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Characterizations of dual curves and dual focal curves in dual Lorentzian space

$$D_1^3$$

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Abstract: In this paper, we have introduced dual Lorentzian connection, bracket and curvature tensor on dual Lorentzian space D_1^3 . We have studied a dual curve in different situations in dual Lorentzian space D_1^3 and have found Bishop Darboux vector and some relations according to this vector field, Bishop frame and focal curve of the present dual curve. It has been shown that Bishop Darboux vector has a similar amount in three different cases of a dual curve and the first dual focal curvature of the aforementioned curve is constant function.

Key words: Bishop frame, Darboux vector, dual Lorentzian space D_1^3 , focal curve

1. Introduction

Dual numbers were first introduced by William Kingdon Clifford (1873) and by Aleksandr Petrovich Kotelnikov (1895) and it has been employed for some specific applications [4]. Many studies have been conducted in relation to dual numbers and dual Lorentzian space D_1^3 . We have gained some benefits from the results provided through these studies. We have presented some preliminaries and a brief summary of the recent results and works in this field. Some characteristics of spacelike curves in the Minkowski space like [2, 5] and dual Lorentzian space have been explored in the previous studies, but it seems these kinds of curves have some characteristics that need to be discovered more. Therefore, we have studied these new characteristics and made an effort to add new findings to this area of the literature. Kocayigit, Bukcu and Pektas in [5] presented some features of spacelike curves according to Bishop frame in the Minkowski 3-space. The results provided from the previous studies have been used in three different situations of dual curve in dual Lorentzian space D_1^3 . Korpınar and Turhan in [7] introduced a characterization of dual focal curves in terms of their focal curvatures. Therefore, with the help of the abovementioned studies, the aim of this study was to investigate focal curve for a dual curve in the different cases.

It is worth mentioning that, in the present study, the calculations and relations are associated to the Bishop frame and Bishop Darboux vector field. The basic idea of the Bishop frame is that the relatively parallel field exists in the space [1]. In the current study, we employed Noether's theorem in a way that is different from the previous existed studies had used such as [9]. In the first section, we have introduced the basic notations that we need to know in this work. There was no definition of dual Lorentzian connection, bracket

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and curvature tensor on dual Lorentzian space D_1^3 , so that an attempt has been made to define these notations for the first time. In the second section, we have put under investigation the results of some other related works to dual Lorentzian space D_1^3 in general. we have calculated Bishop Darboux vector field and Bishop frame in the different situations of a dual curve and then have shown new ways to use of Noether's theorem for our calculations. Moreover, have found the focal curve associated to this dual curve according to Bishop frame and Bishop Darboux vector field for some results.

2. Preliminaries

This section contains basic definitions and notations.

Definition 2.1 A commutative ring D is called a general dual number commutative ring if every element in D is given by $\hat{X} = X + \epsilon X^*$ where $X, X^* \in R$ and R is a commutative ring and ϵ is a nonzero unique new element such that $\epsilon^2 = 0$. (see [4])

In this paper, we consider the real numbers \mathbb{R} as the commutative ring R . Let us denote by

$D = \{\hat{\varphi} : \hat{\varphi} = \varphi + \epsilon\varphi^*; \varphi, \varphi^* \in \mathbb{R}, \epsilon^2 = 0\}$ and $D^3 = \{\hat{\varphi} : \hat{\varphi} = \varphi + \epsilon\varphi^*; \varphi, \varphi^* \in \mathbb{R}^3, \epsilon^2 = 0\}$ set of dual numbers and dual vectors, respectively.

Definition 2.2 Given two dual vectors $\hat{\Omega} = \Omega + \epsilon\Omega^*$ and $\hat{\Psi} = \Psi + \epsilon\Psi^*$ in D^3 , then dual Lorentzian inner product of $\hat{\Omega}$ and $\hat{\Psi}$ is defined by

$$\hat{g}(\hat{\Omega}, \hat{\Psi}) = g(\Omega, \Psi) + \epsilon(g(\Omega, \Psi^*) + g(\Omega^*, \Psi)),$$

where $g(\Omega, \Psi) = -\Omega_1\Psi_1 + \Omega_2\Psi_2 + \Omega_3\Psi_3$, $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ and $\Psi = (\Psi_1, \Psi_2, \Psi_3)$. Now D^3 with the dual Lorentzian inner product \hat{g} is called dual Lorentzian space and denoted by D_1^3 . A dual vector $\hat{\Omega} = \Omega + \epsilon\Omega^*$ is called spacelike [timelike and lightlike (null), respectively] if Ω is spacelike [timelike and lightlike (null), respectively] and $g(\Omega, \Omega^*) = 0$.

Note that a dual curve $\hat{\gamma}(s) = \gamma(s) + \epsilon\gamma^*(s)$ is spacelike [timelike and lightlike (null), respectively] if $\hat{\gamma}'(s) = \gamma'(s) + \epsilon\gamma'^*(s)$ is spacelike [timelike and lightlike (null), respectively].

Now, we define essential notations like dual Lorentzian connection, dual Lorentzian bracket and dual Lorentzian curvature tensor as results in the dual Lorentzian space D_1^3 .

Definition 2.3 A dual Lorentzian connection $\hat{\nabla}$ on the dual Lorentzian space D_1^3 is a map $(\hat{X}, \hat{Y}) \mapsto \hat{\nabla}_{\hat{X}}\hat{Y}$ associated to two given differentiable vector fields $\hat{X} = X + \epsilon X^*$ and $\hat{Y} = Y + \epsilon Y^*$ which

$$\hat{\nabla}_{\hat{X}}\hat{Y} = \nabla_X Y + \epsilon(\nabla_X Y^* + \nabla_{X^*} Y),$$

where ∇ is the Riemannian connection on \mathbb{R}^3 .

The dual Lorentzian connection $\hat{\nabla}$ has properties of the Riemannian connection ∇ in [8].

Definition 2.4 For two given differentiable vector fields $\hat{X} = X + \epsilon X^*$ and $\hat{Y} = Y + \epsilon Y^*$ in the dual Lorentzian space D_1^3 , the expression

$$[\hat{X}, \hat{Y}]_1 = [X, Y] + \epsilon([X, Y^*] + [X^*, Y])$$

is called dual Lorentzian bracket of \hat{X} and \hat{Y} where $[\cdot, \cdot]$ is the lie bracket of two vector fields of \mathbb{R}^3 . The dual Lorentzian bracket $[\cdot, \cdot]_1$ has properties of the lie bracket $[\cdot, \cdot]$ in [8].

Definition 2.5 For three given differentiable vector fields $\hat{X} = X + \epsilon X^*$, $\hat{Y} = Y + \epsilon Y^*$ and $\hat{Z} = Z + \epsilon Z^*$ in the dual Lorentzian space D_1^3 , the expression

$$\hat{\mathcal{R}}(\hat{X}, \hat{Y})\hat{Z} = \mathcal{R}(X, Y)Z + \epsilon(\mathcal{R}(X, Y^*)Z + \mathcal{R}(X^*, Y)Z)$$

is called dual Lorentzian curvature tensor $\hat{\mathcal{R}}$ where \mathcal{R} is the curvature tensor on \mathbb{R}^3 . The dual Lorentzian curvature tensor $\hat{\mathcal{R}}$ has properties of the curvature tensor \mathcal{R} in [8].

