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## Remarks on the treatments of nonsolvable potentials

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**Abstract:** A recently introduced scheme is extended to propose an algebraic nonperturbative approach for the analytical treatment of Schrödinger equations with nonsolvable potentials involving an exactly solvable potential form together with an additional potential term. As an illustration the procedure is successfully applied to the Cornell potential by means of very simple algebraic manipulations. However, instead of providing numerical eigenvalues for the only consideration of the small strength of the related linear potential as in the previous reports, the present model puts forward a clean route to interpret related experimental or precise numerical results involving a wide range of the linear potential strengths. We hope this new technique will shed some light on the questions concerning the limitations of the traditional perturbation techniques.

**Key words:** Nonsolvable potentials, Cornell potential, perturbation theory

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### 1. Introduction

The Schrödinger equation with the Cornell potential,  $V(r) = -\frac{a}{r} + br$ , also known as the Coulomb plus linear potential, has received a great deal of attention [1–19] as an important nonrelativistic model in both particle physics, or more precisely in the context of meson spectroscopy, where it is used to describe systems of quark and antiquark bound states, and in atomic and molecular physics, where it represents a radial Stark effect in hydrogen. Aside from the physical relevance, the solutions of the Schrödinger equation for the Coulomb plus linear potential have been rigorously investigated with a large number of techniques [5,8,10–19] due to its nontrivial mathematical properties. In addition, this potential has an advantage that leads naturally to two choices of parent Hamiltonian in perturbative treatments, one based on the Coulomb part and the other on the linear term, which can be usefully compared. Although such models have been studied for many years, exact solutions of the Schrödinger equation with this potential are still unknown to a great extent. Most of the earlier work relied either on direct numerical integration of the Schrödinger equation or various techniques for approximating the eigenenergies.

Within this context, and also gaining confidence from the successful applications [20,21] of the recently developed formalism that appeared in a series of papers for the analytical treatment of exactly and quasi/conditionally exactly solvable potentials used in analyzing heavy quarkonium observations, we aim in this paper at indicating how the previous approach in [20,21] can be extended to give a further novel prescription for also nonexactly solvable more realistic potentials employed in different disciplines of physics and engineering as

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well. This work therefore completes the preceding reports in [20,21] with a justification of the flexibility of the model developed and used for various treatments. The method that we propose in this article, however, does not suggest a new prescription for the energy eigenvalues of such potentials, unlike the previous works. Approaching such problems from a different perspective, we will try here, contrary to the previous reports, to suggest an unusual but simple expression for an explicit analysis of the results obtained exactly, but without providing a deep physical interpretation of the findings, such as precise numerical treatments or reliable experimental observations for any physical problem that may be modeled by this class of potential. More specifically, the present communication emphasizes an interesting relationship between the critical  $r$ -values, leading to exact energies, in the problem considered and the limitations of the known perturbation theories, which depend on finding an appropriate parent Hamiltonian for which no general procedure is available even though the choice may be crucial to the success of such approximate schemes. In this respect, the comments on our tabulated results provide intuition on this crucial choice, which seems to have been insufficiently appreciated to date.

The paper is organized as follows. In Section 2, we extend the scenario in [20,21] to  $N$ -dimensional space for the treatment of nonsolvable potentials. The main features of the formalism used and its application to Cornell-type potentials are given in Section 3 and in Section 4 we draw our conclusions.

## 2. Formalism

Here, for the stringent test of the present model with the others available in the literature, the usual one-dimensional solution of the nonrelativistic Schrödinger equation is extended to arbitrary dimensions considering the framework in [20,21].

The radial Schrödinger equation for a spherically symmetric potential in  $N$ -dimensional space ( $\hbar = 1$ ) reads as follows [22]:

$$\frac{d^2 R}{dr^2} + \frac{N-1}{r} \frac{dR}{dr} = 2m \left[ \left( V(r) + \frac{\ell(\ell+1)}{2mr^2} \right) - E \right] R, \quad (1)$$

which is transformed to:

$$\frac{d^2 \Psi}{dr^2} = 2m \left[ \left( V(r) + \frac{(M-1)(M-3)}{8mr^2} \right) - E \right] \Psi, \quad (2)$$

where  $\Psi(r) = r^{(N-1)/2} R(r)$ , being the reduced radial wave function, and  $M = N + 2\ell$ . Eq. (2) can also be expressed as:

$$\frac{d^2 \Psi}{dr^2} = 2m \left[ \left( V(r) + \frac{(\Lambda)(\Lambda+1)}{2mr^2} \right) - E \right] \Psi, \quad (3)$$

where  $\Lambda = (M-3)/2$ . We see that the radial Schrödinger equation in  $N$ -dimensions has the same form as the three-dimensional one. Consequently, given that the potential has the same form in any dimension, the solution in three dimensions can be employed to obtain the solution in any dimension simply by using the substitution  $\ell \rightarrow \Lambda$ .

