0 + a_0a_1 = 2a_1a_0$. Thus, we have

$$\begin{aligned} 0 &= a_1 - 2a_1a_0 \\ &= 2a_1\frac{\mathbf{e}}{2} - 2a_1a_0 \\ &= 2a_1\left(\frac{\mathbf{e}}{2} - a_0\right), \end{aligned}$$

which means that

$$2a_1\left(\frac{\mathbf{e}}{2} - a_0\right) = 0. \tag{2.16}$$

Multiplying (2.16) from the left by an arbitrary element $a \in \mathcal{R}$, we have $2a_1a\left(\frac{\mathbf{e}}{2} - a_0\right) = 0$ and since $\mathcal{R}$ is a 2-torsion free prime ring, we obtain that $a_0 = \frac{\mathbf{e}}{2}$ or $a_1 = 0$. We investigate the following three cases:

(1) $a_0 \neq \frac{\mathbf{e}}{2}$ and $a_1 = 0$;
(2) $a_0 = \frac{\mathbf{e}}{2}$ and $a_1 \neq 0$;
(3) $a_0 = \frac{\mathbf{e}}{2}$ and $a_1 = 0$.

Case 1. Suppose that $a_0 \neq \frac{\mathbf{e}}{2}$ and $a_1 = 0$. Therefore, we have $a_2 = a_2a_0 + a_1^2 + a_0a_2 = 2a_2a_0$. Reasoning like above, we see that

$$\begin{aligned} 0 &= a_2 - 2a_2a_0 \\ &= 2a_2\frac{\mathbf{e}}{2} - 2a_2a_0 \\ &= 2a_2\left(\frac{\mathbf{e}}{2} - a_0\right), \end{aligned}$$

which means that

$$2a_2\left(\frac{\mathbf{e}}{2} - a_0\right) = 0. \tag{2.17}$$

Multiplying (2.17) from the left by an arbitrary element $a \in \mathcal{R}$, we have $2a_2a\left(\frac{\mathbf{e}}{2} - a_0\right) = 0$ and since $\mathcal{R}$ is a prime ring, we obtain that $a_0 = \frac{\mathbf{e}}{2}$, which is a contradiction. Therefore, $a_2 = 0$. Continuing this procedure, we get that $a_n = 0$ for all $n \in \mathbb{N}$.

Case 2. Suppose that $a_0 = \frac{\mathbf{e}}{2}$ and $a_1 \neq 0$. We have

$$\begin{aligned} a_2 &= a_2a_0 + a_1^2 + a_0a_2 \\ &= a_2 + a_1^2, \end{aligned}$$

which means that $a_1^2 = 0$. Since $a_n \in Z(\mathcal{R})$ for all $n \in \mathbb{N}$ and $\mathcal{R}$ is prime, we have $a_1 = 0$, which is a contradiction.

Case 3. Suppose that $a_0 = \frac{\mathbf{e}}{2}$ and that $a_1 = 0$. We have

$$\begin{aligned} a_4 &= a_4a_0 + a_1a_3 + a_2^2 + a_3a_1 + a_4a_0 \\ &= a_4\frac{\mathbf{e}}{2} + 0 + a_2^2 + 0 + a_4\frac{\mathbf{e}}{2} \\ &= a_4 + a_2^2, \end{aligned}$$

which means that $a_2^2 = 0$. Since $a_n \in Z(\mathcal{R})$ for all $n \in \mathbb{N}$ and $\mathcal{R}$ is prime, $a_2 = 0$. We show that $a_n = 0$ for all $n \geq 2$. Suppose that $a_1, \ldots, a_{n-1} = 0$ for all $n \in \mathbb{N}_2$. Therefore, we have

$$\begin{aligned} a_{2n} &= a_{2n}a_0 + a_{2n-1}a_1 + \cdots + a_{n+1}a_{n-1} + a_n^2 + a_{n-1}a_{n+1} + \cdots + a_1a_{2n-1} + a_0a_{2n} \\ &= a_{2n}\frac{\mathbf{e}}{2} + a_n^2 + \frac{\mathbf{e}}{2}a_{2n} \\ &= a_{2n} + a_n^2, \end{aligned}$$

which means that $a_n^2 = 0$. Since $a_n \in Z(\mathcal{R})$ for all $n \in \mathbb{N}$ and $\mathcal{R}$ is prime, $a_n = 0$. This proves the lemma completely. □

Now we are ready to prove the following theorem.