Definition 2.6 Let $\hat{\nabla}$ denote the dual Lorentzian connection. Laplacian operator of a dual curve $\hat{\alpha}$ is defined by

$$\hat{\Delta} = -\hat{\nabla}_{\hat{\alpha}'}^2 = -\hat{\nabla}_{\hat{\alpha}'}\hat{\nabla}_{\hat{\alpha}'}. \tag{2.1}$$

Definition 2.7 (Relatively parallel field)

An arbitrary vector field is called relatively parallel if its tangential and normal parts are relatively parallel. We call a normal vector field, along a curve, is relatively parallel if its derivative is tangential and a tangential vector field is relatively parallel if it is a constant multiple of the unit tangent field. (see [1])

Definition 2.8 (Bishop frame)

Define $\gamma : (a, b) \rightarrow \mathbb{R}^3$, where γ is a C^2 -curve and $\|\gamma'(s)\| = 1$. In our definition Bishop frame agrees with the idea of relatively parallel fields. A single normal vector field M_0 at point $\gamma(t_0)$ generates a unique relatively parallel vector field M such that $M(t_0) = M_0$. According to this idea any orthonormal basis $\{T, M_{10}, M_{20}\}$ at point $\gamma(t_0)$ generates a unique C^1 orthonormal frame $\{T, M_1, M_2\}$. Bishop's equations are defined by

$$T' = k_1M_1 + k_2M_2,$$

$$M_1' = -k_1T,$$

$$M_2' = -k_2T.$$

We have $k = \sqrt{k_1^2 + k_2^2}$ where $k = \|T'(s)\|$ is the Frenet curvature and k_1, k_2 are the Bishop curvatures. If $k \neq 0$, then there exists a C^0 -function $\Theta(s)$ on subintervals of (a, b) such that

$$N(s) = \cos \Theta(s)M_1(s) + \sin \Theta(s)M_2(s) = \frac{k_1(s)}{k(s)}M_1(s) + \frac{k_2(s)}{k(s)}M_2(s) \text{ where } N \text{ is the Frenet principal normal.}$$

(see [3])

Noether's theorem:

Let $\hat{\gamma} : I \rightarrow D_1^3$ be a regular curve by the arc length parameter s in the dual Lorentzian space D_1^3 and the dual curvature $\hat{k} = k + \epsilon k^* = \|\hat{T}'\|$ of $\hat{\gamma} = \gamma + \epsilon \gamma^*$ is nonvanishing and $\hat{T} = \hat{\gamma}'$ is the dual unit tangent vector. The elastica minimizes the bending energy

$$\Pi(\hat{\gamma}) = \int_{\hat{\gamma}} \hat{k}(s)^2 ds.$$

Let $\hat{\alpha}_1$ and $\hat{\alpha}_2$ be points in D_1^3 and $\hat{\alpha}_1'$, $\hat{\alpha}_2'$ nonzero vectors. Consider

$$\Xi = \{\hat{\gamma} : \hat{\gamma}(a_i) = \hat{\alpha}_i, \quad \hat{\gamma}'(a_i) = \hat{\alpha}_i'\}$$

as space of smooth curves and

$$\Xi_u = \{\hat{\gamma} \in \Xi : \|\hat{\gamma}'\| = 1\}$$

as the subspace of unit speed curves.

$$\prod^\lambda : \Xi \rightarrow D$$

is defined by

$$\prod^\lambda(\hat{\gamma}) = \frac{1}{2} \int_{\hat{\gamma}} [\|\hat{\gamma}''\| + \hat{\Lambda}(t)(\|\hat{\gamma}'\| - 1)] dt,$$

where $\hat{\Lambda}(t) = \Lambda(t) + \epsilon \Lambda^*(t)$ is a dual multiplier.

Theorem 2.9 *Suppose that $\hat{\gamma}$ is a solution curve and V is an infinitesimal symmetry, then*

$$\hat{\gamma}'' \cdot V' + (\hat{\Lambda}\hat{\gamma}' - \hat{\gamma}''') \cdot V$$

is constant. Moreover for a translational symmetry, V is constant, so

$$(\hat{\Lambda}\hat{\gamma}' - \hat{\gamma}''') \cdot V = \text{constant}.$$

When V ranges over all translations, we get

$$\hat{\Lambda}\hat{\gamma}' - \hat{\gamma}''' = \hat{J} = J + \epsilon J^*$$

is constant field. (see [6])

3. Characterizations of dual curves

In this section, we will study the different situations of a dual curve in the dual Lorentzian space D_1^3 and find some results. At first we calculate dual Bishop frame and dual Bishop Darboux vector field in the different situations.

3.1. Dual spacelike curve with timelike principal normal in the dual Lorentzian space D_1^3

Let $\hat{\gamma} = \gamma + \epsilon\gamma^*(t) : I \subset \mathbb{R} \rightarrow D_1^3$ be a C^4 -dual spacelike curve with timelike principal normal by the arc length parameter s . $\hat{\gamma}' = \hat{T}$ is the dual tangent vector, $\hat{N} = \frac{1}{\hat{k}} \hat{\nabla}_{\hat{T}} \hat{T}$ is the dual principal normal where \hat{k} is never a pure-dual. $\hat{k} = \|\hat{\nabla}_{\hat{T}} \hat{T}\| = k + \epsilon k^*$ is the dual curvature of the dual curve $\hat{\gamma}$ and $\hat{B} = \hat{T} \times \hat{N}$ is the dual binormal of $\hat{\gamma}$. For dual Frenet frame $\{\hat{T}, \hat{N}, \hat{B}\}$, we have

$$\begin{aligned} \hat{\nabla}_{\hat{T}} \hat{T} &= -\hat{k}\hat{N}, \\ \hat{\nabla}_{\hat{T}} \hat{N} &= -\hat{k}\hat{T} + \hat{\tau}\hat{B}, \\ \hat{\nabla}_{\hat{T}} \hat{B} &= \hat{\tau}\hat{N}, \end{aligned} \tag{3.1}$$

where $\hat{\tau} = \tau + \epsilon\tau^*$ is the dual torsion. Also

$$\begin{aligned} \hat{g}(\hat{T}, \hat{T}) &= 1, & \hat{g}(\hat{N}, \hat{N}) &= -1, & \hat{g}(\hat{B}, \hat{B}) &= 1, \\ \hat{g}(\hat{T}, \hat{N}) &= \hat{g}(\hat{T}, \hat{B}) = \hat{g}(\hat{N}, \hat{B}) &= 0. \end{aligned} \tag{3.2}$$

Now, we present the dual Bishop frame or parallel transport frame $\{\hat{T}, \hat{M}_1, \hat{M}_2\}$ as follows

$$\begin{aligned} \hat{\nabla}_{\hat{T}} \hat{T} &= -\hat{k}_1 \hat{M}_1 + \hat{k}_2 \hat{M}_2, \\ \hat{\nabla}_{\hat{T}} \hat{M}_1 &= -\hat{k}_1 \hat{T}, \\ \hat{\nabla}_{\hat{T}} \hat{M}_2 &= -\hat{k}_2 \hat{T}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \hat{g}(\hat{T}, \hat{T}) &= 1, & \hat{g}(\hat{M}_1, \hat{M}_1) &= -1, & \hat{g}(\hat{M}_2, \hat{M}_2) &= 1, \\ \hat{g}(\hat{T}, \hat{M}_1) &= \hat{g}(\hat{T}, \hat{M}_2) = \hat{g}(\hat{M}_1, \hat{M}_2) &= 0. \end{aligned} \tag{3.4}$$

\hat{k}_1 and \hat{k}_2 are dual Bishop curvatures and we have $\hat{k} = \sqrt{|\hat{k}_2^2 - \hat{k}_1^2|}$ where $\hat{k} = \|\hat{\nabla}_{\hat{T}} \hat{T}\| = k + \epsilon k^*$ is the dual Frenet curvature. Thus there exists a function $\hat{\Theta}(s)$ on subintervals of I such that

$$\hat{k}_1(s) = \hat{k}(s) \sinh \hat{\Theta}(s), \quad \hat{k}_2(s) = \hat{k}(s) \cosh \hat{\Theta}(s).$$

