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On some sums at the a-points of the k-th derivatives of the Dirichlet L-functions

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Abstract: Let $L^{(k)}(s,\chi)$ be the k-th derivative of the Dirichlet L-function associated with a primitive character χ mod q and a be a complex number. The solutions of $L^{(k)}(s,\chi)=a$ are called a-points. In this paper, we give an asymptotic formula for the sums

$$\sum_{\rho_{0,\chi}^{(k)}:0<\gamma_{0,\chi}^{(k)}<T} L^{(j)}(\rho_{0,\chi}^{(k)},\chi) \quad and \quad \sum_{\rho_{a,\chi}^{(k)}:1<\gamma_{a,\chi}^{(k)}<T} L^{(j)}(\rho_{a,\chi}^{(k)},\chi) \quad as \qquad T\to\infty$$

where j and k are nonnegative integers and $\rho_{a,\chi}^{(k)}$ denotes an a-point of the k-th derivative $L^{(k)}(s,\chi)$ and $\gamma_{a,\chi}^{(k)} = \mathrm{Im}(\rho_{a,\chi}^{(k)})$. This work continues the investigations of Kaptan, Karabulut, and Yildirim [7, 10] and Mazhouda and Onozuka [12].

Key words: Dirichlet L-function, a-points, value-distribution

1. Introduction

Let $L(s,\chi)$ be the Dirichlet L-function associated with a primitive character χ mod q and a be a complex number. The zeros of $L(s,\chi)-a$, which will be denoted by $\rho_{a,\chi}=\beta_{a,\chi}+i\gamma_{a,\chi}$ are called the a-points of $L(s,\chi)$. First, we note that there is an a-point near any trivial zero $s=-2n$ if $\chi(-1)=1$ and $s=-2n-1$ if $\chi(-1)=-1$ for sufficiently large n. Apart from these a-points, there are only finitely many other a-points in the half-plane $Re(s)=\sigma\le 0$. The a-points with $\beta_{a,\chi}\le 0$ are said to be trivial. All other a-points lie in a strip $0<Re(s)<A$, where A is a constant depending on a; these numbers are called the nontrivial a-points. The number of these a-points satisfies a Riemann-von Mangoldt type formula (we refer to [14, chapter 7.2] for the proof of this formula which is stated for functions in a subclass of the Selberg class including the Dirichlet L-functions $L(s,\chi)$), namely

$$N_{a,\chi}(T)=\sum_{\substack{\rho_{a,\chi}\,:\,0<\gamma_{a,\chi}\le T\\ \beta_{a,\chi}>0}} 1=\frac{T}{2\pi}\log\left(\frac{qT}{2\pi c_a e}\right)+O\left(\log T\right), \tag{1.1}$$

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where $c_a = m$ if $a = 1$ and $c_a = 1$, otherwise, with $m = \min\{n \geq 2, \chi(n) \neq 0\}$. Here and in the sequel the error term depends on q; however, the main term is essentially independent of a. Moreover, $N_{a,\chi}(T) \sim N_\chi(T)$ as $T \longrightarrow \infty$, where $N_\chi(T) = N_{0,\chi}(T)$ denotes the number of nontrivial zeros $\rho_\chi = \beta_\chi + i\gamma_\chi$ of $L(s,\chi)$ satisfying $0 < \gamma_\chi < T$.

In [1], Conrey and Ghosh suggested the problem of estimating the average $\sum_{0<\gamma^{(k)}<T} \zeta^{(j)}(\rho^{(k)})$ for nonnegative integers j and k, where $\rho^{(k)} = \beta^{(k)} + i\gamma^{(k)}$ denote a zero of the k-th derivative $\zeta^{(k)}(s)$. One of the first results on this topic was given by Fujii [3]. He gave an asymptotic formula of the sum $\sum_{0<\gamma<T} \zeta'(\rho)X^\rho$ for a rational number $X > 0$. The $k = 0$ case was treated by Kaptan et al. [7]. Garunkštis and Steuding [4] gave a generalization of Fujii's asymptotic formula with $X = 1$ that if $T \longrightarrow \infty$, we have

$$\begin{aligned}\sum_{\substack{\rho_a : 0 < \gamma_a \leq T \\ \beta_a > 0}} \zeta'(\rho_a) &= \left(\frac{1}{2} - a\right)\frac{T}{2\pi}\log^2\left(\frac{T}{2\pi}\right) + (c_0 - 1 + 2a)\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) \\ &+ (1 - c_0 - c_0^2 + 3c_1 - 2a)\frac{T}{2\pi} + O\left(Te^{-C\sqrt{\log T}}\right), \end{aligned} \tag{1.2}$$

where C is some positive constant and c_n are the Stieltjes constants given by the Laurent series expansion of $\zeta(s)$ at $s = 1$,

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} c_n (s-1)^n \tag{1.3}$$

Recently, Mazhouda and Onozuka [12] proved that for $j, k \in \mathbb{Z}_{\geq 0}$ and large T,

$$\sum_{1<\gamma_a^{(k)}<T} \zeta^{(j)}\left(\rho_a^{(k)}\right) = (-1)^j\left(\delta_{j,0} + a\delta_{k,0} + B(j,k)\right)\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{j+1} + O_{j,k}\left(T(\log T)^j\right), \tag{1.4}$$

where the implicit constant in the error terms may depend on a. To do so, they used the following result of Karabulut and Yildirim [10] for fixed $j, k \in \mathbb{Z}_{\geq 0}$ and large T, one has

$$\sum_{0<\gamma^{(k)}<T} \zeta^{(j)}(\rho^{(k)}) = (-1)^j(\delta_{j,0} + B(j,k))\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{j+1} + O_{j,k}(T\log^j T), \tag{1.5}$$

where $\delta_{j,0} = 1$ if $j = 0$ and 0 otherwise,

$$B(j,k) = -\frac{k+1}{j+1} - j!\sum_{r=1}^{k}\frac{e^{-z_r}}{z_r^{j+1}}P_k(z_r) + j!\sum_{r=1}^{k}\frac{1}{z_r^{j+1}}, \tag{1.6}$$

the sum over r being void in the case $k = 0$ and z_r $(r = 1, ..., k)$ being the zeros of $P_k(z) = \sum_{j=0}^{k}\dfrac{z^j}{j!}$.

Let $\rho_{a,\chi}^{(k)} = \beta_{a,\chi}^{(k)} + i\gamma_{a,\chi}^{(k)}$ denote an a-point of $L^{(k)}(s,\chi)$. Similar to the a-points of $L(s,\chi)$, there is an a-point of $L^{(k)}(s,\chi)$ near any trivial zero $s = -2n - \left(\frac{1-\chi(-1)}{2}\right)$ for sufficiently large n and apart from these

a-points, there are only finitely many other a-points in the half-plane $\sigma \leq C$ for any $C < 0$ (see Lemma 2.1 below).

In this paper, first we give an asymptotic formula for the sum

$$\sum_{\rho_{0,\chi}^{(k)}:0<\gamma_{0,\chi}^{(k)}<T} L^{(j)}(\rho_{0,\chi}^{(k)}, \chi) \tag{1.7}$$

and as a consequence, we obtain an estimate for

$$\sum_{\rho_{a,\chi}^{(k)}:1<\gamma_{a,\chi}^{(k)}<T} L^{(j)}(\rho_{a,\chi}^{(k)}, \chi) \tag{1.8}$$

where a is a complex number. The first sum extend Karabulut and Yildirim's result to the k-th derivative of the Dirichlet L-functions and is evaluated in the following theorem.