Theorem 2.5 *Let $\mathcal{A}$ be a unital, prime complex normed $*$-algebra such that $\mathfrak{O}_{\mathcal{A}}$ is norm dense in $\mathcal{A}_{sa}$ and let $\{d_n\}_{n=0}^{\infty}$ be a sequence of continuous linear mappings satisfying $d_n(p) = \sum_{k=0}^{n} d_{n-k}(p)d_k(p)$ for all $p \in P(\mathcal{A})$ and all $n \in \mathbb{N}$. If $d_n(p) \in Z(\mathcal{A})$ for all $p \in P(\mathcal{A})$ and all $n \in \mathbb{N}$, then $d_n = 0$ for all $n \in \mathbb{N}$.*

Proof Putting $a_n = d_n(p)$, where p is an arbitrary projection of $\mathcal{A}$, we get that $a_n = \sum_{k=0}^{n} a_{n-k}a_k$ for all $n \in \mathbb{N}$. Now, all the conditions of Lemma 2.4 are fulfilled, and we achieve that $0 = a_n = d_n(p)$ for all $p \in P(\mathcal{A})$ and all $n \in \mathbb{N}$. Let a be an arbitrary algebraic element of $\mathcal{A}$. Then, $a = \sum_{i=1}^{m} \lambda_i p_i$ for some mutually orthogonal projections $p_1, p_2, \dots, p_m$ of $\mathcal{A}$. Hence, we have

$$d_n(a) = d_n(\sum_{i=1}^{m} \lambda_i p_i) = \sum_{i=1}^{m} \lambda_i d_n(p_i) = 0.$$

Since every self-adjoint element in $\mathcal{A}$ can be approximated in norm by algebraic elements and every d_n is a continuous linear mapping, we have $d_n(a) = 0$ for all $a \in \mathcal{A}_{sa}$. Let a be an arbitrary element of $\mathcal{A}$. Then there exist two self-adjoint elements a_1 and a_2 of $\mathcal{A}$ such that $a = a_1 + ia_2$. Hence, $d_n(a) = d_n(a_1 + ia_2) = d_n(a_1) + id_n(a_2) = 0$ for all $a \in \mathcal{A}$, which means that $d_n = 0$ for all $n \in \mathbb{N}$, as desired. □

The following corollary provides the conditions under which a σ-derivation is zero.

Corollary 2.6 *Let $\mathcal{A}$ be a unital, prime complex normed $*$-algebra such that $\mathfrak{O}_{\mathcal{A}}$ is norm dense in $\mathcal{A}_{sa}$ and let $d : \mathcal{A} \to \mathcal{A}$ be a continuous σ-derivation such that $d\sigma = \sigma d = d$ and $\sigma^2 = \sigma$. If $d(p) \in Z(\mathcal{A})$ for all $p \in P(\mathcal{A})$, then d is identically zero.*

Proof One can easily prove that for each $n \in \mathbb{N}$,

$$d^n(ab) = \sum_{k=0}^{n} \binom{n}{k} d^{n-k}(a)d^k(b)$$

for all $a, b \in \mathcal{A}$, where $d^0 = \sigma$. If $D_n = \frac{d^n}{n!}$, then $D_n(ab) = \sum_{k=0}^{n} D_{n-k}(a)D_k(b)$ for all $a, b \in \mathcal{A}$. Therefore, $D_n(p) = \sum_{k=0}^{n} D_{n-k}(p)D_k(p)$ for all $p \in P(\mathcal{A})$ and all $n \in \mathbb{N}$. Since d is continuous and $d(p) \in Z(\mathcal{A})$ for all $p \in P(\mathcal{A})$ and also since $Z(\mathcal{A})$ is a closed subalgebra of $\mathcal{A}$, $d(\mathcal{A}) \subseteq Z(\mathcal{A})$. Now, it follows from Theorem 2.5 that D_n is identically zero for all $n \in \mathbb{N}$ and consequently $d = 0$. □

For more details on σ-derivations, we refer the reader to [6] and the references therein. In the following theorem, it is supposed that $d_0 = I$, where I is the identity mapping on $\mathcal{A}$.

Theorem 2.7 *Let $\mathcal{A}$ be a complex normed $*$-algebra such that $\mathfrak{O}_{\mathcal{A}}$ is norm dense in $\mathcal{A}_{sa}$.*

(i) If $\{d_n\}_{n=0}^{\infty}$ is a bounded left higher derivation (i.e. d_n is a bounded linear mapping for any $n \in \mathbb{N}_0$), then $d_n = 0$ for all $n \in \mathbb{N}$.

(ii) If $\{d_n\}_{n=0}^{\infty}$ is a bounded higher derivation such that $d_n(p) \in Z(\mathcal{A})$ for all $p \in P(\mathcal{A})$ and all $n \in \mathbb{N}$, then $d_n = 0$ for all $n \in \mathbb{N}$.

