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On quasi-affinity and reducing subspaces of multiplication operator on a certain closed subspace

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Abstract: Let H denote a certain closed subspace of the Bergman space $A^2_\alpha(\mathbb{B}_n)(\alpha > -1)$ of the unit ball in $\mathbb{C}^n$. In this paper, we prove that the operator $\bigoplus\limits_1^m M_{z^{(s_1,\cdots,s_n)}}$ is quasi-affine to the multiplication operator $M_{z^{(ms_1,\cdots,ms_n)}}$ on H. Furthermore, the reducing subspaces of $M_{z^{(ms_1,\cdots,ms_n)}}$ are characterized on H.

Key words: Bergman space, multiplication operator, quasi-affinity, reducing subspaces

1. Introduction

Let $\mathbb{C}$ denote the set of complex numbers and $\mathbb{C}^n = \mathbb{C}\times\mathbb{C}\times\cdots\times\mathbb{C}$ denote the Euclidean space of complex dimension n. The open unit ball in $\mathbb{C}^n$ is the set $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$. The boundary of $\mathbb{B}_n$ is the set $\mathbb{S}_n = \{z \in \mathbb{C}^n : |z| = 1\}$. For $\alpha > -1$, the weighted Bergman space $A^2_\alpha(\mathbb{B}_n)$ consists of holomorphic functions f in $L^2(\mathbb{B}_n, dv_\alpha)$. The weighted Lebesgue measure dv_α is defined by $dv_\alpha(z) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}(1-|z|^2)^\alpha dv(z)$. The Bergman space $A^2_\alpha(\mathbb{B}_n)$ is a Hilbert space with the reproducing kernel $K^\alpha(z,w) = \frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}}$, $z, w \in \mathbb{B}_n$. If $f, g \in A^2_\alpha(\mathbb{B}_n)$, the inner product of f and g is defined by $\langle f, g\rangle_\alpha = \int_{\mathbb{B}_n} f(z)\overline{g(z)}dv_\alpha(z)$.

For a bounded linear operator S on a Hilbert space H, let $\mathcal{A}'(S)$ denote the commutant of S, i.e. $\mathcal{A}'(S) = \{T \in \mathcal{L}(H)|TS = ST\}$. The characterization of the commutant of S should help in understanding the structure of S. From the information of the commutant of a given operator, people research the similar or unitary equivalence and the reducing subspaces of the operator. These problems on function spaces such as the Hardy space and the Bergman space have been studied extensively in the literature. We mention here that the papers [1–3, 5, 6, 8–11, 13–15] and the books [4, 17] include a lot of analysis of the operator theory associated with the Hardy and the Bergman spaces. J. A. Ball (see [2]) and E. Nordgren (see [13]) studied the problem of determining the reducing subspaces for an analytic Toeplitz operator on the Hardy space. In [14], M. Stessin and K. H. Zhu described the properties of the commutant of analytic Toeplitz operators with inner function symbols on the Hardy space and the Bergman space. In [7], Gu characterized reducing subspaces of nonanalytic Toeplitz operators on weighted Hardy and Dirichlet spaces of the bidisk. In 2010, Lu and Zhou researched the invariant subspaces and reducing subspaces of weighted Bergman space over bidisk (see [12]). In 2011, Douglas and Kim in [5] studied the reducing subspaces of an analytic multiplication operator M_{z^n} on

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the Bergman space $A^2_\alpha(A_r)$ of the annulus A_r.

Based on the above works, in this paper, we prove that the operator $\bigoplus_1^m M_{z^{(s_1,\cdots,s_n)}}$ is quasi-affine to the multiplication operator $M_{z^{(ms_1,\cdots,ms_n)}}$ on a certain closed subspace H which depends on $s=(s_1,\cdots,s_n)$ of the Bergman space $A^2_\alpha(\mathbb{B}_n)$. Then we characterize the reducing subspaces of $M_{z^{(ms_1,\cdots,ms_n)}}$ on H.