Theorem 1.1 *Let $k, j \in \mathbb{N}$ be fixed and χ be a primitive character modulo q. Then as $T \to \infty$, we have*

$$\sum_{\rho_{\chi}^{(k)};\ 0<\gamma_{\chi}^{(k)}\leq T} L^{(j)}\left(\rho_{\chi}^{(k)}, \chi\right) = (-1)^j (\delta_{j,0} + B(j,k)) \frac{T}{2\pi} \left(\log \frac{qT}{2\pi}\right)^{j+1} + O_{j,k}\left(T \left(\log qT\right)^j\right), \tag{1.9}$$

where $B(j,k)$ is defined by (1.6).

From Theorem 1.1, we get our main result

Theorem 1.2 *Let $k, j \in \mathbb{N}$ be fixed, a be a complex number and χ be a primitive character modulo q. Then as $T \to \infty$, we have*

$$\sum_{\rho_{a,\chi}^{(k)};\ 1<\gamma_{a,\chi}^{(k)}\leq T} L^{(j)}\left(\rho_{a,\chi}^{(k)}, \chi\right) = (-1)^j (\delta_{j,0} + a\delta_{k,0} + B(j,k)) \frac{T}{2\pi} \left(\log \frac{qT}{2\pi}\right)^{j+1} + O_{j,k}\left(T \left(\log qT\right)^j\right) \tag{1.10}$$

Here and in the sequel, the implicit constant in the error terms may depend on a.

Remark. By Theorem 1.2, we deduce the average value of $L^{(j)}(\rho_a^{(k)}, \chi)$ over the a-points $\rho_{a,\chi}^{(k)}$ of $L^{(k)}(s, \chi)$ with $1 < \mathrm{Im}(\rho_{a,\chi}^{(k)}) < T$, i.e.

$$\frac{1}{N_{k,\chi}(a,T)} \sum_{1<\gamma_{a,\chi}^{(k)}<T} L^{(j)}(\rho_{a,\chi}^{(k)}, \chi),$$

where $N_{k,\chi}(a,T)$ is the number of terms in the above sum. By the same argument as in [13], we have an asymptotic formula for $N_{k,\chi}(a,T)$ which is $\sim (T/2\pi)\log \frac{qT}{2\pi}$ (see [15] for the asymptotic formula of $N_{k,\chi}(0,T)$). Hence, the average is $(-1)^j \left(\delta_{j,0} + a\delta_{k,0} + B(j,k)\right) \left(\log \frac{qT}{2\pi}\right)^j$. Thus, this tells us about the size of $L^{(j)}(s,\chi)$ at certain points (namely the a-points of $L^{(k)}(s,\chi)$).

2. Preliminary lemmas and equations

In this section, we give some lemmas and formulas useful for the proof of our Theorems. We start with well-known results on the Dirichlet L-function $L(s,\chi)$ (see Davenport book [2]) and its k-th derivative. If χ mod q is a primitive character, then

$$L(s,\chi)=\Lambda(s,\chi)L(1-s,\overline{\chi}), \tag{2.1}$$

where

$$\Lambda(s,\chi)=\frac{2\tau(\chi)}{i^{\kappa}q}\left(\frac{2\pi}{q}\right)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi}{2}(s+\kappa)\right), \tag{2.2}$$

with $\tau(\chi)=\sum_{r=1}^{q}\chi(r)e^{\frac{2\pi i r}{q}}$ and $\kappa=\frac{1}{2}(1-\chi(-1))$. From (2.2) and by Stirling's formula (see[9, page 13]), we get

$$\begin{aligned}&\Lambda(1-s,\chi)\\ =\;&\frac{\tau(\chi)}{i^{\kappa}\sqrt{q}}\exp\left\{it\log\left(\frac{q|t|}{2\pi e}\right)-sgn(t)(\frac{i\pi}{2})(\frac{1}{2}-\kappa)\right\}\left(\frac{q|t|}{2\pi}\right)^{\sigma-\frac{1}{2}}\left(1+O\left(\frac{1}{|t|}\right)\right)\end{aligned} \tag{2.3}$$

in any fixed halfstrip $\alpha\leq\sigma\leq\beta,|t|\geq 1$. Moreover, for any fixed $\sigma,j\geq 0$ and $|t|\geq 1$, we have

$$\frac{\Lambda'}{\Lambda}(s,\chi)=-\log\frac{q|t|}{2\pi}+O\left(\frac{1}{|t|}\right),\left(\frac{d}{ds}\right)^{j}\frac{\Lambda'}{\Lambda}(s,\chi)\ll|t|^{-j} \tag{2.4}$$

and

$$\Lambda^{(j)}(1-s,\chi)=\Lambda(1-s,\chi)\left(-\log\frac{q|t|}{2\pi}\right)^{j}+O\left(q^{\sigma-\frac{1}{2}}|t|^{\sigma-\frac{3}{2}}\left(\log q|t|\right)^{j-1}\right). \tag{2.5}$$

Using equations (2.3)–(2.5) with upon j-fold differentiation of the functional equation (2.1), we obtain

$$L^{(j)}(1-s,\chi)\;=\;(-1)^{j}\Lambda(1-s,\chi)\left(1+O\left(\frac{1}{t}\right)\right)\sum_{m=0}^{j}\binom{j}{m}\ell^{j-m}L^{(m)}(s,\overline{\chi}), \tag{2.6}$$

where σ is fixed, $|t|\geq 1$ and $\ell=\log\left(\frac{q|t|}{2\pi}\right)$. Furthermore, for any fixed $\sigma,k\in\mathbb{Z}_{\geq 0}$ and $t\geq 1$, we have

$$\begin{aligned}\frac{L^{(k+1)}}{L^{(k)}}(1-s,\chi)\;&=\;-\left(1+O\left(\frac{1}{t}\right)\right)\left(\ell+\frac{\sum_{v=0}^{k}\binom{k}{v}\ell^{k-v}\frac{L^{(v+1)}}{L}(s,\overline{\chi})}{\sum_{w=0}^{k}\binom{k}{w}\ell^{k-w}\frac{L^{(w)}}{L}(s,\overline{\chi})}\right)\\ &=\;-\left(1+O\left(\frac{1}{t}\right)\right)\left(\ell+\frac{\sum_{v=0}^{k}\binom{k}{v}\frac{1}{\ell^{v}}\frac{L^{(v+1)}}{L}(s,\overline{\chi})}{1+\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^{w}}\frac{L^{(w)}}{L}(s,\overline{\chi})}\right)\\ &=\;-\left(1+O\left(\frac{1}{t}\right)\right)\left(\ell+\frac{G'_k}{G_k}(s,\ell,\overline{\chi})\right),\end{aligned} \tag{2.7}$$

with the differentiation in G' is respect to s. Since $\frac{L^{(w)}}{L}(s,\overline{\chi}) \ll_w 1$ when $\sigma \geq 1+\delta$, for sufficiently large t, we get

$$\sum_{w=1}^{k} \binom{k}{w} \frac{1}{\ell^w} \frac{L^{(w)}}{L}(s,\overline{\chi}) \ll_k \frac{1}{\log qt}. \tag{2.8}$$