Proof (i) If $\{d_n\}_{n=0}^{\infty}$ is a higher left derivation, then it follows from [7] that

$$d_n(p) = \sum_{i+j=n,\quad i \leq j} \Big[d_i(p)d_j(p) + c_{ij}d_i(p)d_j(p) \Big],$$

where

$$c_{ij} = \begin{cases} 0 & if\ i = j, \\ 1 & if\ i \neq j. \end{cases}$$

for all $p \in P(\mathcal{A})$. We have $d_1(p) = 2pd_1(p)$. It follows from [5, Theorem 2.11] that d_1 is identically zero. Moreover, we see that $d_2(p) = pd_2(p) + d_1(p)d_1(p) + pd_2(p) = 2pd_2(p)$, and reusing [5, Theorem 2.11] implies that $d_2 = 0$. Continuing this procedure, we get that $d_n = 0$ for all $n \in \mathbb{N}$.
(ii) The proof is straightforward. □

We now intend to prove another fundamental result of this paper. The following auxiliary result is needed to prove that theorem. In what follows, $\mathbb{P}$ stands for the set of all prime numbers.

Lemma 2.8 *Let $\mathcal{R}$ be a unital 2-torsion free prime ring containing the element $\frac{\mathbf{e}}{2}$ and let $\{a_n\}_{n=0}^{\infty}$ be a sequence of $\mathcal{R}$ satisfying $a_n = \sum_{k|n} a_{\frac{n}{k}} a_k$ for all $n \in \mathbb{N}_2$. If $a_n \in Z(\mathcal{R})$ for all $n \in \mathbb{N}_2$, then*
(i) $a_{p^n} = 0$ for all $n \in \mathbb{N}$ and all $p \in \mathbb{P}$.
(ii) $a_{p_1p_2} = 0$ and $a_{p_1^2p_2} = 0$ for all prime numbers $p_1, p_2 \in \mathbb{P}$.
(iii) $a_{p_1p_2p_3} = 0$ for all prime numbers $p_1, p_2, p_3 \in \mathbb{P}$.
(iv) $a_{p_1\dots p_n} = 0$ for all $n \in \mathbb{N}$ and all prime numbers $p_1, \dots, p_n \in \mathbb{P}$.
(v) $a_n = 0$ for all $n \in \mathbb{N}_2$.

Proof (i) Let p be an arbitrary prime number. By putting $A_n = a_{p^n}$ and getting idea from [10, Lemma 2.2], we have

$$A_n = a_{p^n} = \sum_{k|p^n} a_{\frac{p^n}{k}} a_k = \sum_{i=0}^{n} a_{\frac{p^n}{p^i}} a_{p^i} = \sum_{i=0}^{n} a_{p^{n-i}} a_{p^i} = \sum_{i=0}^{n} A_{n-i} A_i$$

for all $n \in \mathbb{N}$. It follows from Lemma 2.4 that $A_n = 0$. Consequently, $a_{p^n} = 0$ for all $n \in \mathbb{N}$ and all $p \in \mathbb{P}$.

(ii) Let p_1 and p_2 be two arbitrary prime numbers. Applying part (i) of the current theorem, we have

$$\begin{aligned} a_{p_1p_2} &= \sum_{k|p_1p_2} a_{\frac{p_1p_2}{k}} a_k \\ &= a_{p_1p_2} a_1 + a_{p_2} a_{p_1} + a_{p_1} a_{p_2} + a_1 a_{p_1p_2} \\ &= 2a_{p_1p_2} a_1, \end{aligned}$$

which means that

$$a_{p_1p_2}(\mathbf{e} - 2a_1) = 0. \tag{2.18}$$

Multiplying (2.18) from the left by an arbitrary element $a \in \mathcal{R}$, we have $2a_{p_1p_2} a \left(\frac{\mathbf{e}}{2} - a_1\right) = 0$, and since $\mathcal{R}$ is a 2-torsion free prime ring, we obtain that $a_1 = \frac{\mathbf{e}}{2}$ or $a_{p_1p_2} = 0$. If $a_{p_1p_2} = 0$, then there is nothing to prove.