2. The quasi-affinity of $M_{z^{(ms_1,\cdots,ms_n)}}$ $(m\geq 2)$

Recall that the functions $e_l(z)=\sqrt{\frac{\Gamma(n+|l|+\alpha+1)}{l!\Gamma(n+\alpha+1)}}z^l$ forms an orthonormal basis of the Bergman space $A^2_\alpha(\mathbb{B}_n)$, where $l=(l_1,l_2,\cdots,l_n)$ runs over all multiindexes of nonnegative integers. And $|l|=l_1+l_2+\cdots+l_n$, $l!=(l_1)!\cdots(l_n)!$, $z^l=z_1^{l_1}\cdots z_n^{l_n}$. It is well known that the multiplication operator M_{w^m} is similar to $\bigoplus_1^m M_w$ on the Bergman space $A^2_\alpha(\mathbb{D})$ (see [11]). Now, we will investigate the situation on the Bergman space $A^2_\alpha(\mathbb{B}_n)$. Because of the complication of multiindexes map, we consider a certain closed subspace H of the Bergman space $A^2_\alpha(\mathbb{B}_n)$. That is, $H=\bigvee_{k=0}^{\infty}\{e_{(ks_1,\cdots,ks_n)}\}$. Without loss of generality, we assume that $s_i\geq 1\,(i=1,2,\cdots,n)$.

In the following lemma we give a decomposition of the space H.

Lemma 2.1 *If $H_j=span\{e_{((mk+j)s_1,\cdots,(mk+j)s_n)}|k\geq 0\}$ $(j=0,1,\cdots,m-1)$, then we have*
(i) $\{e_{((mk+j)s_1,\cdots,(mk+j)s_n)}\}_{k=0}^{\infty}$ forms an orthonormal basis of H_j.
(ii) $H=H_0\bigoplus H_1\bigoplus\cdots\bigoplus H_{m-1}$.
(iii) H_j is a reducing subspace of $M_{z^{(ms_1,\cdots,ms_n)}}$.

Proof (i) Simple computation shows that

$$\langle e_{((mk_1+j)s_1,\cdots,(mk_1+j)s_n)},e_{((mk_2+j)s_1,\cdots,(mk_2+j)s_n)}\rangle=\begin{cases}1, & k_1=k_2,\\ 0, & k_1\neq k_2.\end{cases}\tag{2.1}$$

(ii) It is easy to prove that $H_j\perp H_t$, $0\leq j\neq t\leq m-1$.

Next, for $f\in H$, we have that f has the form

$$f=\sum_{k=0}^{\infty}a_{0k}e_{(mks_1,\cdots,mks_n)}+\cdots+\sum_{k=0}^{\infty}a_{m-1,\,k}e_{((mk+m-1)s_1,\cdots,(mk+m-1)s_n)}.$$

Suppose that $f=0$. Then from

$$\langle\sum_{k=0}^{\infty}\sum_{j=0}^{m-1}a_{jk}e_{((mk+j)s_1,\cdots,(mk+j)s_n)},e_{(ls_1,\cdots,ls_n)}\rangle=0\ (l=0,1,\cdots),\tag{2.2}$$

we obtain that $a_{jk}=0\,(j=0,\cdots,m-1,\,k=0,1,\cdots)$. That is, $0=\overbrace{0\oplus 0\oplus\cdots\oplus 0}^{m}$.
Therefore, $H=H_0\bigoplus H_1\bigoplus\cdots\bigoplus H_{m-1}$.
(iii) It is easy to see that both H_j and $H_j^{\perp}$ are invariant subspaces of $M_{z^{(ms_1,\cdots,ms_n)}}$. $\square$

Let H and K be complex Hilbert spaces. An operator X in $\mathcal{L}(H,K)$ is said to be quasi-invertible if X has zero kernel and dense range. Recall that for $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$, A is quasi-affine to B if there exists a quasi-invertible operator S in $\mathcal{L}(H,K)$ such that $SA = BS$ (see [16]).

Now we state our main theorem of this section.

Theorem 2.2 *The operator* $\bigoplus_1^m M_{z^{(s_1,\cdots,s_n)}}$ *is quasi-affine to the multiplication operator* $M_{z^{(ms_1,\cdots,ms_n)}}$ *on* H.

Proof In the following, for simplicity, we set $\gamma_k = \sqrt{\frac{\Gamma(n+k|s|+\alpha+1)}{\prod_{i=1}^{n}(ks_i)!\Gamma(n+\alpha+1)}}$.