Now, we can obtain dual Bishop Darboux vector. Let

$$\hat{W} = \lambda_1 \hat{T} + \lambda_2 \hat{M}_1 + \lambda_3 \hat{M}_2 \tag{3.5}$$

be the dual Bishop Darboux vector. We find λ_1, λ_2 and λ_3 reaching the dual Bishop Darboux vector: From the cross product of (3.5) with \hat{T} , we have

$$\hat{T}' = \lambda_2 \hat{M}_2 + \lambda_3 \hat{M}_1,$$

now we use (3.3) and obtain

$$-\hat{k}_1 \hat{M}_1 + \hat{k}_2 \hat{M}_2 = \lambda_2 \hat{M}_2 + \lambda_3 \hat{M}_1,$$

then $\lambda_3 = -\hat{k}_1$ and $\lambda_2 = \hat{k}_2$. Alike the abovementioned method

$$\hat{M}'_1 = -\lambda_1 \hat{M}_2 + \lambda_3 \hat{T}$$

and

$$-\hat{k}_1 \hat{T} = -\lambda_1 \hat{M}_2 + \lambda_3 \hat{T},$$

then $\lambda_1 = 0$ and $\lambda_3 = -\hat{k}_1$. Also

$$\hat{M}'_2 = -\lambda_1 \hat{M}_1 - \lambda_2 \hat{T}$$

and

$$-\hat{k}_2\hat{T} = -\lambda_1\hat{M}_1 - \lambda_2\hat{T}.$$

Then $\lambda_1 = 0$ and $\lambda_2 = \hat{k}_2$. Therefore

$$\hat{W} = \hat{k}_2\hat{M}_1 - \hat{k}_1\hat{M}_2. \tag{3.6}$$

3.2. Dual spacelike curve with spacelike principal normal in the dual Lorentzian space D_1^3

Let $\hat{\gamma} = \gamma + \epsilon\gamma^*(t) : I \subset \mathbb{R} \rightarrow D_1^3$ be a C^4 -dual spacelike curve with spacelike principal normal by the arc length parameter s . $\hat{\gamma}' = \hat{T}$ is the dual tangent vector, $\hat{N} = \frac{1}{\hat{k}}\hat{\nabla}_{\hat{T}}\hat{T}$ is the dual principal normal where \hat{k} is never a pure-dual. $\hat{k} = \|\hat{\nabla}_{\hat{T}}\hat{T}\| = k + \epsilon k^*$ is the dual curvature of the dual curve $\hat{\gamma}$ and $\hat{B} = \hat{T} \times \hat{N}$ is the dual binormal of $\hat{\gamma}$.

The dual Bishop frame is expressed as

$$\begin{aligned} \hat{\nabla}_{\hat{T}}\hat{T} &= \hat{k}_1\hat{M}_1 - \hat{k}_2\hat{M}_2, \\ \hat{\nabla}_{\hat{T}}\hat{M}_1 &= -\hat{k}_1\hat{T}, \\ \hat{\nabla}_{\hat{T}}\hat{M}_2 &= -\hat{k}_2\hat{T}, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \hat{g}(\hat{T}, \hat{T}) &= 1, \quad \hat{g}(\hat{M}_1, \hat{M}_1) = 1, \quad \hat{g}(\hat{M}_2, \hat{M}_2) = -1, \\ \hat{g}(\hat{T}, \hat{M}_1) &= \hat{g}(\hat{T}, \hat{M}_2) = \hat{g}(\hat{M}_1, \hat{M}_2) = 0. \end{aligned} \tag{3.8}$$

\hat{k}_1 and \hat{k}_2 are dual Bishop curvatures and we have $\hat{k} = \sqrt{|k_1^2 - k_2^2|}$ where $\hat{k} = \|\hat{\nabla}_{\hat{T}}\hat{T}\| = k + \epsilon k^*$ is the dual Frenet curvature. Thus there exists a function $\hat{\Theta}(s)$ on subintervals of I such that

$$\hat{k}_1(s) = \hat{k}(s) \cosh \hat{\Theta}(s), \quad \hat{k}_2(s) = \hat{k}(s) \sinh \hat{\Theta}(s).$$

Similar to the previous section, we can obtain

$$\hat{W} = \hat{k}_2\hat{M}_1 - \hat{k}_1\hat{M}_2. \tag{3.9}$$

3.3. Dual timelike curve with spacelike principal normal in the dual Lorentzian space D_1^3

Let $\hat{\gamma} = \gamma + \epsilon\gamma^*(t) : I \subset \mathbb{R} \rightarrow D_1^3$ be a C^4 -dual timelike curve with spacelike principal normal by the arc length parameter s . $\hat{\gamma}' = \hat{T}$ is the dual tangent vector, $\hat{N} = \frac{1}{\hat{k}}\hat{\nabla}_{\hat{T}}\hat{T}$ is the dual principal normal where \hat{k} is never a pure-dual. $\hat{k} = \|\hat{\nabla}_{\hat{T}}\hat{T}\| = k + \epsilon k^*$ is the dual curvature of the dual curve $\hat{\gamma}$ and $\hat{B} = \hat{T} \times \hat{N}$ is the dual binormal of $\hat{\gamma}$.

The dual Bishop frame is expressed as

$$\begin{aligned}\hat{\nabla}_{\hat{T}}^T &= \hat{k}_1 \hat{M}_1 + \hat{k}_2 \hat{M}_2, \\ \hat{\nabla}_{\hat{T}}^{\hat{M}_1} &= \hat{k}_1 \hat{T}, \\ \hat{\nabla}_{\hat{T}}^{\hat{M}_2} &= \hat{k}_2 \hat{T},\end{aligned}\tag{3.10}$$

where

$$\begin{aligned}\hat{g}(\hat{T}, \hat{T}) &= -1, \quad \hat{g}(\hat{M}_1, \hat{M}_1) = 1, \quad \hat{g}(\hat{M}_2, \hat{M}_2) = 1, \\ \hat{g}(\hat{T}, \hat{M}_1) &= \hat{g}(\hat{T}, \hat{M}_2) = \hat{g}(\hat{M}_1, \hat{M}_2) = 0.\end{aligned}\tag{3.11}$$

\hat{k}_1 and \hat{k}_2 are dual Bishop curvatures and we have $\hat{k} = \sqrt{\hat{k}_1^2 + \hat{k}_2^2}$ where $\hat{k} = \|\hat{\nabla}_{\hat{T}}^T\| = k + \epsilon k^*$ is the dual Frenet curvature. Thus there exists a function $\hat{\Theta}(s)$ on subintervals of I such that

$$\hat{k}_1(s) = \hat{k}(s) \cos \hat{\Theta}(s), \quad \hat{k}_2(s) = \hat{k}(s) \sin \hat{\Theta}(s).$$

Similar to the previous sections, we have obtained the dual Bishop Darboux vector of a dual timelike curve in the dual Lorentzian space D_1^3 as follows

$$\hat{W} = \hat{k}_2 \hat{M}_1 - \hat{k}_1 \hat{M}_2.\tag{3.12}$$

Finding the dual Bishop frame’s elements:

Case 3.1:

With respect to (3.3) and (3.6) it has been obtained :

$$\begin{aligned}\hat{\nabla}_{\hat{T}} \hat{W} &= \hat{k}'_2 \hat{M}_1 - \hat{k}'_1 \hat{M}_2, \\ \hat{\nabla}_{\hat{T}}^2 \hat{W} &= (\hat{k}'_1 \hat{k}_2 - \hat{k}_2 \hat{k}'_1) \hat{T} + \hat{k}''_2 \hat{M}_1 - \hat{k}''_1 \hat{M}_2.\end{aligned}$$

So

$$\begin{aligned}\hat{M}_1 &= \frac{\hat{k}'_1}{\hat{k}'_1 \hat{k}_2 - \hat{k}_1 \hat{k}'_2} \hat{W} - \frac{\hat{k}_1}{\hat{k}'_1 \hat{k}_2 - \hat{k}_1 \hat{k}'_2} \hat{\nabla}_{\hat{T}} \hat{W}, \\ \hat{M}_2 &= \frac{\hat{k}'_2}{\hat{k}'_1 \hat{k}_2 - \hat{k}_1 \hat{k}'_2} \hat{W} - \frac{\hat{k}_2}{\hat{k}'_1 \hat{k}_2 - \hat{k}_1 \hat{k}'_2} \hat{\nabla}_{\hat{T}} \hat{W}, \\ \hat{T} &= \frac{(\hat{k}''_1 \hat{k}'_2 - \hat{k}'_2 \hat{k}''_1)}{(\hat{k}'_1 \hat{k}_2 - \hat{k}_1 \hat{k}'_2)^2} \hat{W} + \frac{(\hat{k}_1 \hat{k}''_2 - \hat{k}_2 \hat{k}''_1)}{(\hat{k}'_1 \hat{k}_2 - \hat{k}_1 \hat{k}'_2)^2} \hat{\nabla}_{\hat{T}} \hat{W} + \frac{1}{(\hat{k}'_1 \hat{k}_2 - \hat{k}_1 \hat{k}'_2)} \hat{\nabla}_{\hat{T}}^2 \hat{W}.\end{aligned}\tag{3.13}$$

Case 3.2:

With respect to (3.7) and (3.9), we can obtain:

$$\begin{aligned}\hat{\nabla}_{\hat{T}} \hat{W} &= \hat{k}'_2 \hat{M}_1 - \hat{k}'_1 \hat{M}_2, \\ \hat{\nabla}_{\hat{T}}^2 \hat{W} &= (\hat{k}'_1 \hat{k}_2 - \hat{k}_2 \hat{k}'_1) \hat{T} + \hat{k}''_2 \hat{M}_1 - \hat{k}''_1 \hat{M}_2.\end{aligned}$$

So

$$\begin{aligned} \hat{M}_1 &= \frac{\hat{k}'_1}{\hat{k}'_1\hat{k}_2 - \hat{k}_1\hat{k}'_2} \hat{W} - \frac{\hat{k}_1}{\hat{k}'_1\hat{k}_2 - \hat{k}_1\hat{k}'_2} \hat{\nabla}_{\hat{T}} \hat{W}, \\ \hat{M}_2 &= \frac{\hat{k}'_2}{\hat{k}'_1\hat{k}_2 - \hat{k}_1\hat{k}'_2} \hat{W} - \frac{\hat{k}_2}{\hat{k}'_1\hat{k}_2 - \hat{k}_1\hat{k}'_2} \hat{\nabla}_{\hat{T}} \hat{W}, \\ \hat{T} &= \frac{(\hat{k}''_1\hat{k}'_2 - \hat{k}'_2\hat{k}''_1)}{(\hat{k}'_1\hat{k}_2 - \hat{k}_1\hat{k}'_2)^2} \hat{W} + \frac{(\hat{k}_1\hat{k}''_2 - \hat{k}_2\hat{k}''_1)}{(\hat{k}'_1\hat{k}_2 - \hat{k}_1\hat{k}'_2)^2} \hat{\nabla}_{\hat{T}} \hat{W} + \frac{1}{(\hat{k}'_1\hat{k}_2 - \hat{k}_1\hat{k}'_2)} \hat{\nabla}_{\hat{T}}^2 \hat{W}. \end{aligned} \tag{3.14}$$

Case 3.3:

With respect to (3.10) and (3.12), we found out:

$$\begin{aligned} \hat{\nabla}_{\hat{T}} \hat{W} &= \hat{k}'_2 \hat{M}_1 - \hat{k}'_1 \hat{M}_2, \\ \hat{\nabla}_{\hat{T}}^2 \hat{W} &= -(\hat{k}'_1\hat{k}_2 - \hat{k}'_2\hat{k}_1) \hat{T} + \hat{k}''_2 \hat{M}_1 - \hat{k}''_1 \hat{M}_2. \end{aligned}$$

So

$$\begin{aligned} \hat{M}_1 &= \frac{\hat{k}'_1}{\hat{k}'_1\hat{k}_2 - \hat{k}_1\hat{k}'_2} \hat{W} - \frac{\hat{k}_1}{\hat{k}'_1\hat{k}_2 - \hat{k}_1\hat{k}'_2} \hat{\nabla}_{\hat{T}} \hat{W}, \\ \hat{M}_2 &= \frac{\hat{k}'_2}{\hat{k}'_1\hat{k}_2 - \hat{k}_1\hat{k}'_2} \hat{W} - \frac{\hat{k}_2}{\hat{k}'_1\hat{k}_2 - \hat{k}_1\hat{k}'_2} \hat{\nabla}_{\hat{T}} \hat{W}, \\ \hat{T} &= \frac{(\hat{k}''_2\hat{k}'_1 - \hat{k}'_1\hat{k}''_2)}{(\hat{k}'_1\hat{k}_2 - \hat{k}_1\hat{k}'_2)^2} \hat{W} + \frac{(\hat{k}_2\hat{k}''_1 - \hat{k}_1\hat{k}''_2)}{(\hat{k}'_1\hat{k}_2 - \hat{k}_1\hat{k}'_2)^2} \hat{\nabla}_{\hat{T}} \hat{W} - \frac{1}{(\hat{k}'_1\hat{k}_2 - \hat{k}_1\hat{k}'_2)} \hat{\nabla}_{\hat{T}}^2 \hat{W}. \end{aligned} \tag{3.15}$$

We can now formulate our results.

Theorem 3.1 Suppose three different situations of dual curve $\hat{\gamma} = \gamma + \epsilon\gamma^*(t) : I \subset \mathbb{R} \rightarrow D_1^3$ in the dual Lorentzian space D_1^3 as follows:

- (i) Dual spacelike curve with timelike principal normal,
- (ii) Dual spacelike curve with spacelike principal normal,
- (iii) Dual timelike curve with spacelike principal normal,

under the above assumptions we have:

1. In three cases (i), (ii) and (iii) the dual Bishop Darboux vector is

$$\hat{W} = \hat{k}_2 \hat{M}_1 - \hat{k}_1 \hat{M}_2.$$

2. If we denote $\{\hat{T}, \hat{M}_1, \hat{M}_2\}$ as the dual Bishop frame in the cases (i) and (ii) and consider $\{\hat{A}, \hat{B}, \hat{C}\}$ as the dual Bishop frame in the case (iii), then

$$\hat{A} = -\hat{T}, \quad \hat{B} = \hat{M}_1, \quad \hat{C} = \hat{M}_2$$

where

$$\begin{aligned} \hat{T} &= \frac{(\hat{k}_1''\hat{k}_2' - \hat{k}_2''\hat{k}_1')}{(\hat{k}_1'\hat{k}_2 - \hat{k}_1\hat{k}_2')^2}\hat{W} + \frac{(\hat{k}_1\hat{k}_2'' - \hat{k}_2\hat{k}_1'')}{(\hat{k}_1'\hat{k}_2 - \hat{k}_1\hat{k}_2')^2}\hat{\nabla}_{\hat{T}}\hat{W} + \frac{1}{(\hat{k}_1'\hat{k}_2 - \hat{k}_1\hat{k}_2')}\hat{\nabla}_{\hat{T}}^2\hat{W}, \\ \hat{M}_1 &= \frac{\hat{k}_1'}{\hat{k}_1'\hat{k}_2 - \hat{k}_1\hat{k}_2'}\hat{W} - \frac{\hat{k}_1}{\hat{k}_1'\hat{k}_2 - \hat{k}_1\hat{k}_2'}\hat{\nabla}_{\hat{T}}\hat{W}, \\ \hat{M}_2 &= \frac{\hat{k}_2'}{\hat{k}_1'\hat{k}_2 - \hat{k}_1\hat{k}_2'}\hat{W} - \frac{\hat{k}_2}{\hat{k}_1'\hat{k}_2 - \hat{k}_1\hat{k}_2'}\hat{\nabla}_{\hat{T}}\hat{W}. \end{aligned}$$

3.4. Noether's theorem

We want to calculate \hat{J} in Noether's theorem according to the dual Bishop Darboux vector \hat{W} in three cases that we considered in our work.

Case 3.1:

By using Noether's theorem, we have

$$\hat{J} = \hat{\Lambda}\hat{\gamma}' - \hat{\gamma}'''.$$

Then

$$\hat{J} = (\hat{\Lambda} - (\hat{k}_1^2 - \hat{k}_2^2))\hat{T} + \hat{k}_1'\hat{M}_1 - \hat{k}_2'\hat{M}_2. \tag{3.16}$$

Now with respect to (3.13), we can obtain:

$$\begin{aligned} \hat{J} &= \frac{(\hat{\Lambda} - (\hat{k}_1^2 - \hat{k}_2^2))(\hat{k}_1''\hat{k}_2' - \hat{k}_2''\hat{k}_1') + (\hat{k}_1'^2 - \hat{k}_2'^2)(\hat{k}_1'\hat{k}_2 - \hat{k}_2'\hat{k}_1)}{(\hat{k}_1'\hat{k}_2 - \hat{k}_2'\hat{k}_1)^2}\hat{W} \\ &+ \frac{(\hat{\Lambda} - (\hat{k}_1^2 - \hat{k}_2^2))(\hat{k}_2''\hat{k}_1 - \hat{k}_1''\hat{k}_2) - (\hat{k}_1\hat{k}_1' - \hat{k}_2\hat{k}_2')(\hat{k}_1'\hat{k}_2 - \hat{k}_2'\hat{k}_1)}{(\hat{k}_1'\hat{k}_2 - \hat{k}_2'\hat{k}_1)^2}\hat{\nabla}_{\hat{T}}\hat{W} \\ &+ \frac{(\hat{\Lambda} - (\hat{k}_1^2 - \hat{k}_2^2))}{(\hat{k}_1'\hat{k}_2 - \hat{k}_2'\hat{k}_1)}\hat{\nabla}_{\hat{T}}^2\hat{W}. \end{aligned}$$