By expanding the denominator of (2.7) as a power series, we obtain

$$\begin{aligned} \left(1+\sum_{w=1}^{k} \binom{k}{w} \frac{1}{\ell^w} \frac{L^{(w)}}{L}(s,\overline{\chi})\right)^{-1} &= \sum_{u=0}^{\infty}(-1)^u \left(\sum_{w=1}^{k} \binom{k}{w} \frac{1}{\ell^w} \frac{L^{(w)}}{L}(s,\overline{\chi})\right)^u \\ &= \sum_{u \leq \frac{\log A}{\log\log A}}^{\infty} (-1)^u \left(\sum_{w=1}^{k} \binom{k}{w} \frac{1}{\ell^w} \frac{L^{(w)}}{L}(s,\overline{\chi})\right)^u + O\left(\frac{1}{A}\right) \end{aligned} \tag{2.9}$$

where $\sigma \geq 1+\delta$ and $t \geq A$ for large A. By the functional equation (2.1) and the Phragmén–Lindel$\ddot{o}$f principle, we deduce that

$$L(s,\chi) \ll_\epsilon \begin{cases} |qt|^{\frac{1}{2}-\sigma+\epsilon} & \sigma < 0, \\ |qt|^{\frac{1}{2}(1-\sigma)+\epsilon} & 0 \leq \sigma \leq 1, \\ |qt|^{\epsilon} & \sigma > 1, \end{cases} \tag{2.10}$$

as $|t| \to \infty$ and where ϵ is an arbitrarily small positive number. Moreover, by Cauchy's integral formula, we get

$$L^{(k)}(s,\chi) = \frac{k!}{2\pi i} \int_{\mathbf{C}} \frac{L(w,\chi)}{(w-s)^{k+1}} ds,$$

where $\mathbf{C}$ is any arbitrarily small circle centered at s. Using the last bound of $L(s,\chi)$, it follows that

$$L^{(k)}(s,\chi) \ll_\epsilon \begin{cases} |qt|^{\frac{1}{2}-\sigma+\epsilon} & \sigma < 0, \\ |qt|^{\frac{1}{2}(1-\sigma)+\epsilon} & 0 \leq \sigma \leq 1, \\ |qt|^{\epsilon} & \sigma > 1. \end{cases} \tag{2.11}$$

Now, using the same argument as in [13, Lemma 2.6], we get easily

$$\frac{L^{(k+1)}(s,\chi)}{L^{(k)}(s,\chi)-a} = \sum_{|\gamma_{a,\chi}^{(k)}-t|<1} \frac{1}{s-\rho_{a,\chi}^{(k)}} + O\left(\log qt\right), \tag{2.12}$$

for any constants α, β and $s \in \mathbb{C}$ with $\alpha \leq \sigma \leq \beta$ and large t.

Lemma 2.1 *Let k be a positive integer, χ be a primitive character modulo q and $a \in \mathbb{C}$. Then, there exist real numbers $E_1 = E_1(k,a,q) \leq 0$ and $E_2 = E_2(k,a,q) \geq 1$ such that there is no a-point of $L^{(k)}(s,\chi)$ for $\{s \in \mathbb{C}, \sigma \leq E_1, |t| \geq 1\}$ and $\{s \in \mathbb{C}, \sigma \geq E_2\}$.*

Proof The case $a=0$ was treated by Yildirim in [16]. Hence, we consider only the case $a\neq 0$. From equation (2.1) and by differentiating k times, we obtain

$$\begin{aligned} L^{(k)}(1-s,\chi) &= (-1)^k \frac{2\tau(\chi)}{i^\kappa q}\left(\frac{2\pi}{q}\right)^{-s}\sum_{j=0}^{k}\Gamma^{(j)}(s)R_{j,k}(s) \\ &= (-1)^k \frac{2\tau(\chi)}{i^\kappa q}\left(\frac{2\pi}{q}\right)^{-s}\left\{\Gamma^{(k)}(s)\cos\left(\frac{\pi}{2}(s-\kappa)\right)L(s,\overline{\chi})+\sum_{j=0}^{k-1}\Gamma^{(j)}(s)R_{j,k}(s)\right\}, \end{aligned} \tag{2.13}$$

where

$$R_{j,k}(s) = P_{j,k}(s)\cos\left(\frac{\pi}{2}(s-\kappa)\right)+Q_{j,k}(s)\sin\left(\frac{\pi}{2}(s-\kappa)\right), \tag{2.14}$$

$$P_{j,k}(s) = \sum_{n=0}^{k} a_{j,k,n}L^{(n)}(s,\overline{\chi}) \tag{2.15}$$

and

$$Q_{j,k}(s)=\sum_{n=0}^{k} b_{j,k,n}L^{(n)}(s,\overline{\chi}), \tag{2.16}$$

where $a_{j,k,n}$ and $b_{j,k,n}$ are constants that may depend on q. Using [16, Equation(13)], derivatives of the Gamma function can be estimated as follows:

$$\Gamma^{(j)}(s)=\Gamma(s)\left(\log s\right)^j\left(1+O\left(\frac{1}{s\log s}\right)\right) \tag{2.17}$$

in the region $\{s\in\mathbb{C},\sigma\geq 1+\delta,|t|\geq 1\}$. Using the last estimate and the fact that in the same region $L(s,\chi)\asymp 1$ and $L^{(j)}(s,\chi)=\sum_{n\geq 2}\frac{\chi(n)(-\log n)^j}{n^s}\ll 1$, we get

$$\left|\Gamma^{(k)}(s)\cos\left(\frac{\pi}{2}(s-\kappa)\right)L(s,\overline{\chi})\right| \asymp \left|\Gamma(s)\log^k(s)e^{\pi\frac{|t|}{2}}\right| \tag{2.18}$$

and

$$\left|\sum_{j=0}^{k-1}\Gamma^{(j)}(s)R_{j,k}(s)\right| \ll \left|\Gamma(s)\log^{k-1}(s)e^{\pi\frac{|t|}{2}}\right|. \tag{2.19}$$

As a consequence, one has

$$\begin{aligned} &L^{(k)}(1-s,\chi) \\ = &(-1)^k\frac{2\tau(\chi)}{i^\kappa q}\left(\frac{2\pi}{q}\right)^{-s}\Gamma(s)\log^k(s)\cos\left(\frac{\pi}{2}(s-\kappa)\right)L(s,\overline{\chi})\left(1+O\left(\frac{1}{|\log s|}\right)\right) \end{aligned} \tag{2.20}$$

in the region $\{s\in\mathbb{C},\sigma\geq 1+\delta,|t|\geq 1\}$. It follows from (2.20) that $L^k(1-s,\chi)\to\infty$ as $\sigma\to\infty$. Thus, there exists $E_1=E_1(k,a,q)\leq 0$ such that $|L^{(k)}(s,\chi)|>|a|$ for $\sigma\leq E_1$ and $|t|\geq 1$. Next, since

$L^{(k)}(s,\chi) = \sum_{n\geq 2} \frac{\chi(n)(-\log n)^k}{n^s} \to 0$ as $\sigma \to \infty$ there exists $E_2 = E_2(k,a,q) \geq 1$ such that $|L^{(k)}(s,\chi)| < |a|$.
□

Remark. It can also be seen by Rouché's theorem that there is $N_k = N_k(a,q) < 0$ such that $L^{(k)}(s,\chi) = a$ has only one zero in the region $\{s \in \mathbb{C},\ \ -1-2n-\kappa < \sigma < 1-2n-\kappa,\ \ -1 < t < 1\}$ for $-n < N_k$. Moreover, apart from these a-points, there are only finitely many other a-points in the half-plane $\sigma \leq C$ for any $C < 0$.