Note that

$$M_{z^{(s_1,\cdots,s_n)}} e_{(ks_1,\cdots,ks_n)} = z^{(s_1,\cdots,s_n)}\gamma_k z^{(ks_1,\cdots,ks_n)} = \frac{\gamma_k}{\gamma_{k+1}} e_{((k+1)s_1,\cdots,(k+1)s_n)}. \tag{2.3}$$

Set $M_j = M_{z^{(ms_1,\cdots,ms_n)}}|_{H_j}$ $(j = 0, 1, \cdots, m-1)$. Then

$$\begin{aligned} & M_j e_{((mk+j)s_1,\cdots,(mk+j)s_n)} \\ = \ & z^{(ms_1,\cdots,ms_n)}\gamma_{mk+j} z^{((mk+j)s_1,\cdots,(mk+j)s_n)} \\ = \ & \frac{\gamma_{mk+j}}{\gamma_{mk+m+j}} e_{((mk+m+j)s_1,\cdots,(mk+m+j)s_n)}. \end{aligned} \tag{2.4}$$

Define $X_j \colon H \to H_j$ such that $X_j e_{(ks_1,\cdots,ks_n)} = c_{kj} e_{((mk+j)s_1,\cdots,(mk+j)s_n)}$. The coefficients c_{kj} are to be determined later. Then we have

$$X_j M_{z^{(s_1,\cdots,s_n)}} e_{(ks_1,\cdots,ks_n)} = M_j X_j e_{(ks_1,\cdots,ks_n)}.$$

In fact,

$$X_j \frac{\gamma_k}{\gamma_{k+1}} e_{((k+1)s_1,\cdots,(k+1)s_n)} = M_j c_{kj} e_{((mk+j)s_1,\cdots,(mk+j)s_n)},$$

and

$$\begin{aligned} & \tfrac{\gamma_k}{\gamma_{k+1}} c_{k+1,j} e_{((mk+m+j)s_1,\cdots,(mk+m+j)s_n)} \\ = \ & c_{kj} \tfrac{\gamma_{mk+j}}{\gamma_{mk+m+j}} e_{((mk+m+j)s_1,\cdots,(mk+m+j)s_n)}. \end{aligned}$$

From

$$\tfrac{c_{k+1,j}}{c_{kj}} = \tfrac{\gamma_{k+1}\gamma_{mk+j}}{\gamma_k\gamma_{mk+m+j}}, \tag{2.5}$$

we obtain

$$c_{kj} = \tfrac{\gamma_k\gamma_j}{\gamma_{mk+j}} \ (k \geq 0). \tag{2.6}$$

Next, we will compute the limit of sequence $\{c_{kj}\}$ as $k \to +\infty$.

$$\begin{aligned} & \lim_{k\to+\infty}(c_{kj})^2 \\ = \ & \lim_{k\to+\infty}(\gamma_j)^2 \frac{\Gamma(n+k|s|+\alpha+1)\prod_{i=1}^{n}((mk+j)s_i)!}{\Gamma(n+(mk+j)|s|+\alpha+1)\prod_{i=1}^{n}(ks_i)!}. \end{aligned} \tag{2.7}$$

Hence, we only need to compute the limit of

$$d_{kj} = \frac{\prod\limits_{i=1}^{n}[(km+j)s_i]!\Gamma(n+k|s|+\alpha+1)}{\prod\limits_{i=1}^{n}(ks_i)!\Gamma(n+(km+j)|s|+\alpha+1)}. \tag{2.8}$$

Applying the Stirling's formula $n! \sim n^{n+\frac{1}{2}}e^{-n}\sqrt{2\pi}$, we have

$$\begin{aligned}
& d_{kj} \\
= \;& \frac{\prod\limits_{i=1}^{n}[(km+j)s_i]!\Gamma(n+k|s|+\alpha+1)}{\prod\limits_{i=1}^{n}(ks_i)!\Gamma(n+(km+j)|s|+\alpha+1)} \\
\sim \;& \frac{\prod\limits_{i=1}^{n}[(km+j)s_i]^{(km+j)s_i+\frac{1}{2}}}{\prod\limits_{i=1}^{n}(ks_i)^{ks_i+\frac{1}{2}}} \times \frac{(k|s|+n+\alpha)^{k|s|+n+\alpha+\frac{1}{2}}}{[(km+j)|s|+n+\alpha]^{(km+j)|s|+n+\alpha+\frac{1}{2}}} \\
= \;& \prod_{i=1}^{n}\left[\frac{(km+j)s_i}{ks_i}\right]^{ks_i+\frac{1}{2}} \times \prod_{i=1}^{n}[(km+j)s_i]^{(km+j)s_i-ks_i} \times \left[\frac{k|s|+n+\alpha}{(km+j)|s|+n+\alpha}\right]^{k|s|+n+\alpha+\frac{1}{2}} \\
& \times\left[\frac{1}{(km+j)|s|+n+\alpha}\right]^{(km+j)|s|-k|s|} \\
= \;& \prod_{i=1}^{n}\left(m+\frac{j}{k}\right)^{ks_i+\frac{1}{2}} \times \prod_{i=1}^{n} s_i^{(km+j)s_i-ks_i} \times \left[\frac{k|s|+n+\alpha}{(km+j)|s|+n+\alpha}\right]^{k|s|+n+\alpha+\frac{1}{2}} \\
& \times\left[\frac{km+j}{(km+j)|s|+n+\alpha}\right]^{(km+j)|s|-k|s|} \\
= \;& m^{k|s|+\frac{1}{2}n}\left(1+\frac{j}{km}\right)^{k|s|+\frac{1}{2}n}\prod_{i=1}^{n} s_i^{(km+j)s_i-ks_i}\left(1+\frac{n+\alpha}{k|s|}\right)^{k|s|+n+\alpha+\frac{1}{2}} \\
& \times\left(\frac{1}{m}\right)^{k|s|+n+\alpha+\frac{1}{2}}\left(\frac{1}{1+\frac{j|s|+n+\alpha}{km|s|}}\right)^{k|s|+n+\alpha+\frac{1}{2}}\left(\frac{1}{|s|}\right)^{(km+j)|s|-k|s|}\left[\frac{1}{1+\frac{n+\alpha}{(km+j)|s|}}\right]^{(km+j)|s|-k|s|}
\end{aligned}$$