Now, suppose that $a_1 = \frac{\mathbf{e}}{2}$. We have

$$\begin{aligned} a_{(p_1p_2)^2} &= a_{(p_1p_2)^2}a_1 + a_{p_1p_2^2}a_{p_1} + a_{p_1}a_{p_1p_2^2} + a_{p_1^2p_2}a_{p_2} + a_{p_2}a_{p_1^2p_2} + a_{p_1p_2}a_{p_1p_2} \\ &\quad + a_{p_1^2}a_{p_2^2} + a_{p_2^2}a_{p_1^2} + a_1 a_{(p_1p_2)^2} \\ &= a_{(p_1p_2)^2}\frac{\mathbf{e}}{2} + (a_{p_1p_2})^2 + \frac{\mathbf{e}}{2}a_{(p_1p_2)^2} \\ &= a_{(p_1p_2)^2} + (a_{p_1p_2})^2, \end{aligned}$$

which implies that $(a_{p_1p_2})^2 = 0$. From this equation and using the assumption that $\mathcal{R}$ is a prime ring, we obtain that $a_{p_1p_2} = 0$. Now, we show that $a_{p_1^2p_2} = 0$. We have

$$\begin{aligned} a_{p_1^2p_2} &= \sum_{k|p_1^2p_2} a_{\frac{p_1^2p_2}{k}}a_k \\ &= a_{p_1^2p_2}a_1 + a_{p_2}a_{p_1^2} + a_{p_1^2}a_{p_2} + a_{p_1p_2}a_{p_1} + a_{p_1}a_{p_1p_2} + a_1a_{p_1^2p_2} \\ &= 2a_{p_1^2p_2}a_1, \end{aligned}$$

which means that

$$a_{p_1^2p_2}(\mathbf{e} - 2a_1) = 0. \tag{2.19}$$

Multiplying (2.19) from the left by an arbitrary element $a \in \mathcal{R}$, we infer that $a_{p_1^2p_2} = 0$ or $a_1 = \frac{\mathbf{e}}{2}$. If $a_{p_1^2p_2} = 0$, then there is nothing to prove. Now, suppose that $a_1 = \frac{\mathbf{e}}{2}$. Thus, we have

$$\begin{aligned} a_{(p_1^2p_2)^2} &= a_{p_1^4p_2^2} = a_{p_1^4p_2^2}a_1 + a_1a_{p_1^4p_2^2} + a_{p_1^4}a_{p_2^2} + a_{p_2^2}a_{p_1^4} + a_{p_1^4p_2}a_{p_2} + a_{p_2}a_{p_1^4p_2} + a_{p_1^3}a_{p_1p_2^2} + a_{p_1p_2^2}a_{p_1^3} \\ &\quad + a_{p_1^2}a_{p_1^2p_2^2} + a_{p_1^2p_2^2}a_{p_1^2} + a_{p_1}a_{p_1^3p_2^2} + a_{p_1^3p_2^2}a_{p_1} + a_{p_1^3p_2}a_{p_1p_2} + a_{p_1p_2}a_{p_1^3p_2} + a_{p_1^2p_2}a_{p_1^2p_2} \\ &= a_{p_1^4p_2^2}\frac{\mathbf{e}}{2} + \frac{\mathbf{e}}{2}a_{p_1^4p_2^2} + (a_{p_1^2p_2})^2 \\ &= a_{p_1^4p_2^2} + (a_{p_1^2p_2})^2, \end{aligned}$$

which implies that $(a_{p_1^2p_2})^2 = 0$. This equation along with the assumption that $\mathcal{R}$ is a prime ring implies that $a_{p_1^2p_2} = 0$.

(iii) Let p_1, p_2, and p_3 be arbitrary prime numbers. Then

$$\begin{aligned} a_{p_1p_2p_3} &= \sum_{k|p_1p_2p_3} a_{\frac{p_1p_2p_3}{k}}a_k \\ &= a_{p_1p_2p_3}a_1 + a_1a_{p_1p_2p_3} + a_{p_1p_2}a_{p_3} + a_{p_3}a_{p_1p_2} + a_{p_2p_3}a_{p_1} + a_{p_1}a_{p_2p_3} + a_{p_1p_3}a_{p_2} + a_{p_2}a_{p_1p_3} \\ &= 2a_{p_1p_2p_3}a_1, \end{aligned}$$

which means that

$$a_{p_1p_2p_3}(\mathbf{e} - 2a_1) = 0. \tag{2.20}$$

Multiplying (2.20) from the left by an arbitrary element $a \in \mathcal{R}$, we get that $a_{p_1p_2p_3} = 0$ or $a_1 = \frac{\mathbf{e}}{2}$. If