From Lemma 2.1, equation (2.11) and by Jensen's formula, we deduce easily the following lemma.

Lemma 2.2 *For any complex number a and any sufficiently large T, we have*

$$N_{k,\chi}(a;1,T+1) - N_{k,\chi}(a;1,T) \ll \log(qT), \tag{2.21}$$

where

$$N_{k,\chi}(a;1,T) = \sum_{\rho_{a,\chi}^{(k)}:1<\gamma_{a,\chi}^{(k)}<T} 1. \tag{2.22}$$

3. Proof of Theorem 1.1

To prove Theorem 1.1, we use the same argument as in [10]. For this purpose, we need to extend some lemmas for k-th derivative of Dirichlet L-function $L^k(s,\chi)$. The case $k=0$ was already proved by Kaptan et al. [6], so here we assume $k \neq 0$.

Lemma 3.1 *Let $(b_n)_n$ be a sequence of complex numbers such that $b_n \ll n^\epsilon$ for any $\epsilon > 0$. Let $a > 1$ and m be an integer. Then, for $1 \leq T_1 \leq T$ and $|m| = O(T)$ as $T \to \infty$, one has*

$$\frac{1}{2\pi}\int_{T_1}^{T} \Lambda(1-a-it,\chi)\left(\log\left(\frac{qt}{2\pi}\right)\right)^m \sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}dt$$
$$= \frac{\tau(\chi)}{q}\sum_{1\leq n\leq \frac{qT}{2\pi}} b_n e^{-\frac{2\pi i n}{q}}(\log n)^m + O\left((qT)^{a-\frac{1}{2}}(\log qT)^m\right) + O\left(q^{2a-1}(\log q)^m\right).$$

Proof The case m nonnegative is treated by Kaptan in [8, Lemma 2.14] which is based on [5, Lemma 2] (see also [10, Lemma 2.2]). For the case when m is negative, we use the same argument of Kaptan and [11, Lemma 3.5] to obtain the result. □

An elementary computation yields the following lemma.

Lemma 3.2 *For $k, i_1, i_2, ..., i_k, m \in \mathbb{N}, v \in \{0,1,...,k\}$, $\sigma > 1$ and χ be a Dirichlet character modulo q, let us define*

$$\sum_{n=1}^{\infty}\frac{c_n(i_1,i_2,...,i_k;v;m;\chi)}{n^s} := \frac{L^{(v+1)}}{L}(s,\chi)L^{(m)}(s,\chi)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\chi)\right)^{i_w}.$$

We have

$$\chi'(n)c_n(i_1,i_2,...,i_k;v;m;\chi) = c_n(i_1,i_2,...,i_k;v;m;\chi'\chi),$$

for every Dirichlet character χ' *modulo* q, *with*

$$|c_n(i_1,i_2,...,i_k;v;m;\chi)| \le (\log n)^{K+m+1},$$

where

$$K := i_1 + 2i_2 + ... + ki_k + v.$$

Lemma 3.3 *Let* χ *be a Dirichlet character modulo* q. *Let* $k, i_1, i_2, ..., i_k, m \in \mathbb{N}, v \in \{0,1,...,k\}$. *For fixed* k, *if* $i_1 + i_2 + ... + i_k \le \frac{\log x}{\log\log x}$, *then as* $T \to \infty$, *we have*

$$\sum_{n \le x} c_n(i_1,i_2,...,i_k;v;m;\chi) = O_k\left(x(\log x)^{K+m}\right),$$

if χ *is nonprincipal and*

$$\sum_{n \le x} c_n(i_1,i_2,...,i_k;v;m;\chi) = \frac{\varphi(q)}{q} S(i_1,i_2,...,i_k;v;m) x(\log x)^{K+m+1} + O_k\left(x(\log x)^{K+m}\right)$$

if χ *is the principal character, where*

$$S(i_1,i_2,...,i_k;v;m) = \frac{(-1)^{K+m+1}(v+1)!m!\prod_{w=1}^{k}(w!)^{i_w}}{(K+m+1)!}.$$

Proof Let χ be a nonprincipal character modulo q. Lemma 3.2 with Perron's formula [15, chapter 3.12], yields

$$\begin{aligned}\sum_{n \le x} c_n(i_1,i_2,...,i_k;v;m;\chi) &= \int_{1+\frac{1}{\log x}-iU}^{1+\frac{1}{\log x}+iU} \frac{L^{(v+1)}}{L}(s,\chi)L^{(m)}(s,\chi)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\chi)\right)^{i_w}\frac{x^s}{s}ds \\ &+ O_k\left(\frac{x}{U}(\log x)^{K+m+2}\right),\end{aligned}$$

where $q \le U \le x$. Now, let C be the rectangle with vertices $1+\frac{1}{\log x}-iU$, $1+\frac{1}{\log x}+iU$, σ_0+iU, and $\sigma_0 - iU$.

Case 1. Assume that $L(s,\chi)$ has no exceptional zero. We take $\sigma_0 = 1-\frac{c}{5\log qU}$, where c is the constant such that $L(s,\chi) \ne 0$ for $\sigma > 1-\frac{c}{\log qU}$ (see [2, page 93]). Thus, the integrand is analytic on and inside C and we have the bound $\frac{L^{(w)}}{L}(s,\chi) \ll (\log qU)^w$. Then, we have by Cauchy's formula

$$M = \frac{1}{2\pi i}\int_C \frac{L^{(v+1)}}{L}(s,\chi)L^{(m)}(s,\chi)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\chi)\right)^{i_w}\frac{x^s}{s}ds = 0.$$

Now, using that $L^{(m)}(s,\chi) \ll (qU)^{\frac{1}{2}(1-\sigma)+\epsilon}$, we get

$$\begin{aligned}\int_{1+\frac{1}{\log x}+iU}^{1-\frac{1}{5\log qU}+iU} \frac{L^{(v+1)}}{L}(s,\chi)L^{(m)}(s,\chi)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\chi)\right)^{i_w}\frac{x^s}{s}ds &\ll (\log qU)^{K+1}\int_{1+\frac{1}{\log x}}^{1-\frac{1}{5\log qU}} (qU)^{\frac{1}{2}(1-\sigma)+\epsilon}\frac{x^\sigma}{|\sigma+iU|}d\sigma \\ &\ll x(\log qU)^{K+1}(qU)^{\frac{c}{10\log qU}}\left(\frac{1}{\log x}+\frac{c}{5\log qU}\right) \\ &\ll \frac{x}{U^{1-\epsilon}}(\log qU)^K.\end{aligned}$$