$$\begin{aligned}
= \;& \left(\frac{1}{m}\right)^{\frac{1}{2}n+\alpha+\frac{1}{2}}\left[\left(1+\frac{j}{km}\right)^{\frac{1}{2}n}\left(1+\frac{n+\alpha}{k|s|}\right)^{n+\alpha+\frac{1}{2}}\left(\frac{1}{1+\frac{j|s|+n+\alpha}{km|s|}}\right)^{n+\alpha+\frac{1}{2}}\right]\frac{\prod\limits_{i=1}^{n} s_i^{(km+j)s_i-ks_i}}{|s|^{(km+j)|s|-k|s|}} \\
& \times\left[\left(1+\frac{j}{km}\right)^{k|s|}\left(\frac{1}{1+\frac{j|s|+n+\alpha}{km|s|}}\right)^{k|s|}\left(1+\frac{n+\alpha}{(km+j)|s|}\right)^{k|s|}\right] \\
& \times\left(1+\frac{n+\alpha}{k|s|}\right)^{k|s|}\left[\frac{1}{1+\frac{n+\alpha}{(km+j)|s|}}\right]^{(km+j)|s|}.
\end{aligned} \tag{2.9}$$

Note that

$$\lim_{k\to\infty}\left[\left(1+\frac{j}{km}\right)^{\frac{1}{2}n}\left(1+\frac{n+\alpha}{k|s|}\right)^{n+\alpha+\frac{1}{2}}\left(\frac{1}{1+\frac{j|s|+n+\alpha}{km|s|}}\right)^{n+\alpha+\frac{1}{2}}\right] = 1.$$

For

$$\lim_{k\to\infty}\frac{\prod\limits_{i=1}^{n} s_i^{(km+j)s_i-ks_i}}{|s|^{(km+j)|s|-k|s|}},$$

we set

$$a = \max\{s_1, \cdots, s_n\}, b = \min\{s_1, \cdots, s_n\}.$$

Then

$$\frac{b^{(km+j)|s|-k|s|}}{|s|^{(km+j)|s|-k|s|}} \leq \frac{s_1^{(km+j)s_1-ks_1} \cdots s_n^{(km+j)s_n-ks_n}}{|s|^{(km+j)|s|-k|s|}} \leq \frac{a^{(km+j)|s|-k|s|}}{|s|^{(km+j)|s|-k|s|}}.$$

From

$$\lim_{k\to\infty} \frac{b^{(km+j)|s|-k|s|}}{|s|^{(km+j)|s|-k|s|}} = 0, \lim_{k\to\infty} \frac{a^{(km+j)|s|-k|s|}}{|s|^{(km+j)|s|-k|s|}} = 0,$$

we obtain

$$\lim_{k\to\infty} \frac{s_1^{(km+j)s_1-ks_1} \cdots s_n^{(km+j)s_n-ks_n}}{|s|^{(km+j)|s|-k|s|}} = 0.$$

Clearly, when $j = 0$,

$$\lim_{k\to\infty} \left[\left(1 + \frac{j}{km}\right)^{k|s|} \left(\frac{1}{1 + \frac{j|s|+n+\alpha}{km|s|}} \right)^{k|s|} \left(1 + \frac{n+\alpha}{(km+j)|s|}\right)^{k|s|} \right] = 1.$$