Other hand from (3.16), we have

$$\begin{aligned} \hat{\nabla}_{\hat{T}}\hat{J} &= (\hat{\Lambda}' - 3(\hat{k}_1'\hat{k}_1 - \hat{k}_2'\hat{k}_2))\hat{T} + (-(\hat{\Lambda} - (\hat{k}_1^2 - \hat{k}_2^2))\hat{k}_1 + \hat{k}_1'')\hat{M}_1 \\ &+ ((\hat{\Lambda} - (\hat{k}_1^2 - \hat{k}_2^2))\hat{k}_2 - \hat{k}_2'')\hat{M}_2, \\ \hat{\nabla}_{\hat{T}}^2\hat{J} &= (\hat{\Lambda}'' + (\hat{\Lambda} - (\hat{k}_1^2 - \hat{k}_2^2))(\hat{k}_1^2 - \hat{k}_2^2) - 4(\hat{k}_1''\hat{k}_1 - \hat{k}_2''\hat{k}_2) - 3(\hat{k}_1'^2 - \hat{k}_2'^2))\hat{T} \\ &+ (-(2\hat{k}_1\hat{\Lambda}' + \hat{k}_1'\hat{\Lambda}) + 5(\hat{k}_1'\hat{k}_1 - \hat{k}_2'\hat{k}_2)\hat{k}_1 + (\hat{k}_1^2 - \hat{k}_2^2)\hat{k}_1' + \hat{k}_1''')\hat{M}_1 \\ &+ ((2\hat{k}_2\hat{\Lambda}' + \hat{k}_2'\hat{\Lambda}) - 5(\hat{k}_1'\hat{k}_1 - \hat{k}_2'\hat{k}_2)\hat{k}_2 - (\hat{k}_1^2 - \hat{k}_2^2)\hat{k}_2' - \hat{k}_2''')\hat{M}_2. \end{aligned}$$

We claim that $\{\hat{J}, \hat{\nabla}_{\hat{T}}\hat{J}, \hat{\nabla}_{\hat{T}}^2\hat{J}\}$ are linearly independent. Proving the current claim, we assume that

$$\lambda_3 \hat{\nabla}_{\hat{T}}^2 \hat{J} + \lambda_2 \hat{\nabla}_{\hat{T}} \hat{J} + \lambda_1 \hat{J} = 0.$$

By ussing \hat{J} , $\hat{\nabla}_{\hat{T}} \hat{J}$ and $\hat{\nabla}_{\hat{T}}^2 \hat{J}$, we produce the following results

$$\begin{cases} \left(\hat{\Lambda}'' + f\hat{\Lambda} - 4f'' - f^2 + g \right) \lambda_3 + \left(\hat{\Lambda}' - \frac{3}{2}f' \right) \lambda_2 + \left(\hat{\Lambda} - f \right) \lambda_1 & = 0 \\ \left(-(\hat{\Lambda} - f)\hat{k}'_1 + (-2\hat{\Lambda}' + \frac{5}{2}f')\hat{k}_1 + \hat{k}'''_1 \right) \lambda_3 + \left(-(\hat{\Lambda} - f)\hat{k}_1 + \hat{k}''_1 \right) \lambda_2 + \hat{k}'_1 \lambda_1 & = 0 \\ \left((\hat{\Lambda} - f)\hat{k}'_2 + (2\hat{\Lambda}' - \frac{5}{2}f')\hat{k}_2 - \hat{k}'''_2 \right) \lambda_3 + \left((\hat{\Lambda} - f)\hat{k}_2 - \hat{k}''_2 \right) \lambda_2 - \hat{k}'_2 \lambda_1 & = 0 \end{cases}$$

where

$$f = \hat{k}_1^2 - \hat{k}_2^2, \quad g = \hat{k}'_1{}^2 - \hat{k}'_2{}^2.$$

Therefore

$$\lambda_3 = \lambda_2 = \lambda_1 = 0.$$

So $\{\hat{J}, \hat{\nabla}_{\hat{T}} \hat{J}, \hat{\nabla}_{\hat{T}}^2 \hat{J}\}$ are linearly independent.

In a similar way being used for the cases 3.2 and 3.3, we have

Case 3.2:

$$\begin{aligned} \hat{J} &= \left(\hat{\Lambda} + (\hat{k}_1^2 - \hat{k}_2^2) \right) \hat{T} - \hat{k}'_1 \hat{M}_1 + \hat{k}'_2 \hat{M}_2, \\ \hat{\nabla}_{\hat{T}} \hat{J} &= \left(\hat{\Lambda}' + 3(\hat{k}'_1 \hat{k}_1 - \hat{k}'_2 \hat{k}_2) \right) \hat{T} + \left((\hat{\Lambda} + (\hat{k}_1^2 - \hat{k}_2^2)) \hat{k}_1 - \hat{k}''_1 \right) \hat{M}_1 \\ &\quad + \left(-(\hat{\Lambda} + (\hat{k}_1^2 - \hat{k}_2^2)) \hat{k}_2 + \hat{k}''_2 \right) \hat{M}_2, \\ \hat{\nabla}_{\hat{T}}^2 \hat{J} &= \left(\hat{\Lambda}'' - (\hat{\Lambda} + (\hat{k}_1^2 - \hat{k}_2^2))(\hat{k}_1^2 - \hat{k}_2^2) + 4(\hat{k}''_1 \hat{k}_1 - \hat{k}''_2 \hat{k}_2) + 3(\hat{k}'_1{}^2 - \hat{k}'_2{}^2) \right) \hat{T} \\ &\quad + \left((2\hat{k}_1 \hat{\Lambda}' + \hat{k}'_1 \hat{\Lambda}) + 5(\hat{k}'_1 \hat{k}_1 - \hat{k}'_2 \hat{k}_2) \hat{k}_1 + (\hat{k}_1^2 - \hat{k}_2^2) \hat{k}'_1 - \hat{k}'''_1 \right) \hat{M}_1 \\ &\quad - \left((2\hat{k}_2 \hat{\Lambda}' + \hat{k}'_2 \hat{\Lambda}) + 5(\hat{k}'_1 \hat{k}_1 - \hat{k}'_2 \hat{k}_2) \hat{k}_2 + (\hat{k}_1^2 - \hat{k}_2^2) \hat{k}'_2 - \hat{k}'''_2 \right) \hat{M}_2. \end{aligned}$$

And

$$\begin{aligned} \hat{J} &= \frac{\left(\hat{\Lambda} + (\hat{k}_1^2 - \hat{k}_2^2) \right) \left(\hat{k}'_1 \hat{k}'_2 - \hat{k}'_2 \hat{k}'_1 \right) - \left(\hat{k}_1^2 - \hat{k}_2^2 \right) \left(\hat{k}'_1 \hat{k}_2 - \hat{k}'_2 \hat{k}_1 \right)}{\left(\hat{k}'_1 \hat{k}_2 - \hat{k}'_2 \hat{k}_1 \right)^2} \hat{W} \\ &\quad + \frac{\left(\hat{\Lambda} + (\hat{k}_1^2 - \hat{k}_2^2) \right) \left(\hat{k}''_2 \hat{k}_1 - \hat{k}''_1 \hat{k}_2 \right) + \left(\hat{k}_1 \hat{k}'_1 - \hat{k}_2 \hat{k}'_2 \right) \left(\hat{k}'_1 \hat{k}_2 - \hat{k}'_2 \hat{k}_1 \right)}{\left(\hat{k}'_1 \hat{k}_2 - \hat{k}'_2 \hat{k}_1 \right)^2} \hat{\nabla}_{\hat{T}} \hat{W} \\ &\quad + \frac{\left(\hat{\Lambda} + (\hat{k}_1^2 - \hat{k}_2^2) \right)}{\left(\hat{k}'_1 \hat{k}_2 - \hat{k}'_2 \hat{k}_1 \right)} \hat{\nabla}_{\hat{T}}^2 \hat{W}. \end{aligned}$$