Analogously, we have

$$\int_{1-\frac{1}{5\log qU}-iU}^{1+\frac{1}{\log x}-iU}\frac{L^{(v+1)}}{L}(s,\chi)L^{(m)}(s,\chi)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\chi)\right)^{i_w}\frac{x^s}{s}ds\ll\frac{x}{U^{1-\epsilon}}(\log qU)^K$$

and

$$\begin{aligned}\int_{1-\frac{c}{5\log qU}+iU}^{1-\frac{1}{5\log qU}-iU}\frac{L^{(v+1)}}{L}(s,\chi)L^{(m)}(s,\chi)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\chi)\right)^{i_w}\frac{x^s}{s}ds &\ll (\log qU)^{K+1}\int_{-U}^{U}(|qt|)^{\frac{1}{2}(1-\sigma_0)+\epsilon}\frac{x^{\sigma_0}}{|\sigma_0+it|}dt\\ &\ll x(\log qU)^{K+1}(qU)^{\frac{c}{10\log qU}}x^{\sigma_0}\int_{-U}^{U}\frac{1}{|\sigma_0+it|}dt\\ &\ll xU^{\epsilon}(\log qU)^{K+1}\exp\left(\frac{-c\log x}{10\log qU}\right).\end{aligned}$$

Let $U=(\log x)^2$. Then, from all above estimates, we obtain

$$\sum_{n\le x}c_n(i_1,i_2,...,i_k;v;m;\chi)=O_k\left(x(\log x)^{K+m}\right)$$

Case 2. Suppose that there is an exceptional zero β, with $\beta\ge 1-\frac{c}{4\log qU}$. Therefore, we take $\sigma_0=1-\frac{c}{3\log qU}$. Thus, the integrand has a pole at β of order $L+1$, where $L=i_1+i_2+...+i_k$. Hence,

$$\begin{aligned}M &= \frac{1}{L!}\frac{d^L}{ds^L}\left\{(s-\beta)^{L+1}\frac{L^{(v+1)}}{L}(s,\chi)L^{(m)}(s,\chi)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\chi)\right)^{i_w}\frac{x^s}{s}\right\}_{s=\beta}\\ &= \frac{1}{L!}\sum_{j_1+j_2+j_3=L}\left(\begin{array}{c}L\\ j_1,j_2,j_3\end{array}\right)\frac{d^{j_1}}{ds^{j_1}}\left\{(s-\beta)^{L+1}\frac{L^{(v+1)}}{L}(s,\chi)L^{(m)}(s,\chi)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\chi)\right)^{i_w}\right\}_{s=\beta}\\ &\times \frac{d^{j_2}}{ds^{j_2}}\{x^s\}_{s=\beta}\frac{d^{j_3}}{ds^{j_3}}\left\{\frac{1}{s}\right\}_{s=\beta}\\ &= (-1)L\frac{x^\beta}{\beta^{L+1}}\sum_{j_1=0}^{L}\frac{(-1)^{j_1}}{j_1!}\beta^{j_1}\frac{d^{j_1}}{ds^{j_1}}\left\{(s-\beta)^{L+1}\frac{L^{(v+1)}}{L}(s,\chi)L^{(m)}(s,\chi)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\chi)\right)^{i_w}\right\}_{s=\beta}\\ &\times \sum_{j_2=0}^{L-j_1}\frac{(-1)^{j_2}}{j_2!}\beta^{j_2}(\log x)^{j_2}.\end{aligned}$$

By Cauchy's formula on a disk of radius 1 centered at $s=\beta$, we deduce

$$\begin{aligned}\left|\frac{d^{j_1}}{ds^{j_1}}\left\{(s-\beta)^{L+1}\frac{L^{(v+1)}}{L}(s,\chi)L^{(m)}(s,\chi)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\chi)\right)^{i_w}\right\}\right| &\le j_1!\max_{|s-\beta|=1}\left|\frac{L^{(v+1)}}{L}(s,\chi)L^{(m)}(s,\chi)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\chi)\right)^{i_w}\right|\\ &\ll_k j_1!.\end{aligned}$$

The last equation yields to

$$\begin{aligned} M &\ll_k \frac{x^\beta}{\beta^{L+1}}\sum_{j_1=0}^{L}\beta^{j_1}\sum_{j_2=0}^{L-j_1}\frac{\beta^{j_2}}{j_2!}(\log x)^{j_2} \\ &\ll_k \frac{x^\beta}{\beta}(\log x)^L \\ &\ll_k x(\log x)^L. \end{aligned}$$

As above, we obtain

$$\sum_{n\le x} c_n(i_1,i_2,...,i_k;v;m;\chi) = O_k\left(x(\log x)^{K+m}\right)$$

Case 3. Suppose the existence of an exceptional zero β, with $\beta < 1-\frac{c}{4\log qU}$. Therefore, proceeding similarly as in case 1, we get

$$\sum_{n\le x} c_n(i_1,i_2,...,i_k;v;m;\chi) = O_k\left(x(\log x)^{K+m}\right).$$

The proof of Lemma 3.3 when χ is principal is closely similar to that in [10, Lemma 2.4]. □

Lemma 3.4 *Let χ be a Dirichlet character modulo q. Let $k,i_1,i_2,...,i_k,m\in\mathbb{N}$ and $v\in\{0,1,...,k\}$. For fixed k, if $i_1+i_2+...+i_k\le\frac{\log x}{\log\log x}$, then as $T\to\infty$, we have*

$$\sum_{n\le x}\frac{c_n(i_1,i_2,...,i_k;v;m;\chi)}{(\log n)^{K-r}} = O_{k,r,m}\left(x(\log x)^{r+m}\right)$$

if χ is nonprincipal and

$$\sum_{n\le x}\frac{c_n(i_1,i_2,...,i_k;v;m;\chi)}{(\log n)^{K-r}} = \frac{\varphi(q)}{q}S(i_1,i_2,...,i_k;v;m)x(\log x)^{r+m+1} + O_{k,r,m}\left(x(\log x)^{r+m}\right)$$

if χ is a principal character.

Proof of Theorem 1.1. The basic idea of the proof is to interpret the sum of $L^{(j)}(\rho_\chi^{(k)},\chi)$ as a sum of residues. By Cauchy's theorem, we have

$$\sum_{\substack{0<\gamma_\chi^{(k)}<T \\ -b<\beta_\chi^{(k)}<a}} L^{(j)}(\rho_\chi^{(k)},\chi) = \frac{1}{2\pi i}\int_R L^{(j)}(s,\chi)\frac{L^{(k+1)}}{L^{(k)}}(s,\chi)ds$$

where the integration is taken over a rectangular contour in counterclockwise direction denoted by R with vertices $-b+ic$, $a+ic$, $a+iT$, $-b+iT$ with some constants a, b, $c>0$ such that $\frac{1}{L^{(k)}(a+it,\chi)}\ll_k 1$,