$a_{p_1p_2p_3} = 0$, then there is nothing to prove. Now, suppose that $a_1 = \frac{\mathbf{e}}{2}$. Therefore, we have

$$\begin{aligned}
a_{(p_1p_2p_3)^2} = a_{p_1^2p_2^2p_3^2} &= a_{p_1^2p_2^2p_3^2}a_1 + a_1a_{p_1^2p_2^2p_3^2} + a_{p_1p_2^2p_3^2}a_{p_1} + a_{p_1}a_{p_1p_2^2p_3^2} \\
&+ a_{p_1^2p_2p_3^2}a_{p_2} + a_{p_2}a_{p_1^2p_2p_3^2} + a_{p_1^2p_2^2p_3}a_{p_3} + a_{p_3}a_{p_1^2p_2^2p_3} \\
&+ a_{p_1^2p_2^2}a_{p_3^2} + a_{p_3^2}a_{p_1^2p_2^2} + a_{p_1^2p_3^2}a_{p_2^2} + a_{p_2^2}a_{p_1^2p_3^2} \\
&+ a_{p_2^2p_3^2}a_{p_1^2} + a_{p_1^2}a_{p_2^2p_3^2} + a_{p_1p_2}a_{p_1p_2p_3^2} + a_{p_1p_2p_3^2}a_{p_1p_2} \\
&+ a_{p_1p_3}a_{p_1p_2^2p_3} + a_{p_1p_2^2p_3}a_{p_1p_3} + a_{p_2p_3}a_{p_1^2p_2p_3} + a_{p_1^2p_2p_3}a_{p_2p_3} \\
&+ a_{p_1^2p_2}a_{p_2p_3^2} + a_{p_2p_3^2}a_{p_1^2p_2} + a_{p_1^2p_3}a_{p_2^2p_3} + a_{p_3p_2^2}a_{p_1^2p_3} + (a_{p_1p_2p_3})^2 \\
&= a_{p_1^2p_2^2p_3^2}\frac{\mathbf{e}}{2} + \frac{\mathbf{e}}{2}a_{p_1^2p_2^2p_3^2} + (a_{p_1p_2p_3})^2 \\
&= a_{p_1^2p_2^2p_3^2} + (a_{p_1p_2p_3})^2,
\end{aligned}$$

which implies that $(a_{p_1p_2p_3})^2 = 0$. This equation along with the assumption that $\mathcal{R}$ is a prime ring implies that $a_{p_1p_2p_3} = 0$.

(iv) Let $p_1, \ldots, p_n$ be arbitrary prime numbers. We know that $a_{p_i} = 0$ and $a_{p_ip_j} = 0$ for all $i, j \in \{1, \ldots, n\}$. Suppose that $a_{p_{i_1}\ldots p_{i_{n-1}}} = 0$ for any $i_1, \ldots, i_{n-1} \in \{1, \ldots, n\}$. Thus, we have

$$\begin{aligned}
a_{p_1\ldots p_n} &= a_{p_1\ldots p_n}a_1 + a_1a_{p_1\ldots p_n} + \sum a_{p_{i_1}\ldots p_{i_k}}a_{p_{i_{k+1}}\ldots p_{i_n}} \\
&= a_{p_1\ldots p_n}a_1 + a_1a_{p_1\ldots p_n} + 0 \\
&= 2a_{p_1\ldots p_n}a_1,
\end{aligned}$$

such that $p_{i_1}\ldots p_{i_k}.p_{i_{k+1}}\ldots p_{i_n} = p_1\ldots p_n$ and further, $p_{i_1}\ldots p_{i_k}, p_{i_{k+1}}\ldots p_{i_n} \neq p_1\ldots p_n$. Hence,

$$a_{p_1\ldots p_n}(\mathbf{e} - 2a_1) = 0. \tag{2.21}$$

Multiplying (2.21) from the left by an arbitrary element $a \in \mathcal{R}$, we get that $a_{p_1\ldots p_n} = 0$ or $a_1 = \frac{\mathbf{e}}{2}$. If $a_{p_1\ldots p_n} = 0$, then there is nothing to prove. Now, suppose that $a_1 = \frac{\mathbf{e}}{2}$. Therefore, we have

$$\begin{aligned}
a_{(p_1\ldots p_n)^2} &= a_{(p_1\ldots p_n)^2}a_1 + a_1a_{(p_1\ldots p_n)^2} + (a_{p_1\ldots p_n})^2 + \sum a_{p_{i_1}\ldots p_{i_k}}a_{p_{j_1}\ldots p_{j_r}} \\
&= a_{(p_1\ldots p_n)^2}\frac{\mathbf{e}}{2} + \frac{\mathbf{e}}{2}a_{(p_1\ldots p_n)^2} + (a_{p_1\ldots p_n})^2 \\
&= a_{(p_1\ldots p_n)^2} + (a_{p_1\ldots p_n})^2,
\end{aligned}$$

such that

$$\begin{cases}
p_{i_1}\ldots p_{i_k}.p_{j_1}\ldots p_{j_r} = (p_1\ldots p_n)^2, \\
p_{i_1}\ldots p_{i_k} \neq p_1\ldots p_n, \\
p_{j_1}\ldots p_{j_r} \neq p_1\ldots p_n, \\
i_1, \ldots, i_k, j_1, \ldots, j_r \in \{1, \ldots, 2n\}.
\end{cases}$$

Therefore, $(a_{p_1 \dots p_n})^2 = 0$. This equation along with the assumption that $\mathcal{R}$ is a prime ring implies that $a_{p_1 \dots p_n} = 0$.