We assume that

$$\lambda_3 \hat{\nabla}_{\hat{T}}^2 \hat{J} + \lambda_2 \hat{\nabla}_{\hat{T}} \hat{J} + \lambda_1 \hat{J} = 0.$$

By ussing \hat{J} , $\hat{\nabla}_{\hat{T}} \hat{J}$ and $\hat{\nabla}_{\hat{T}}^2 \hat{J}$, we demonstrate

$$\begin{cases} \left(\hat{\Lambda}'' - f\hat{\Lambda} + 4f'' - f^2 - g \right) \lambda_3 + \left(\hat{\Lambda}' + \frac{3}{2}f' \right) \lambda_2 + \left(\hat{\Lambda} + f \right) \lambda_1 & = 0 \\ \left((\hat{\Lambda} + f)\hat{k}'_1 + (2\hat{\Lambda}' + \frac{5}{2}f')\hat{k}_1 - \hat{k}'''_1 \right) \lambda_3 + \left((\hat{\Lambda} + f)\hat{k}_1 - \hat{k}''_1 \right) \lambda_2 - \hat{k}'_1 \lambda_1 & = 0 \\ \left((-\hat{\Lambda} - f)\hat{k}'_2 + (-2\hat{\Lambda}' - \frac{5}{2}f')\hat{k}_2 + \hat{k}'''_2 \right) \lambda_3 + \left((-\hat{\Lambda} - f)\hat{k}_2 + \hat{k}''_2 \right) \lambda_2 + \hat{k}'_2 \lambda_1 & = 0 \end{cases}$$

where

$$f = \hat{k}_1^2 - \hat{k}_2^2, \quad g = \hat{k}_1'^2 - \hat{k}_2'^2.$$

Therefore

$$\lambda_3 = \lambda_2 = \lambda_1 = 0.$$

So $\{\hat{J}, \hat{\nabla}_{\hat{T}} \hat{J}, \hat{\nabla}_{\hat{T}}^2 \hat{J}\}$ are linearly independent.

Case 3.3:

$$\begin{aligned} \hat{J} &= \left(\hat{\Lambda} - (\hat{k}_1^2 + \hat{k}_2^2) \right) \hat{T} - \hat{k}'_1 \hat{M}_1 - \hat{k}'_2 \hat{M}_2, \\ \hat{\nabla}_{\hat{T}} \hat{J} &= \left(\hat{\Lambda}' - 3(\hat{k}'_1 \hat{k}_1 + \hat{k}'_2 \hat{k}_2) \right) \hat{T} + \left((\hat{\Lambda} - (\hat{k}_1^2 + \hat{k}_2^2))\hat{k}_1 - \hat{k}''_1 \right) \hat{M}_1 \\ &\quad + \left((\hat{\Lambda} - (\hat{k}_1^2 + \hat{k}_2^2))\hat{k}_2 - \hat{k}''_2 \right) \hat{M}_2, \\ \hat{\nabla}_{\hat{T}}^2 \hat{J} &= \left(\hat{\Lambda}'' + (\hat{\Lambda} - (\hat{k}_1^2 + \hat{k}_2^2))(\hat{k}_1^2 + \hat{k}_2^2) - 4(\hat{k}''_1 \hat{k}_1 + \hat{k}''_2 \hat{k}_2) - 3(\hat{k}_1'^2 + \hat{k}_2'^2) \right) \hat{T} \\ &\quad + \left((2\hat{k}_1 \hat{\Lambda}' + \hat{k}'_1 \hat{\Lambda}) - 5(\hat{k}'_1 \hat{k}_1 + \hat{k}'_2 \hat{k}_2)\hat{k}_1 - (\hat{k}_1^2 + \hat{k}_2^2)\hat{k}'_1 - \hat{k}'''_1 \right) \hat{M}_1 \\ &\quad + \left((2\hat{k}_2 \hat{\Lambda}' + \hat{k}'_2 \hat{\Lambda}) - 5(\hat{k}'_1 \hat{k}_1 + \hat{k}'_2 \hat{k}_2)\hat{k}_2 - (\hat{k}_1^2 + \hat{k}_2^2)\hat{k}'_2 - \hat{k}'''_2 \right) \hat{M}_2. \end{aligned}$$

And

$$\begin{aligned} \hat{J} &= \frac{\left(\hat{\Lambda} - (\hat{k}_1^2 + \hat{k}_2^2) \right) \left(\hat{k}'_1 \hat{k}'_2 - \hat{k}'_2 \hat{k}'_1 \right) - \left(\hat{k}_1'^2 + \hat{k}_2'^2 \right) \left(\hat{k}'_1 \hat{k}_2 - \hat{k}'_2 \hat{k}_1 \right)}{\left(\hat{k}'_1 \hat{k}_2 - \hat{k}'_2 \hat{k}_1 \right)^2} \hat{W} \\ &\quad + \frac{\left(\hat{\Lambda} - (\hat{k}_1^2 + \hat{k}_2^2) \right) \left(\hat{k}'_1 \hat{k}_2 - \hat{k}'_2 \hat{k}_1 \right) + \left(\hat{k}_1 \hat{k}'_1 + \hat{k}_2 \hat{k}'_2 \right) \left(\hat{k}'_1 \hat{k}_2 - \hat{k}'_2 \hat{k}_1 \right)}{\left(\hat{k}'_1 \hat{k}_2 - \hat{k}'_2 \hat{k}_1 \right)^2} \hat{\nabla}_{\hat{T}} \hat{W} \\ &\quad - \frac{\left(\hat{\Lambda} - (\hat{k}_1^2 + \hat{k}_2^2) \right)}{\left(\hat{k}'_1 \hat{k}_2 - \hat{k}'_2 \hat{k}_1 \right)} \hat{\nabla}_{\hat{T}}^2 \hat{W}. \end{aligned}$$

We assume that

$$\lambda_3 \hat{\nabla}_{\hat{T}}^2 \hat{J} + \lambda_2 \hat{\nabla}_{\hat{T}} \hat{J} + \lambda_1 \hat{J} = 0.$$

By using \hat{J} , $\hat{\nabla}_{\hat{T}} \hat{J}$ and $\hat{\nabla}_{\hat{T}}^2 \hat{J}$, we come to following results

$$\begin{cases} \left(\hat{\Lambda}'' + f\hat{\Lambda} - 4f'' - f^2 + g \right) \lambda_3 + \left(\hat{\Lambda}' - \frac{3}{2}f' \right) \lambda_2 + (\hat{\Lambda} - f) \lambda_1 & = 0 \\ \left((\hat{\Lambda} - f)\hat{k}'_1 + (2\hat{\Lambda}' - \frac{5}{2}f')\hat{k}_1 - \hat{k}'''_1 \right) \lambda_3 + \left((\hat{\Lambda} - f)\hat{k}_1 - \hat{k}''_1 \right) \lambda_2 - \hat{k}'_1 \lambda_1 & = 0 \\ \left((\hat{\Lambda} - f)\hat{k}'_2 + (2\hat{\Lambda}' - \frac{5}{2}f')\hat{k}_2 - \hat{k}'''_2 \right) \lambda_3 + \left((\hat{\Lambda} - f)\hat{k}_2 - \hat{k}''_2 \right) \lambda_2 - \hat{k}'_2 \lambda_1 & = 0 \end{cases}$$

where

$$f = \hat{k}_1^2 + \hat{k}_2^2, \quad g = \hat{k}'_1{}^2 + \hat{k}'_2{}^2.$$

Therefore

$$\lambda_3 = \lambda_2 = \lambda_1 = 0.$$

So $\{\hat{J}, \hat{\nabla}_{\hat{T}} \hat{J}, \hat{\nabla}_{\hat{T}}^2 \hat{J}\}$ are linearly independent.

According to all of the calculations, we found out:

Theorem 3.2 *If $\hat{\gamma}(s)$ is a dual curve under assumptions of theorem 3.1, then $\{\hat{J}, \hat{\nabla}_{\hat{T}} \hat{J}, \hat{\nabla}_{\hat{T}}^2 \hat{J}\}$ are linearly independent.*

3.5. Dual focal curve of dual curve in D_1^3

In this section, we want to find dual focal curve for dual curve in the dual Lorentzian space according to the dual Bishop Darboux vector field.