$0 < b < \frac{1}{8}$ and $L^{(k)}(s,\chi)$ has no zero on the lines $t = T$ and $t = c$. From [16, Theorem 3], we deduce that there are finitely many zeros of $L^{(k)}(s,\chi)$ in the region $\sigma < -b$ and $t > c$, then we have

$$\begin{aligned}
\sum_{0<\gamma_\chi^{(k)}<T} L^{(j)}(\rho^{(k)},\chi) &= \frac{1}{2\pi i}\int_R L^{(j)}(s,\chi)\frac{L^{(k+1)}}{L^{(k)}}(s,\chi)ds + O(1) \\
&= \frac{1}{2\pi i}\left\{\int_{-b+ic}^{a+ic} + \int_{a+ic}^{a+iT} + \int_{a+iT}^{-b+iT} + \int_{-b+iT}^{-b+ic}\right\} L^{(j)}(s,\chi)\frac{L^{(k+1)}}{L^{(k)}}(s,\chi)ds + O(1) \\
&= I_1 + I_2 + I_3 + I_4 + O(1)
\end{aligned}$$

The first integral I_1 is independent of T, so $I_1 = O(1)$. Next, we consider I_2, using that $\dfrac{1}{L^{(k)}(a+it,\chi)} \ll_k 1$ and $L^{(j)}(s,\chi) \ll 1$, we get $I_2 = O(T)$. Now, using equation (2.12) and take the horizontal sides of the rectangular contour to be a distance $\gg \dfrac{1}{\log qT}$ from any zero of $L^{(k)}(s,\chi)$, one has

$$\begin{aligned}
I_3 &= \frac{1}{2\pi i}\int_{a+iT}^{b+iT} \sum_{|\gamma_\chi^{(k)}-t|<1} \frac{L^{(j)}(s,\chi)}{s-\rho_\chi^{(k)}}ds + O\left(\int_{a+iT}^{b+iT} \log(qt)L^{(j)}(s,\chi)ds\right) \\
&= O\left((qT)^{\frac{1}{2}+b+\epsilon}\log qT \sum_{|\gamma_\chi^{(k)}-T|<1} 1\right) + O\left((qT)^{\frac{1}{2}+b+\epsilon}\log qT\right).
\end{aligned}$$

By Lemma 2.2, we obtain

$$I_3 = O\left((q)T^{\frac{1}{2}+b+\epsilon}(\log qT)^2\right).$$

This leads $I_3 \ll T$, since $0 < b < \frac{1}{8}$. For the fourth integral I_4, by using equations (2.6), (2.7), and (2.9), we obtain

$$\begin{aligned}
\overline{I_4} &= -\frac{1}{2\pi i}\int_{1+b+ic}^{1+b+iT} L^{(j)}(1-s,\overline{\chi})\frac{L^{(k+1)}}{L^{(k)}}(1-s,\overline{\chi})ds \\
&= \frac{(-1)^j}{2\pi i}\sum_{m=0}^{j}\begin{pmatrix} j \\ m \end{pmatrix}\int_{1+b+ic}^{1+b+iT} \Lambda(1-s,\overline{\chi})\ell^{j-m+1}L^{(m)}(s,\chi)ds \\
&\quad + \frac{(-1)^j}{2\pi i}\sum_{m=0}^{j}\begin{pmatrix} j \\ m \end{pmatrix}\int_{1+b+ic}^{1+b+iT} \Lambda(1-s,\overline{\chi})\ell^{j-m}\frac{G_k'}{G_k}(s,\ell,\chi)L^{(m)}(s,\chi)ds + O(T) \\
&= S_1 + S_2 + O(T).
\end{aligned}$$

Lemma 3.1 gives

$$\begin{aligned} S_1 &= (-1)^j \sum_{m=0}^{j} \binom{j}{m} \frac{\tau(\overline{\chi})}{q} \sum_{1\le n\le \frac{qT}{2\pi}} (-1)^m \chi(n) e^{-\frac{2\pi i n}{q}} (\log n)^{j+1} + O\left(T^{b+\frac{1}{2}} (\log qT)^{j+1}\right) \\ &= (-1)^j \frac{\tau(\overline{\chi})}{q} \sum_{1\le n\le \frac{qT}{2\pi}} \chi(n) e^{-\frac{2\pi i n}{q}} (\log n)^{j+1} \sum_{m=0}^{j} \binom{j}{m} (-1)^m + O\left(T^{b+\frac{1}{2}} (\log qT)^{j+1}\right) \\ &= \begin{cases} O\left(T^{b+\frac{1}{2}} (\log qT)^{j+1}\right) & if \quad j \ge 1, \\ \frac{\tau(\overline{\chi})}{q} \sum_{1\le n\le \frac{qT}{2\pi}} \chi(n) e^{-\frac{2\pi i n}{q}} \log n + O\left(T^{b+\frac{1}{2}} \log qT\right) & if \quad j = 0. \end{cases} \end{aligned}$$

Recall that (see [2, page 146])

$$e^{-\frac{2\pi i n}{q}} = \frac{1}{\varphi(q)} \sum_{\chi' \equiv q} \tau(\overline{\chi'}) \chi'(-n),$$

when $(n, q) = 1$. The last formula yields

$$\begin{aligned} \frac{\tau(\overline{\chi})}{q} \sum_{1\le n\le \frac{qT}{2\pi}} \chi(n) e^{-\frac{2\pi i n}{q}} \log n &= \frac{\tau(\overline{\chi})}{q\varphi(q)} \sum_{\chi' \equiv q} \tau(\overline{\chi'}) \chi'(-1) \sum_{1\le n\le \frac{qT}{2\pi}} \chi(n)\chi'(n) \log n \\ &= \sum_{\chi' \ne \overline{\chi}} \frac{\tau(\overline{\chi})\tau(\overline{\chi'})\chi'(-1)}{q\varphi(q)} \sum_{1\le n\le \frac{qT}{2\pi}} \chi(n)\chi'(n) \log n \\ &+ \frac{\tau(\overline{\chi})\tau(\chi)\overline{\chi(-1)}}{q\varphi(q)} \sum_{1\le n\le \frac{qT}{2\pi}} \chi_0(n) \log n. \end{aligned}$$

Using the following estimate

$$\sum_{1\le n\le x} \chi_0(n) \log n = \frac{\varphi(q)}{q} x \log(x) + O(\frac{\varphi(q)}{q} x) + O(q^{\epsilon} \log(x))$$

and Pólya-Vinogradov inequality

$$\sum_{n\le x} \chi(n) \ll 2\sqrt{q} \log q$$

for every nonprincipal character modulo q, we obtain

$$S_1 = \begin{cases} O\left(T^{b+\frac{1}{2}} (\log qT)^{j+1}\right) & if \quad j \ge 1, \\ \frac{T}{2\pi} \log(\frac{qT}{2\pi}) + O\left(T^{b+\frac{1}{2}} \log qT\right) & if \quad j = 0. \end{cases}$$