When $j = 1, \cdots, m-1$, we have

$$\begin{aligned}
&\lim_{k\to\infty} \left[\left(1 + \frac{j}{km}\right)^{k|s|} \left(\frac{1}{1 + \frac{j|s|+n+\alpha}{km|s|}} \right)^{k|s|} \left(1 + \frac{n+\alpha}{(km+j)|s|}\right)^{k|s|} \right] \\
= &\lim_{k\to\infty} \left(1 + \frac{j}{km}\right)^{k|s|} \lim_{k\to\infty} \left[\frac{km|s|}{km|s| + j|s| + n + \alpha} \times \frac{(km+j)|s| + n + \alpha}{(km+j)|s|} \right]^{k|s|} \\
= &\ 1.
\end{aligned}$$

And

$$\lim_{k\to\infty} \left(1 + \frac{n+\alpha}{k|s|}\right)^{k|s|} \left[\frac{1}{1 + \frac{n+\alpha}{(km+j)|s|}} \right]^{(km+j)|s|} = 1.$$

Moreover

$$\lim_{k\to\infty} d_{kj} = 0.$$

Therefore

$$\lim_{k\to\infty} (c_{kj})^2 = \lim_{k\to\infty} \gamma_j^2 d_{kj} = 0.$$

Suppose that $f \in \ker X_j$, and $f = \sum_{k=0}^{\infty} \xi_k e_{(ks_1, \cdots, ks_n)}$, $\xi_k \in \mathbb{C}$. Then, from

$$\begin{aligned}
0 &= \langle X_j f, e_{((mk+j)s_1, \cdots, (mk+j)s_n)} \rangle \\
&= \langle \sum_{k=0}^{\infty} \xi_k c_{kj} e_{((mk+j)s_1, \cdots, (mk+j)s_n)}, e_{((mk+j)s_1, \cdots, (mk+j)s_n)} \rangle,
\end{aligned}$$

we deduce that $\xi_k = 0$ $(k = 0, 1, \cdots)$. So $\ker X_j = \{0\}$.

Next, for $g \in \ker X_j^*$, and $g = \sum_{k=0}^{\infty} \eta_k e_{((mk+j)s_1,\cdots,(mk+j)s_n)}$, $\eta_k \in \mathbb{C}$, from

$$\begin{aligned} 0 &= \langle X_j^* g, e_{(ks_1,\cdots,ks_n)}\rangle = \langle g, X_j e_{(ks_1,\cdots,ks_n)}\rangle \\ &= \langle \sum_{k=0}^{\infty} \eta_k e_{((mk+j)s_1,\cdots,(mk+j)s_n)}, c_{kj} e_{((mk+j)s_1,\cdots,(mk+j)s_n)}\rangle, \end{aligned}$$

we have $\eta_k = 0\,(k = 0, 1, \cdots)$. So $\ker X_j^* = (\operatorname{ran} X_j)^{\perp} = \{0\}$. That is, $\overline{\operatorname{ran} X_j} = H_j$. This implies that X_j is quasi-invertible. Therefore, $M_{z^{(s_1,\cdots,s_n)}}$ is quasi-affine to M_j. Since

$$M_{z^{(ms_1,\cdots,ms_n)}}\,|_H = M_0 \bigoplus M_1 \bigoplus \cdots \bigoplus M_{m-1},$$

we conclude that $\bigoplus_1^m M_{z^{(s_1,\cdots,s_n)}}$ is quasi-affine to $M_{z^{(ms_1,\cdots,ms_n)}}$. □

3. The reducing subspaces of $M_{z^{(ms_1,\cdots,ms_n)}}$ on H

From Lemma 2.1, we know that for any $f \in H$, there exists a unique orthogonal decomposition $f = f_0 + f_1 + \cdots + f_{m-1}$, where $f_j \in H_j\,(j = 0, 1, \cdots, m-1)$. In this section, we will characterize the reducing subspaces of $M_{z^{(ms_1,\cdots,ms_n)}}$ on H.

Lemma 3.1 *If $M_g : H_j \longrightarrow H$ is a bounded multiplication operator by a function g on $\mathbb{B}_n$, then $g \in H^{\infty}(\mathbb{B}_n)$, and $\|g\|_{\infty} \leq \|M_g\|$.*