Case 3.1:

Let $\hat{\gamma}(s)$ be a dual spacelike curve with timelike principal normal in the dual Lorentzian space D_1^3 and $\hat{F}(s) = \hat{\gamma}(s) + \hat{f}_1(s)\hat{M}_1 + \hat{f}_2(s)\hat{M}_2$ be its dual focal curve where \hat{f}_1 and \hat{f}_2 are first and second dual focal curvatures of $\hat{\gamma}$, respectively. By differentiating of \hat{F} and by using (3.3), we have

$$\hat{F}'(s) = (1 - \hat{k}_1\hat{f}_1 - \hat{k}_2\hat{f}_2) \hat{T} + \hat{f}'_1 \hat{M}_1 + \hat{f}'_2 \hat{M}_2.$$

Then $1 - \hat{k}_1\hat{f}_1 - \hat{k}_2\hat{f}_2 = 0$ and $\hat{f}'_1 = 0$. By solving these equations, we obtain

$$\begin{aligned} f_1 &= c, \\ f_1^* &= c^*, \\ f_2 &= \frac{1 - k_1c}{k_2}, \\ f_2^* &= -\frac{k_2(k_1c^* + k_1^*c) + k_2^*(1 - k_1c)}{k_2^2}. \end{aligned}$$

Therefore

$$\hat{F}(s) = \hat{\gamma}(s) + (c + \epsilon c^*) \hat{M}_1 + \left(\frac{1 - k_1c}{k_2} - \epsilon \frac{k_2(k_1c^* + k_1^*c) + k_2^*(1 - k_1c)}{k_2^2} \right) \hat{M}_2.$$

And by using (3.13), we obtain

$$\begin{aligned} \hat{F}(s) = & \hat{\gamma}(s) + \left[\frac{ck_2k'_1 + (1 - k_1c)k'_2}{(k'_1k_2 - k'_2k_1)k_2} + \epsilon \left(\frac{c^*k'_1}{(k'_1k_2 - k'_2k_1)} \right. \right. \\ & + \frac{c(k_1^*(k'_1k_2 - k'_2k_1) - k'_1(k_1^*k_2 + k_2^*k'_1 - k_2^*k_1 - k_1^*k'_2))}{(k'_1k_2 - k'_2k_1)^2} \\ & + \frac{(1 - k_1c)(k_2^*(k'_1k_2 - k'_2k_1) - k_2'(k_1^*k_2 + k_2^*k'_1 - k_2^*k_1 - k_1^*k'_2))}{k_2(k'_1k_2 - k'_2k_1)^2} \\ & \left. \left. - \frac{k'_2(k_2(k_1c^* + k_1^*c) + (1 - k_1c)k_2^*)}{k_2^2(k'_1k_2 - k'_2k_1)} \right) \right] \hat{W} \\ & + \left[\frac{-1}{(k'_1k_2 - k'_2k_1)} + \epsilon \left(\frac{-c^*k_1}{(k'_1k_2 - k'_2k_1)} \right. \right. \\ & - \frac{c(k_1^*(k'_1k_2 - k'_2k_1) - k_1(k_1^*k_2 + k_2^*k'_1 - k_2^*k_1 - k_1^*k'_2))}{(k'_1k_2 - k'_2k_1)^2} \\ & - \frac{(1 - k_1c)(k_2^*(k'_1k_2 - k'_2k_1) - k_2(k_1^*k_2 + k_2^*k'_1 - k_2^*k_1 - k_1^*k'_2))}{k_2(k'_1k_2 - k'_2k_1)^2} \\ & \left. \left. + \frac{k_2(k_1c^* + k_1^*c) + (1 - k_1c)k_2^*}{k_2(k'_1k_2 - k'_2k_1)} \right) \right] \hat{\nabla}_{\hat{T}} \hat{W}. \end{aligned}$$

Case 3.2:

Let $\hat{\gamma}(s)$ be a dual spacelike curve with spacelike principal normal in the dual Lorentzian space D_1^3 and $\hat{F}(s) = \hat{\gamma}(s) + \hat{f}_1(s)\hat{M}_1 + \hat{f}_2(s)\hat{M}_2$ be its dual focal curve where \hat{f}_1 and \hat{f}_2 are first and second dual focal curvatures of $\hat{\gamma}$, respectively. By differentiating of \hat{F} and by using (3.7), we have

$$\hat{F}'(s) = (1 - \hat{k}_1\hat{f}_1 - \hat{k}_2\hat{f}_2) \hat{T} + \hat{f}_1' \hat{M}_1 + \hat{f}_2' \hat{M}_2.$$

Then $1 - \hat{k}_1\hat{f}_1 - \hat{k}_2\hat{f}_2 = 0$ and $\hat{f}_1' = 0$. By solving these equations, we obtain

$$\begin{aligned} f_1 &= c, \\ f_1^* &= c^*, \\ f_2 &= \frac{1 - k_1c}{k_2}, \\ f_2^* &= -\frac{k_2(k_1c^* + k_1^*c) + k_2^*(1 - k_1c)}{k_2^2}. \end{aligned}$$

Therefore

$$\hat{F}(s) = \hat{\gamma}(s) + (c + \epsilon c^*) \hat{M}_1 + \left(\frac{1 - k_1c}{k_2} - \epsilon \frac{k_2(k_1c^* + k_1^*c) + k_2^*(1 - k_1c)}{k_2^2} \right) \hat{M}_2.$$

And by using (3.14), we have

$$\begin{aligned} \hat{F}(s) = & \hat{\gamma}(s) + \left[\frac{ck_2k'_1 + (1 - k_1c)k'_2}{(k'_1k_2 - k'_2k_1)k_2} + \epsilon \left(\frac{c^*k'_1}{(k'_1k_2 - k'_2k_1)} \right. \right. \\ & + \frac{c(k'_1(k'_1k_2 - k'_2k_1) - k'_1(k'_1{}^*k_2 + k_2{}^*k'_1 - k'_2{}^*k_1 - k_1{}^*k'_2))}{(k'_1k_2 - k'_2k_1)^2} \\ & + \frac{(1 - k_1c)(k_2{}^*(k'_1k_2 - k'_2k_1) - k_2(k'_1{}^*k_2 + k_2{}^*k'_1 - k'_2{}^*k_1 - k_1{}^*k'_2))}{k_2(k'_1k_2 - k'_2k_1)^2} \\ & \left. \left. - \frac{k'_2(k_2(k_1c^* + k_1{}^*c) + (1 - k_1c)k_2{}^*)}{k_2^2(k'_1k_2 - k'_2k_1)} \right) \right] \hat{W} \\ & + \left[\frac{-1}{(k'_1k_2 - k'_2k_1)} + \epsilon \left(\frac{-c^*k_1}{(k'_1k_2 - k'_2k_1)} \right. \right. \\ & - \frac{c(k_1{}^*(k'_1k_2 - k'_2k_1) - k_1(k'_1{}^*k_2 + k_2{}^*k'_1 - k'_2{}^*k_1 - k_1{}^*k'_2))}{(k'_1k_2 - k'_2k_1)^2} \\ & - \frac{(1 - k_1c)(k_2{}^*(k'_1k_2 - k'_2k_1) - k_2(k'_1{}^*k_2 + k_2{}^*k'_1 - k'_2{}^*k_1 - k_1{}^*k'_2))}{k_2(k'_1k_2 - k'_2k_1)^2} \\ & \left. \left. + \frac{k_2(k_1c^* + k_1{}^*c) + (1 - k_1c)k_2{}^*}{k_2(k'_1k_2 - k'_2k_1)} \right) \right] \hat{\nabla}_{\hat{T}} \hat{W}. \end{aligned}$$

Case 3.3:

Let $\hat{\gamma}(s)$ be a dual timelike curve in the dual Lorentzian space D_1^3 and $\hat{F}(s) = \hat{\gamma}(s) + \hat{f}_1(s)\hat{M}_1 + \hat{f}_2(s)\hat{M}_2$ be its dual focal curve where \hat{f}_1 and \hat{f}_2 are first and second dual focal curvatures of $\hat{\gamma}$, respectively. By differentiating of \hat{F} and by using (3.10), we have

$$\hat{F}'(s) = (1 + \hat{k}_1\hat{f}_1 + \hat{k}_2\hat{f}_2) \hat{T} + \hat{f}'_1 \hat{M}_1 + \hat{f}'_2 \hat{M}_2.$$