Now, we estimate S_2. We have

$$S_2 = \frac{(-1)^j}{2\pi i}\sum_{m=0}^{j}\binom{j}{m}\int_{1+b+ic}^{1+b+iT}\Lambda(1-s,\overline{\chi})\ell^{j-m}\frac{G_k'}{G_k}(s,\ell,\chi)L^{(m)}(s,\chi)ds+O(T)$$

$$= (-1)^j\sum_{m=0}^{j}\binom{j}{m}\sum_{u\leq\frac{\log T}{\log\log T}}\sum_{v=0}^{k}(-1)^u\binom{k}{v}\sum_{i_1+i_2+\ldots+i_k=u}\binom{u}{i_1,i_2,\ldots,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w}$$

$$\times \frac{1}{2\pi}\int_c^T\Lambda(-b-it,\overline{\chi})\ell^{j-K-m}L^{(m)}(1+b+it,\chi)\frac{L^{(v+1)}}{L}(1+b+it,\chi)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(1+b+it,\chi)\right)^{i_w}dt$$

$$+ O_{j,k}\left(T^{\frac{1}{2}+b+\epsilon}\right).$$

From Lemma 3.1, we get

$$S_2 = (-1)^j\sum_{m=0}^{j}\binom{j}{m}\sum_{u\leq\frac{\log T}{\log\log T}}\sum_{v=0}^{k}(-1)^u\binom{k}{v}\sum_{i_1+i_2+\ldots+i_k=u}\binom{u}{i_1,i_2,\ldots,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w}$$

$$\times \frac{\tau(\overline{\chi})}{q}\sum_{1\leq n\leq\frac{qT}{2\pi}}C_n(i_1,i_2,\ldots,i_k;v;m;\chi)e^{-\frac{2\pi i n}{q}}(\log n)^{j-K-m}+O_{j,k}\left(T^{\frac{1}{2}+b+\epsilon}\right).$$

Since

$$e^{-\frac{2\pi i n}{q}} = \frac{1}{\varphi(q)}\sum_{\chi'\equiv q}\tau(\overline{\chi'})\chi'(-n)$$

when $(n,q)=1$, we obtain

$$S_2 = (-1)^j\sum_{m=0}^{j}\binom{j}{m}\sum_{u\leq\frac{\log T}{\log\log T}}\sum_{v=0}^{k}(-1)^u\binom{k}{v}\sum_{i_1+i_2+\ldots+i_k=u}\binom{u}{i_1,i_2,\ldots,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w}$$

$$\times \left\{\sum_{\chi'\neq\overline{\chi}}\frac{\tau(\overline{\chi})\tau(\chi')\chi'(-1)}{q\varphi(q)}\sum_{1\leq n\leq\frac{qT}{2\pi}}\frac{\chi'(n)c_n(i_1,i_2,\ldots,i_k;v;m;\chi)}{(\log n)^{K+m-j}}+\frac{\tau(\overline{\chi})\tau(\chi)\overline{\chi}(-1)}{q\varphi(q)}\sum_{1\leq n\leq\frac{qT}{2\pi}}\frac{\overline{\chi}(n)c_n(i_1,i_2,\ldots,i_k;v;m;\chi)}{(\log n)^{K+m-j}}\right\}$$

$$+ O_{j,k}\left(T^{\frac{1}{2}+b+\epsilon}\right).$$

By Lemma 3.4, we deduce

$$S_2 = (-1)^j\frac{T}{2\pi}\left(\log\frac{qT}{2\pi}\right)^{j+1}\sum_{m=0}^{j}\binom{j}{m}\sum_{u\leq\frac{\log T}{\log\log T}}\sum_{v=0}^{k}(-1)^u\binom{k}{v}$$

$$\times \sum_{i_1+i_2+\ldots+i_k=u}\binom{u}{i_1,i_2,\ldots,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w}\frac{(-1)^{K+m+1}(v+1)!m!\prod_{w=1}^{k}(w!)^{i_w}}{(K+m+1)!}+O_{j,k}\left(T(\log qT)^j\right)$$

This last sum S_2 was evaluated by Karabulut and Yildirim in [10]

$$S_2 = (-1)^j\frac{T}{2\pi}\left(\log\frac{qT}{2\pi}\right)^{j+1}B(j,k)+O_{j,k}\left(T(\log qT)^j\right)$$

Combining S_1 and S_2, we obtain

$$I_4 = (-1)^j(\delta_{j,0} + B(j,k))\frac{T}{2\pi}\left(\log\frac{qT}{2\pi}\right)^{j+1} + O_{j,k}\left(T\left(\log qT\right)^j\right).$$

Finally, theorem 1.1 follows from estimates of I_1, I_2, I_3, and I_4.

4. Proof of Theorem 1.2

Let a be a complex number. We write $s = \sigma + it$, $\rho_{a,\chi}^{(k)} = \beta_{a,\chi}^{(k)} + i\gamma_{a,\chi}^{(k)}$ with real numbers $\sigma, t, \beta_{a,\chi}^{(k)}$ and $\gamma_{a,\chi}^{(k)}$. The case $a = 0$ was already proven in Theorem 1.1, so here we assume $a \neq 0$. By the residue theorem, for a sufficiently large constant B and constant $b \in (1, 9/8)$, we have

$$\sum_{\substack{1<\gamma_{a,\chi}^{(k)}<T \\ 1-b<\beta_{a,\chi}^{(k)}<B}} L^{(j)}\left(\rho_{a,\chi}^{(k)},\chi\right) = \frac{1}{2\pi i}\int_{\mathbf{R}} L^{(j)}(s,\chi)\frac{L^{(k+1)}(s,\chi)}{L^{(k)}(s,\chi)-a}ds, \tag{4.1}$$

where the integration is taken over a rectangular contour in counterclockwise direction denoted by $\mathbf{R}$ with vertices $1-b+i$, $B+i$, $B+iT$, $1-b+iT$. Since there are finitely many a-points in $\{s \in \mathbb{C};\ \mathrm{Re}(s) \leq 1-b,\ \mathrm{Im}(s) \geq 1\}$, we have

$$\sum_{1<\gamma_{a,\chi}^{(k)}<T} L^{(j)}\left(\rho_{a,\chi}^{(k)},\chi\right) = \frac{1}{2\pi i}\int_{\mathbf{R}} L^{(j)}(s,\chi)\frac{L^{(k+1)}(s,\chi)}{L^{(k)}(s,\chi)-a}ds + O(1).$$

Hence,

$$\begin{aligned} &\sum_{1<\gamma_{a,\chi}^{(k)}<T} L^{(j)}\left(\rho_{a,\chi}^{(k)},\chi\right) \\ &= \frac{1}{2\pi i}\left\{\int_{1-b+i}^{B+i} + \int_{B+i}^{B+iT} + \int_{B+iT}^{1-b+iT} + \int_{1-b+iT}^{1-b+i}\right\} L^{(j)}(s,\chi)\frac{L^{(k+1)}(s,\chi)}{L^{(k)}(s,\chi)-a}ds + O(1) \\ &:= \frac{1}{2\pi i}(I_1 + I_2 + I_3 + I_4) + O(1). \end{aligned} \tag{4.2}$$