(v) Let n be an arbitrary element of $\mathbb{N}_2$. We know that there exist prime numbers $p_1, \dots, p_m$ such that $n = p_1^{\alpha_1} \dots p_m^{\alpha_m}$ in which $\alpha_1, \dots, \alpha_m$ are positive integers. Therefore, we have

$$a_n = a_{p_1^{\alpha_1} \dots p_m^{\alpha_m}} = \sum_{k | p_1^{\alpha_1} \dots p_m^{\alpha_m}} a_{\frac{p_1^{\alpha_1} \dots p_m^{\alpha_m}}{k}} a_k.$$

Using the facts (i), (ii), (iii), and (iv), one can easily conclude that $a_n = 0$ for all $n \in \mathbb{N}_2$. □

Theorem 2.9 *Let $\mathcal{A}$ be a unital, prime complex normed $*$-algebra such that $\mathfrak{O}_{\mathcal{A}}$ is norm dense in $\mathcal{A}_{sa}$ and let $\{d_n\}$ be a sequence of continuous linear mappings satisfying $d_n(p) = \sum_{k|n} d_{\frac{n}{k}}(p) d_k(p)$ for all $p \in P(\mathcal{A})$ and all $n \in \mathbb{N}_2$. If $d_n(p) \in Z(\mathcal{A})$ for all $p \in P(\mathcal{A})$ and all $n \in \mathbb{N}_2$, then $d_n = 0$ for all $n \in \mathbb{N}_2$.*

Proof Let p be an arbitrary projection of $\mathcal{A}$. Putting $a_n = d_n(p)$, we see that $a_n = \sum_{k|n} a_{\frac{n}{k}} a_k$ for all $n \in \mathbb{N}_2$. Now, all the conditions of Lemma 2.8 (v) are fulfilled and we achieve that $0 = a_n = d_n(p)$ for all $p \in P(\mathcal{A})$ and all $n \in \mathbb{N}_2$. Using similar arguments as used in Theorem 2.5, we can prove that $d_n = 0$ for all $n \in \mathbb{N}_2$. □

In the following, we plan to show that every prime higher derivation is identically zero under certain conditions. The concept of a prime higher derivation was defined by Mirzavaziri [10], which we state here.

Definition 2.10 *Let $\mathcal{A}$ be an algebra. We say that a sequence $\{d_n\}$ of linear mappings from $\mathcal{A}$ into $\mathcal{A}$ is a prime higher derivation if $d_n(ab) = \sum_{k|n} d_{\frac{n}{k}}(a) d_k(b)$ for all $a, b \in \mathcal{A}$ and for each $n \in \mathbb{N}$.*

Theorem 2.11 *Let $\mathcal{A}$ be a complex normed $*$-algebra such that $\mathfrak{O}_{\mathcal{A}}$ is norm dense in $\mathcal{A}_{sa}$ and let $\{d_n\}$ be a sequence of continuous linear mappings with $d_1 = I$ and $d_n(q) = \sum_{k|n} d_{\frac{n}{k}}(q) d_k(q)$ for all $q \in P(\mathcal{A})$ and all $n \in \mathbb{N}_2$. If $d_n(q) \in Z(\mathcal{A})$ for all $q \in P(\mathcal{A})$ and all $n \in \mathbb{N}_2$, then $d_n = 0$ for all $n \in \mathbb{N}_2$.*