Then $1 + \hat{k}_1\hat{f}_1 + \hat{k}_2\hat{f}_2 = 0$ and $\hat{f}'_1 = 0$. By solving these equations, we obtain

$$\begin{aligned} f_1 &= c, \\ f_1^* &= c^*, \\ f_2 &= -\frac{1 + k_1c}{k_2}, \\ f_2^* &= \frac{-k_2(k_1c^* + k_1{}^*c) + k_2{}^*(1 + k_1c)}{k_2^2}. \end{aligned}$$

Therefore

$$\hat{F}(s) = \hat{\gamma}(s) + (c + \epsilon c^*) \hat{M}_1 + \left(-\frac{1 + k_1c}{k_2} + \epsilon \frac{-k_2(k_1c^* + k_1{}^*c) + k_2{}^*(1 + k_1c)}{k_2^2} \right) \hat{M}_2.$$

And by using (3.15), we have

$$\begin{aligned} \hat{F}(s) = & \hat{\gamma}(s) + \left[\frac{ck_2k'_1 - (1 + k_1c)k'_2}{(k'_1k_2 - k'_2k_1)k_2} + \epsilon \left(\frac{c^*k'_1}{(k'_1k_2 - k'_2k_1)} \right. \right. \\ & + \frac{c(k'_1(k'_1k_2 - k'_2k_1) - k'_1(k'_1k_2 + k_2^*k'_1 - k_2^*k_1 - k_1^*k'_2))}{(k'_1k_2 - k'_2k_1)^2} \\ & - \frac{(1 + k_1c)(k_2^*(k'_1k_2 - k'_2k_1) - k_2(k'_1k_2 + k_2^*k'_1 - k_2^*k_1 - k_1^*k'_2))}{k_2(k'_1k_2 - k'_2k_1)^2} \\ & \left. \left. + \frac{k'_2(-k_2(k_1c^* + k_1^*c) + (1 + k_1c)k_2^*)}{k_2^2(k'_1k_2 - k'_2k_1)} \right) \right] \hat{W} \\ & + \left[\frac{1}{(k'_1k_2 - k'_2k_1)} + \epsilon \left(\frac{-c^*k_1}{(k'_1k_2 - k'_2k_1)} \right. \right. \\ & - \frac{c(k_1^*(k'_1k_2 - k'_2k_1) - k_1(k'_1k_2 + k_2^*k'_1 - k_2^*k_1 - k_1^*k'_2))}{(k'_1k_2 - k'_2k_1)^2} \\ & + \frac{(1 + k_1c)(k_2^*(k'_1k_2 - k'_2k_1) - k_2(k'_1k_2 + k_2^*k'_1 - k_2^*k_1 - k_1^*k'_2))}{k_2(k'_1k_2 - k'_2k_1)^2} \\ & \left. \left. - \frac{-k_2(k_1c^* + k_1^*c) + (1 + k_1c)k_2^*}{k_2(k'_1k_2 - k'_2k_1)} \right) \right] \hat{\nabla}_{\hat{T}} \hat{W}. \end{aligned}$$

In this section, we studied the different cases of the dual curve and found their dual focal curves according to the dual Bishop frame and the dual Bishop Darboux vector. So, we can come to the following corollary.

Theorem 3.3 Let $\hat{\gamma}(s)$ be a dual curve under assumptions of theorem 3.1 and $\hat{F}(s) = \hat{\gamma}(s) + \hat{f}_1(s)\hat{M}_1 + \hat{f}_2(s)\hat{M}_2$ be its dual focal curve where \hat{f}_1 and \hat{f}_2 are first and second dual focal curvatures of $\hat{\gamma}$, respectively. Then 1. \hat{f}_1 is constant function of s and defined by

$$\hat{f}_1 = f_1 + \epsilon f_1^* = c + \epsilon c^*,$$

where c and c^* are constant.

2. Dual focal curve is $\hat{F}(s) = \hat{\gamma}(s) + \hat{\rho}_1(s)\hat{W} + \hat{\rho}_2(s)\hat{\nabla}_{\hat{T}}\hat{W}$ where for the cases (i) and (ii) in theorem 3.1

$$\begin{aligned} \hat{\rho}_1 = & \frac{ck_2k'_1 + (1 - k_1c)k'_2}{(k'_1k_2 - k'_2k_1)k_2} + \epsilon \left(\frac{c^*k'_1}{(k'_1k_2 - k'_2k_1)} \right. \\ & + \frac{c(k'_1(k'_1k_2 - k'_2k_1) - k'_1(k'_1k_2 + k_2^*k'_1 - k_2^*k_1 - k_1^*k'_2))}{(k'_1k_2 - k'_2k_1)^2} \\ & + \frac{(1 - k_1c)(k_2^*(k'_1k_2 - k'_2k_1) - k_2(k'_1k_2 + k_2^*k'_1 - k_2^*k_1 - k_1^*k'_2))}{k_2(k'_1k_2 - k'_2k_1)^2} \\ & \left. - \frac{k'_2(k_2(k_1c^* + k_1^*c) + (1 - k_1c)k_2^*)}{k_2^2(k'_1k_2 - k'_2k_1)} \right) \end{aligned}$$

$$\begin{aligned} \hat{\rho}_2 = & \frac{-1}{(k'_1 k_2 - k'_2 k_1)} + \epsilon \left(\frac{-c^* k_1}{(k'_1 k_2 - k'_2 k_1)} \right. \\ & - \frac{c(k_1^* (k'_1 k_2 - k'_2 k_1) - k_1 (k_1^* k_2 + k_2^* k'_1 - k_2^* k_1 - k_1^* k'_2))}{(k'_1 k_2 - k'_2 k_1)^2} \\ & - \frac{(1 - k_1 c)(k_2^* (k'_1 k_2 - k'_2 k_1) - k_2 (k_1^* k_2 + k_2^* k'_1 - k_2^* k_1 - k_1^* k'_2))}{k_2 (k'_1 k_2 - k'_2 k_1)^2} \\ & \left. + \frac{k_2 (k_1 c^* + k_1^* c) + (1 - k_1 c) k_2^*}{k_2 (k'_1 k_2 - k'_2 k_1)} \right) \end{aligned}$$

and for the case (iii)

$$\begin{aligned} \hat{\rho}_1 = & \frac{ck_2 k'_1 - (1 + k_1 c) k'_2}{(k'_1 k_2 - k'_2 k_1) k_2} + \epsilon \left(\frac{c^* k'_1}{(k'_1 k_2 - k'_2 k_1)} \right. \\ & + \frac{c(k_1^* (k'_1 k_2 - k'_2 k_1) - k'_1 (k_1^* k_2 + k_2^* k'_1 - k_2^* k_1 - k_1^* k'_2))}{(k'_1 k_2 - k'_2 k_1)^2} \\ & - \frac{(1 + k_1 c)(k_2^* (k'_1 k_2 - k'_2 k_1) - k'_2 (k_1^* k_2 + k_2^* k'_1 - k_2^* k_1 - k_1^* k'_2))}{k_2 (k'_1 k_2 - k'_2 k_1)^2} \\ & \left. + \frac{k'_2 (-k_2 (k_1 c^* + k_1^* c) + (1 + k_1 c) k_2^*)}{k_2^2 (k'_1 k_2 - k'_2 k_1)} \right) \end{aligned}$$

$$\begin{aligned} \hat{\rho}_2 = & \frac{1}{(k'_1 k_2 - k'_2 k_1)} + \epsilon \left(\frac{-c^* k_1}{(k'_1 k_2 - k'_2 k_1)} \right. \\ & - \frac{c(k_1^* (k'_1 k_2 - k'_2 k_1) - k_1 (k_1^* k_2 + k_2^* k'_1 - k_2^* k_1 - k_1^* k'_2))}{(k'_1 k_2 - k'_2 k_1)^2} \\ & + \frac{(1 + k_1 c)(k_2^* (k'_1 k_2 - k'_2 k_1) - k_2 (k_1^* k_2 + k_2^* k'_1 - k_2^* k_1 - k_1^* k'_2))}{k_2 (k'_1 k_2 - k'_2 k_1)^2} \\ & \left. - \frac{-k_2 (k_1 c^* + k_1^* c) + (1 + k_1 c) k_2^*}{k_2 (k'_1 k_2 - k'_2 k_1)} \right). \end{aligned}$$

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