The integral I_1 is independent of T, so we have $I_1 = O(1)$. Next, we consider I_2. Since $L^{(k)}(s,\chi) \to 0$ as $\sigma \to \infty$ if $k \geq 1$, we choose in this case B such that $|L^{(k)}(B+it,\chi)| < \frac{|a|}{2}$, then we have $\frac{1}{L^{(k)}(B+it,\chi)-a} \ll_k 1$. Using this and $L^{(j)}(s,\chi) \ll 1$, we get

$$I_2 = O(T).$$

For the case $k = 0$, recall that, for $\sigma \to \infty$, we have $L(s,\chi) = 1 + o(1)$ and $L'(s,\chi) \ll 2^{-\sigma}$ uniformly in t. Hence, there are no a-points for sufficiently large σ provided that $a \neq 1$. For the case a $= 1$, we define $m = \min\{n \geq 2, \chi(n) \neq 0\}$. We observe, for $\sigma \to \infty$, $L(s,\chi) - 1 = \frac{\chi(m)}{m^{\sigma+it}}(1 + o(1))$. Hence, we choose B a fixed constant sufficiently large such that there are no a-points of $L(s,\chi)$ in the half-plane $\sigma > B-1$. Therefore, we deduce that

$$I_2 = O(T).$$

From equation (2.12), we get

$$I_3 = \sum_{|\gamma_{a,\chi}^{(k)}-T|<1} \int_{B+iT}^{1-b+iT} \frac{L^{(j)}(s,\chi)}{s-\rho_{a,\chi}^{(k)}}\, ds + O\left(\int_{B+iT}^{1-b+iT} (\log qt) L^{(j)}(s,\chi) ds\right).$$

Now, we change the path of integration. If $\gamma_{a,\chi}^{(k)} < T$, we change the path to the upper semicircle with center $\rho_{a,\chi}^{(k)}$ and radius 1. If $\gamma_{a,\chi}^{(k)} > T$, we change the path to the lower semicircle with center $\rho_{a,\chi}^{(k)}$ and radius 1. Then, we have

$$\frac{1}{s-\rho_{a,\chi}^{(k)}} \ll 1$$

on the new path. This estimate and the bound (21) yields

$$I_3 = O\left((qT)^{b-\frac{1}{2}+\epsilon} \sum_{|\gamma_{a,\chi}^{(k)}-T|<1} 1\right) + O\left((qT)^{b-\frac{1}{2}+\epsilon} \log qT\right).$$

By Lemma 2.2, we obtain

$$I_3 = O\left((qT)^{b-\frac{1}{2}+\epsilon} \log qT\right).$$

This yields $I_3 \ll T$, since $1 < b < 9/8$.

Finally, we estimate I_4. By equation (2.20) and Stirling's formula, for fixed $1 < b < 9/8$ and large $|t| > 2$, we have

$$\left|L^{(k)}(1-b+it,\chi)\right| \asymp |qt|^{b-1/2} \left|\log|t|\right|^k. \tag{4.3}$$

Therefore, there exists a constant A such that

$$\left|\frac{a}{L^{(k)}(1-b+it,\chi)}\right| < 1$$

holds for any $|t| \geq A$. We divide the path of the integral into two parts

$$I_4 = \left(\int_{1-b+iT}^{1-b+iA} + \int_{1-b+iA}^{1-b+i}\right) L^{(j)}(s,\chi) \frac{L^{(k+1)}(s,\chi)}{L^{(k)}(s,\chi)-a} ds.$$

The second term is $O(1)$ since it is independent of T. Since the integrand of the first term has a geometric series, we have

$$I_4 = -\sum_{n=0}^{\infty} a^n \int_{1-b+iA}^{1-b+iT} \frac{L^{(j)}(s,\chi)L^{(k+1)}(s,\chi)}{(L^{(k)}(s,\chi))^{n+1}} ds + O(1).$$

By (4.3), the integrand can be estimated as

$$\frac{L^{(j)}(s,\chi)L^{(k+1)}(s,\chi)}{(L^{(k)}(s,\chi))^{n+1}} \asymp |qt|^{(b-1/2)(1-n)} (\log t)^{-kn+j+1}. \tag{4.4}$$

Hence, each integral can be calculated as

$$\int_{1-b+iA}^{1-b+iT} \frac{L^{(j)}(s,\chi)L^{(k+1)}(s,\chi)}{(L^{(k)}(s,\chi))^{n+1}}ds \ll (qT)^{(b-1/2)(1-n)+1+\varepsilon}$$

for any small $\varepsilon > 0$. It follows from the last estimate that the sum for $n \geq 2$ is bounded as

$$\sum_{n=2}^{\infty} a^n \int_{1-b+iA}^{1-b+iT} \frac{L^{(j)}(s,\chi)L^{(k+1)}(s,\chi)}{(L^{(k)}(s,\chi))^{n+1}}ds \ll T^{-(b-1/2)+1+\varepsilon} \ll T^{1/2}.$$

Therefore, we get

$$\begin{aligned} I_4 &= -\int_{1-b+iA}^{1-b+iT} \frac{L^{(j)}(s,\chi)L^{(k+1)}(s,\chi)}{L^{(k)}(s,\chi)}ds - a\int_{1-b+iA}^{1-b+iT} \frac{L^{(j)}(s,\chi)L^{(k+1)}(s,\chi)}{(L^{(k)}(s,\chi))^2}ds + O\left(T^{1/2}\right) \\ &:= -K_1 - aK_2 + O\left(T^{1/2}\right). \end{aligned}$$

We already studied K_1 in Theorem 1.1 and we get the estimate

$$K_1 = -2\pi i \left\{ \delta_{j,0}\frac{T}{2\pi}\log\frac{qT}{2\pi} + (-1)^j B(j,k)\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{j+1} + O\left(T(\log qT)^j\right)\right\}.$$

It remains to evaluate K_2. By equation (4.4), for $k \geq 1$, one has

$$K_2 \ll \int_{1-b+iA}^{1-b+iT} |\log t|^j \, |ds| \ll T(\log T)^j.$$

In the case $k = 0$, we use equations (2.1) and (2.6) to obtain

$$\frac{L^{(j)}(s,\chi)L'(s,\chi)}{L^2(s,\chi)} = (-1)^{j+1}\ell^{j+1}\left(1 + O\left(\frac{1}{|t|}\right)\right) \tag{4.5}$$

for fixed σ and $|t| \gg 1$, where $\ell := \log(q|t|/2\pi)$. Then, we have

$$\begin{aligned} K_2 &= \int_{1-b+iA}^{1-b+iT} \left((-\ell)^{j+1} + O\left((\log q|t|)^j\right)\right)ds \\ &= (-1)^{j+1} iT\left(\log\frac{qT}{2\pi}\right)^{j+1} + O\left(T(\log qT)^j\right). \end{aligned}$$

Combining estimates of K_1 and K_2, we get

$$I_4 = (-1)^j 2\pi i\left(\delta_{j,0} + a\delta_{k,0} + B(j,k)\right)\frac{T}{2\pi}\left(\log\frac{qT}{2\pi}\right)^{j+1} + O\left(T(\log qT)^j\right).$$

Finally, Theorem 1.2 follows from estimates of I_1, I_2, I_3 and I_4.

5. Concluding remarks

The a-points of an L-function $L(s)$ are the roots of the equation $L(s) = a$. We refer to Steuding's book [14, chapter 7] for some results about a-points of L-functions from the Selberg class. Therefore, it is an interesting question to extend Theorem 1.1 and mainly Theorem 1.2 to the other class of Dirichlet L-functions (the Selberg class with some further condition) and its higher derivative. This problem will be considered in a sequel to this paper since it is done for the Riemann zeta function and its k-th derivative in [6] and [12].

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