Proof Let $p \in \mathbb{P}$ and let q be an arbitrary projection of $\mathcal{A}$. Note that $d_p(q) = q d_p(q) + d_p(q) q = 2q d_p(q)$. Then, $d_p(q) = 0$ and hence it follows from [5, Theorem 2.11] that d_p is identically zero. A simple argument shows that d_{p^2} and d_{p^3} are zero for all $p \in \mathbb{P}$. Let p be an arbitrary prime number. Suppose that $d_{p^k}(q) = 0$ for any $k \in \{1, \dots, n-1\}$ and any projection $q \in \mathcal{A}$. Our goal is to prove that $d_{p^n}(q) = 0$. We have

$$\begin{aligned} d_{p^n}(q) &= \sum_{k|p^n} d_{\frac{p^n}{k}}(q) d_k(q) = \sum_{i=0}^{n} d_{p^{n-i}}(q) d_{p^i}(q) \\ &= d_{p^n}(q) q + q d_{p^n}(q) + \sum_{i=1}^{n-1} d_{p^{n-i}}(q) d_{p^i}(q) \\ &= 2q d_{p^n}(q), \end{aligned}$$

which means that

$$d_{p^n}(q) = 2q d_{p^n}(q).$$

Reusing [5, Theorem 2.11], we obtain that $d_{p^n} = 0$ for all $p \in \mathbb{P}$ and all $n \in \mathbb{N}$. Now, suppose that p_1 and p_2 are arbitrary prime numbers. For any $q \in P(\mathcal{A})$, we have

$$\begin{aligned} d_{p_1p_2}(q) &= d_{p_1p_2}(q)q + qd_{p_1p_2}(q) + d_{p_1}(q)d_{p_2}(q) + d_{p_2}(q)d_{p_1}(q) \\ &= 2qd_{p_1p_2}(q). \end{aligned}$$

Thus, $d_{p_1p_2} = 0$ by [5, Theorem 2.11]. Similarly, we can prove that $d_{p_1p_2p_3} = 0$ for all $p_1, p_2, p_3 \in \mathbb{P}$. We show that $d_{p_1 \ldots p_n} = 0$ for all $p_1, \ldots, p_n \in \mathbb{P}$. To see this, suppose that $d_{p_{i_1} \ldots p_{i_{n-1}}}(q) = 0$ for any $i_1, \ldots, i_{n-1} \in \{1, \ldots, n\}$ and any $q \in P(\mathcal{A})$. Thus, we have

$$\begin{aligned} d_{p_1 \ldots p_n}(q) &= \sum_{k | p_1 \ldots p_n} d_{\frac{p_1 \ldots p_n}{k}}(q) d_k(q) \\ &= d_{p_1 \ldots p_n}(q)q + qd_{p_1 \ldots p_n}(q) + \sum d_{p_{i_1} \ldots p_{i_k}}(q) d_{p_{i_{k+1}} \ldots p_{i_n}}(q) \\ &= d_{p_1 \ldots p_n}(q)q + qd_{p_1 \ldots p_n}(q) + 0 \\ &= 2qd_{p_1 \ldots p_n}(q) \end{aligned}$$

such that

$$\begin{cases} p_{i_1} \ldots p_{i_k} . p_{i_{k+1}} \ldots p_{i_n} = p_1 \ldots p_n, \\ p_{i_1} \ldots p_{i_k} \neq p_1 \ldots p_n, \\ p_{i_{k+1}} \ldots p_{i_n} \neq p_1 \ldots p_n. \end{cases}$$

Hence, we have $d_{p_1 \ldots p_n}(q) = 2qd_{p_1 \ldots p_n}(q)$, and consequently, $d_{p_1 \ldots p_n} = 0$. Let n be an arbitrary element of $\mathbb{N}_2$. We know that there exist prime numbers $p_1, \ldots, p_m$ such that $n = p_1^{\alpha_1} \ldots p_m^{\alpha_m}$ in which $\alpha_1, \ldots, \alpha_m$ are positive integers. For any $q \in P(\mathcal{A})$, we have

$$d_n(q) = d_{p_1^{\alpha_1} \ldots p_m^{\alpha_m}}(q) = \sum_{k | p_1^{\alpha_1} \ldots p_m^{\alpha_m}} d_{\frac{p_1^{\alpha_1} \ldots p_m^{\alpha_m}}{k}}(q) d_k(q).$$

Using the fact that $d_{p^n}(q) = 0 = d_{p_1 \ldots p_n}(q)$, one can easily conclude that $d_n(q) = 0$ for all $n \in \mathbb{N}_2$. Using similar arguments as used in Theorem 2.5, we get that $d_n = 0$ for all $n \in \mathbb{N}_2$. $\square$

An immediate consequence of Theorem 2.11 is the following.

Corollary 2.12 *Let $\mathcal{A}$ be a complex normed $*$-algebra such that $\mathfrak{O}_{\mathcal{A}}$ is norm dense in $\mathcal{A}_{sa}$ and let $\{d_n\}$ be a prime higher derivation of continuous linear mappings with $d_1 = I$. If $d_n(q) \in Z(\mathcal{A})$ for all $q \in P(\mathcal{A})$ and all $n \in \mathbb{N}_2$, then $d_n = 0$ for all $n \in \mathbb{N}_2$.*

Next, we intend to present the conditions under which the linear mappings satisfying a given recursive relation are zero. First we prove the following useful lemma.