Proof Since $g = \frac{M_g z^{ks}}{z^{ks}}$, clearly $M_g z^{ks}$ and z^{ks} are 2 holomorphic functions. Set $\Omega = \{(z_1, z_2, \cdots, z_n) : z_1 z_2 \cdots z_n = 0\}$, then $g \in H(\mathbb{B}_n \backslash \Omega)$. By the Riemann's theorem about removable singular point in the several variable functions, there exists a unique holomorphic function $G \in H(\mathbb{B}_n)$ such that $G|_{\mathbb{B}_n \backslash \Omega} = g$. Let η_{z^s} denote the point evaluation functional on H defined by $\eta_{z^s}(h) = h(z^s)$ for $h \in H$. It is obvious that η_{z^s} is bounded, and for $h_k \in H_k$,

$$|g(z^s)\eta_{z^s}(h_k)| = |g(z^s)h_k(z^s)| = |\eta_{z^s}(M_g(h_k))| \leq \|\eta_{z^s}\|\|M_g\|\|h_k\|.$$

It follows that

$$|g(z^s)|\|\eta_{z^s}\| \leq \|\eta_{z^s}\|\|M_g\|.$$

Therefore, $|g(z^s)| \leq \|M_g\|$, and g is holomorphic on $\mathbb{B}_n$. □

Lemma 3.2 *Let $S \in \mathcal{L}(H)$. Then $S \in \{M_{z^{(ms_1,\cdots,ms_n)}}\}'$ if and only if there exist functions $g_j\,(j = 0, 1, \cdots, m-1)$ in $H^{\infty}(\mathbb{B}_n)$ such that $Sf = \sum_{j=0}^{m-1} g_j f_j$, where $f_j \in H_j$.*

Proof Let $Sf = \sum_{j=0}^{m-1} g_j f_j$, and $M_{g_j} : H \to H$ is the multiplication operator defined by $M_{g_j}(h) = g_j h$ for $h \in H$. Then for any $f \in H$, we have

$$\begin{aligned} & SM_{z^{(ms_1,\cdots,ms_n)}} f \\ =\; & Sz^{(ms_1,\cdots,ms_n)} f = z^{(ms_1,\cdots,ms_n)} \sum_{j=0}^{m-1} g_j f_j \\ =\; & M_{z^{(ms_1,\cdots,ms_n)}} Sf. \end{aligned} \tag{3.1}$$

That is, $SM_{z^{(ms_1,\cdots,ms_n)}} = M_{z^{(ms_1,\cdots,ms_n)}}S$, as desired.

Next, if we suppose that $SM_{z^{(ms_1,\cdots,ms_n)}} = M_{z^{(ms_1,\cdots,ms_n)}}S$, then

$$M^*_{t^{(ms_1,\cdots,ms_n)}-z^{(ms_1,\cdots,ms_n)}}S^* = S^*M^*_{t^{(ms_1,\cdots,ms_n)}-z^{(ms_1,\cdots,ms_n)}},$$

for any $t = (t_1, t_2, \cdots, t_n) \in \mathbb{C}^n$. From [17], we know that the reproducing kernel of H is defined by

$$K^\alpha(z^s, w^s) = \sum_{k=0}^{\infty} \frac{\Gamma(n+k|s|+\alpha+1)}{\prod_{i=1}^{n}(ks_i)!\Gamma(n+\alpha+1)} z^{ks}\bar{w}^{ks}$$

such that for each f in H, $f(w^s) = \langle f(z^s), K^\alpha(z^s, w^s)\rangle$, where we use z^{ks} to denote $z_1^{ks_1}z_2^{ks_2}\cdots z_n^{ks_n}$. We know that $\ker M^*_{t^{(ms_1,\cdots,ms_n)}-z^{(ms_1,\cdots,ms_n)}}$ is generated by the set

$$\begin{aligned} &\{K^\alpha(z^s, (t_1w_j, \cdots, t_nw_j)^s) \\ =\ & \sum_{k=0}^{\infty} \frac{\Gamma(n+k|s|+\alpha+1)}{\prod_{i=1}^{n}(ks_i)!\Gamma(n+\alpha+1)} z^{ks}\overline{tw_j}^{ks} | w_j = (e^{\frac{2j\pi}{ms_1}i}, e^{\frac{2j\pi}{ms_2}i}, \cdots, e^{\frac{2j\pi}{ms_n}i}),\ 0 \le j \le m-1\}. \end{aligned}$$

Note that $S^*K^\alpha(z^s, t^s) \in \ker M^*_{t^{(ms_1,\cdots,ms_n)}-z^{(ms_1,\cdots,ms_n)}}$, and we obtain

$$S^*K^\alpha(z^s, t^s) = \sum_{j=0}^{m-1} \overline{a_j(t^s)} K^\alpha(z^s, (tw_j)^s),\ a_j(t^s) \in \mathbb{C}. \tag{3.2}$$

Thus,

$$\begin{aligned} & Sf(z^s) = \langle Sf, K^\alpha(w^s, z^s)\rangle \\ =\ & \langle f, S^*K^\alpha(w^s, z^s)\rangle = \sum_{j=0}^{m-1} a_j(z^s) f((zw_j)^s) \\ =\ & \sum_{j=0}^{m-1} a_j(z^s)(f_0((zw_j)^s) + f_1((zw_j)^s) + \cdots + f_{m-1}((zw_j)^s)) \\ =\ & \sum_{j=0}^{m-1} a_j(z^s)(f_0(z^s) + w_j^s f_1(z^s) + \cdots + w_j^{(m-1)s} f_{m-1}(z^s)), \end{aligned} \tag{3.3}$$

where the last equality holds because $w_j^{ms} = 1$ $(0 \le j \le m-1)$. Let

$$g_j = \sum_{l=0}^{m-1} a_l(z^s) w_l^{js}\ (0 \le j \le m-1). \tag{3.4}$$

Since $g_j(z^s) = \frac{S(z^{js})}{z^{js}}$, $0 \le j \le m-1$, by Lemma 3.1, we know that $\|g_j\|_\infty \le \|M_{g_j}\|$, and $g_j \in H^\infty(\mathbb{B}_n)$. Thus, we obtain $Sf = \sum_{j=0}^{m-1} g_j f_j$. □

Now we will determine the reducing subspaces of the multiplication operator $M_{z^{(ms_1,\cdots,ms_n)}}$ on the space H.

Theorem 3.3 *Suppose that $A = (A_{jk})_{m\times m}$ is a projection such that $MA = AM$, where* $M = \begin{pmatrix} M_0 & 0 & \cdots & 0 & 0 \\ 0 & M_1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & M_{m-1} \end{pmatrix}$. *Then there exist functions $\varphi_{jk} \in H$ $(0 \le j, k \le m-1)$, such that $A_{jk} = M_{\varphi_{jk}}$, and* $\varphi_{jk} = \begin{cases} c_j, & j = k, \\ 0, & j \ne k, \end{cases}$ *where c_j is a real number.*

Proof By Lemma 3.2, we have $Af=\sum\limits_{j=0}^{m-1}\varphi_j f_j$, and (3.4) yields that $\varphi_j=\sum\limits_{l=0}^{m-1}a_l(z^s)w_l^{js}$, $\varphi_j\in H^\infty(\mathbb{B}_n)$. Set $a_l(z^s)=\sum\limits_{\lambda=0}^{m-1}a_{l\lambda}(z^s)$, and $a_{l\lambda}(z^s)\in H_\lambda$ $(0\le\lambda\le m-1)$. Then, the operator A has the following matrix representation

$$A=\begin{pmatrix}\sum\limits_{l=0}^{m-1}a_{l0}(z^s) & \sum\limits_{l=0}^{m-1}a_{l0}(z^s)w_l^s & \cdots & \sum\limits_{l=0}^{m-1}a_{l0}(z^s)w_l^{(m-1)s}\\ \sum\limits_{l=0}^{m-1}a_{l1}(z^s) & \sum\limits_{l=0}^{m-1}a_{l1}(z^s)w_l^s & \cdots & \sum\limits_{l=0}^{m-1}a_{l1}(z^s)w_l^{(m-1)s}\\ \cdot & \cdot & \cdot & \cdot\\ \sum\limits_{l=0}^{m-1}a_{l,m-1}(z^s) & \sum\limits_{l=0}^{m-1}a_{l,m-1}(z^s)w_l^s & \cdots & \sum\limits_{l=0}^{m-1}a_{l,m-1}(z^s)w_l^{(m-1)s}\end{pmatrix}. \tag{3.5}$$

It follows that $A=(M_{\varphi_{jk}})_{m\times m}$ $(0\le j,k\le m-1)$, and $M_{\varphi_{jk}}:H_k\to H_j$. Now, we will analyze the multiplication operators $M_{\varphi_{jk}}$. For $j=k$, $M_{\varphi_{jj}}:H_j\to H_j$, suppose that $\varphi_{jj}=\sum\limits_{t=0}^{\infty}b_t^{jj}e_{(ts_1,\cdots,ts_n)}$, $b_t^{jj}\in\mathbb{C}$. Note that

$$H_j=span\{e_{((mt+j)s_1,\cdots,(mt+j)s_n)}|t\ge0\},$$

so $b_t^{jj}=0$ $(t\ne lm)$. Thus, $\varphi_{jj}(z)=\sum\limits_{l=0}^{\infty}b_{lm}^{jj}e_{(lms_1,\cdots,lms_n)}$. Since A is a projection, we know that $M_{\varphi_{jj}}=M^*_{\varphi_{jj}}$. From