Lemma 2.13 *Let $\mathcal{A}$ be a unital C^*-algebra and let $\{a_n\}_{n=0}^{\infty}$ be a uniformly bounded sequence of $\mathcal{A}$ satisfying $a_n = \sum_{k=0}^{n} a_{n-k}a_k$ for all $n \in \mathbb{N}$. If the element $\sum_{n=0}^{\infty} a_n t^n - \frac{e}{2}$ is either a positive or negative element of $\mathcal{A}$ for all $t \in (-1, 1)$, then $a_n = 0$ for all $n \in \mathbb{N}$.*

Proof Since $\{a_n\}$ is a uniformly bounded sequence, there exists a positive number M such that $\|a_n\| \leq M$ for all $n \in \mathbb{N}_0$. For any $t \in (-1,1)$, we have

$$\begin{aligned}\Big\|\sum_{n=0}^{\infty} a_n t^n\Big\| &\leq \sum_{n=0}^{\infty} \|a_n t^n\| \\ &= \sum_{n=0}^{\infty} \|a_n\||t^n| \\ &\leq \sum_{n=0}^{\infty} M|t^n| \\ &= \frac{M}{1-|t|} < \infty.\end{aligned}$$

We first define a function $f : (-1,1) \to \mathcal{A}$ by $f(t) = \sum_{n=0}^{\infty} a_n t^n$. According to the above discussion, f is a well-defined function on $(-1,1)$. Moreover, the mth derivative of f exists and is given by the formula $f^{(m)}(t) := \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} a_n t^{n-m}$. Therefore, we have

$$\begin{aligned}f(t)^2 =& \Big(\sum_{n=0}^{\infty} a_n t^n\Big)\Big(\sum_{n=0}^{\infty} a_n t^n\Big) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a_{n-k}a_k\Big)t^n \\ &= a_0^2 + \sum_{n=1}^{\infty}\sum_{k=0}^{n} a_{n-k}a_k t^n \\ &= a_0^2 + \sum_{n=1}^{\infty} a_n t^n \\ &= a_0^2 + \sum_{n=0}^{\infty} a_n t^n - a_0 \\ &= f(t) + a_0^2 - a_0,\end{aligned}$$

which means that

$$f(t)^2 - f(t) = a_0^2 - a_0.$$

Using the previous equation, one can easily observe that

$$\Big(f(t) - \frac{\mathbf{e}}{2}\Big)^2 = \frac{\mathbf{e}}{4} + a_0^2 - a_0. \tag{2.22}$$

According to our assumptions, the element $\sum_{n=0}^{\infty} a_n t^n - \frac{\mathbf{e}}{2}$ is either a positive or negative element of $\mathcal{A}$ for all $t \in (-1,1)$. It follows from (2.22) that

$$f(t) = \frac{\mathbf{e}}{2} \pm \Big(\frac{\mathbf{e}}{4} + a_0^2 - a_0\Big)^{\frac{1}{2}} \qquad (-1 < t < 1).$$

Thus, $a_1 + 2a_2 t + 3a_3 t^2 + \cdots = \frac{df(t)}{dt} = 0$. Putting $t = 0$ in the previous equation, we get that $a_1 = 0$. We also see that $2a_2 + 3a_3 t + \cdots = \frac{d^2 f(t)}{dt^2} = 0$. Putting $t = 0$ in the previous equation, we deduce that $a_2 = 0$. Continuing this procedure, we obtain that $a_n = 0$ for all $n \in \mathbb{N}$. The proof is now completed. □

We conclude our paper with the following theorem.

Theorem 2.14 *Let $\mathcal{A}$ be a unital C^*-algebra and let $\{d_n\}_{n=0}^{\infty}$ be a sequence of continuous linear mappings satisfying $d_n(p) = \sum_{k=0}^{n} d_{n-k}(p)d_k(p)$ for all $p \in P(\mathcal{A})$ and all $n \in \mathbb{N}$. Suppose that $\sum_{n=0}^{\infty} d_n(p)t^n - \frac{e}{2}$ is either a positive or negative element of $\mathcal{A}$ for all $p \in P(\mathcal{A})$ and all $t \in (-1,1)$. Then, $d_n = 0$ for all $n \in \mathbb{N}$.*

Proof Applying Lemma 2.13 and using similar arguments as used in Theorem 2.5, we get the required result. The details are left to the reader. □

Acknowledgment

The authors thank the referees for their careful reading of the article and suggesting valuable comments that improved the quality of this work.

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