$$\begin{aligned}&\langle M_{\varphi_{jj}}e_{(js_1,\cdots,js_n)},e_{((ml+j)s_1,\cdots,(ml+j)s_n)}\rangle\\ =\;&\langle e_{(js_1,\cdots,js_n)},M^*_{\varphi_{jj}}e_{((ml+j)s_1,\cdots,(ml+j)s_n)}\rangle\\ =\;&\langle e_{(js_1,\cdots,js_n)},M_{\varphi_{jj}}e_{((ml+j)s_1,\cdots,(ml+j)s_n)}\rangle,\end{aligned} \tag{3.6}$$

we deduce that $b_{lm}^{jj}=0$ $(l=1,2,\cdots)$. Observe that $M_{\varphi_{jj}}$ is a self-adjoint operator, hence b_0^{jj} is a real number. Set $c_j=b_0^{jj}$, we then have $\varphi_{jj}=c_j$. If $j\ne k$, $M_{\varphi_{jk}}:H_k\to H_j$, without loss of generality, assume that $k<j$, and set $\varphi_{jk}=\sum\limits_{l=0}^{\infty}b_l^{jk}z^{(ls_1,\cdots,ls_n)}$. Similar to the preceding discussion, we know that only the coefficient $b_{(j-k)+ml}^{jk}$ $(l=0,1,\cdots)$ in the Taylor expansion of $\varphi_{jk}(z)$ is nonzero. Thus,

$$\begin{aligned}&\langle M_{\varphi_{jk}}e_{(ks_1,\cdots,ks_n)},e_{((ml+j)s_1,\cdots,(ml+j)s_n)}\rangle\\ =\;&\langle b_{(j-k)+ml}^{jk}z^{((j-k+ml)s_1,\cdots,(j-k+ml)s_n)}e_{(ks_1,\cdots,ks_n)},e_{((ml+j)s_1,\cdots,(ml+j)s_n)}\rangle\\ =\;&b_{(j-k)+ml}^{jk}\sqrt{\frac{\prod_{i=1}^{n}[(j+ml)s_i]!\Gamma(n+k|s|+\alpha+1)}{\prod_{i=1}^{n}(ks_i)!\Gamma(n+(j+ml)|s|+\alpha+1)}}\;(l=0,1,\cdots).\end{aligned} \tag{3.7}$$

On the other hand, since A is a projection, we have

$$\begin{aligned}&\langle M_{\varphi_{jk}}e_{(ks_1,\cdots,ks_n)},e_{((ml+j)s_1,\cdots,(ml+j)s_n)}\rangle\\ =\;&\langle e_{(ks_1,\cdots,ks_n)},M^*_{\varphi_{jk}}e_{((ml+j)s_1,\cdots,(ml+j)s_n)}\rangle\\ =\;&\langle e_{(ks_1,\cdots,ks_n)},M_{\varphi_{kj}}e_{((ml+j)s_1,\cdots,(ml+j)s_n)}\rangle=0.\end{aligned} \tag{3.8}$$

Hence $b_{(j-k)+ml}^{jk}=0$ $(l=0,1,\cdots)$ and $\varphi_{jk}=0\,(j\ne k)$. The proof is complete. □

Theorem 3.4 *If $M_{z^{(ms_1,\cdots,ms_n)}} \in \mathcal{L}(H)$. Then $M_{z^{(ms_1,\cdots,ms_n)}}$ has 2^m reducing subspaces.*

Proof From Theorem 3.3, we know that A has the following form

$$A = \begin{pmatrix} c_0 & 0 & \cdots & 0 \\ 0 & c_1 & \cdots & 0 \\ . & . & . & . \\ 0 & 0 & \cdots & c_{m-1} \end{pmatrix}. \tag{3.9}$$

$A^2 = A$ yields that $c_j = 0$ $(0 \le j \le m-1)$ or 1. Hence,

$$M_{z^{(ms_1,\cdots,ms_n)}} |_H = M_0 \bigoplus M_1 \bigoplus \cdots \bigoplus M_{m-1}$$

has 2^m reducing subspaces

$$c_0 H_0 \oplus c_1 H_1 \oplus \cdots \oplus c_{m-1} H_{m-1},\ c_j = 0\ (0 \le j \le m-1)\ or\ 1. \tag{3.10}$$

The minimal reducing subspaces are $H_0, H_1, \cdots, H_{m-1}$, as required. □

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