At this stage, we extend our previous formalism, assuming in Eq. (3) that  $\Psi(r) = F[g(r)]f(r)$ , where  $F(g)$  yields an algebraic closed solution for an exactly solvable potential with  $F(g)$  being a special function satisfying a second-order differential equation:

$$\frac{d^2 F}{dg^2} + Q(g) \frac{dF}{dg} + R(g)F(g) = 0, \quad (4)$$

while  $f(r)$  is the moderating function in connection with a perturbing/modifying term of the entire potential given. The form of  $Q(g)$  and  $R(g)$  is already well defined [23] for any special function  $F(g)$  when dealing with analytically solvable potentials. However, in the case of a realistic nonsolvable problem, one should derive a reliable scheme to obtain the corrections more accurately due to the moderating part of the potential in light of the exact part of the calculations. This is the significant point in the framework of the new formalism to reach physically meaningful solutions.

After some algebra, Eq. (3) turns out to be:

$$\frac{F''}{F} (g')^2 + 2 \frac{F'}{F} \frac{f'}{f} g' + \frac{F'}{F} g'' + \frac{f''}{f} = 2m (V_{eff} - E), \quad (5)$$

where  $V_{eff}(r) = V(r) + \frac{\Lambda(\Lambda+1)}{2mr^2}$  and the primes denote derivatives with respect to  $r$ , except the ones related to  $F$  as  $F' = \partial F / \partial g$ . To make a relation between Eqs. (4) and (5), we transform the above equation to the form of:

$$F'' + F' \left( \frac{2}{g'} \frac{f'}{f} + \frac{g''}{(g')^2} \right) + F \left[ \frac{f''}{f (g')^2} + \frac{2m}{(g')^2} (E - V_{eff}) \right] = 0. \quad (6)$$

Bearing in mind that the full spectrum of the system underlined is given by  $E = E_{ES} + \Delta E$ , Eq. (6) can be transformed easily to a coupled equation [20,21], leading to the explicit solutions for exactly solvable part of the potential:

$$E_{ES} - \left[ V_{ES}(r) + \frac{\Lambda(\Lambda+1)}{2mr^2} \right] = \frac{(g')^2}{2m} R(g) \quad (7)$$

and

$$2 \left( \frac{F' g'}{F} \right) \frac{f'}{f} + \frac{f''}{f} = 2m [\Delta V(r) - \Delta E], \quad (8)$$

which is responsible for the computation of the rectifications. More clearly, the solution of Eq. (8) in either an explicit or approximate form depending on the structure of  $\Delta V$  allows us to see the modifications ( $\Delta E, f$ ) to the exact solutions ( $E_{ES}, F(g)$ ) due to the consideration of a modifying/perturbing interaction ( $\Delta V$ ) in a realistic application.

### 3. Application

As the main aim of this article is in general to obtain a physically reliable scheme for more accurate solutions of nonsolvable potentials, and also to illustrate the flexible applicability of the model, we focus here on a specifically chosen example that is the Cornell potential in arbitrary dimensions:

$$V_{eff}(r) = V_{ES}(r) + \Delta V(r) = \left( -\frac{a}{r} + \frac{\Lambda(\Lambda+1)}{2mr^2} \right) + br, \quad (9)$$

in which the first term is an exactly solvable Coulomb-like potential that has the well-known solutions [21] along the frame of Eq. (7), which provides a safe basis for the computation of the connected solutions

$$E_n^{ES} = -\frac{m a^2}{2(n + \Lambda + 1)^2}, \Psi_n^{ES}(r) \sim F_n(g) = \left[ e^{-g/2} g^{(\alpha+1)/2} L_n^\alpha(g) \right], \quad (10)$$

for which we have an appropriate choice [23] such as

$$Q(g) = 0, R(g) = \frac{n + \Lambda + 1}{g} - \frac{\Lambda(\Lambda + 1)}{g^2} - \frac{1}{4}, \quad n = 0, 1, 2, \dots, \tag{11}$$

where  $g(r) = \frac{2mar}{n+\Lambda+1}$ ,  $\alpha = 2\Lambda + 1$ , and finally  $L_n^\alpha(g)$  is the generalized Laguerre polynomial. For the relation between  $f(r)$ ,  $g(r)$ , and  $Q(g)$ , the reader is referred to [20,21].

Obviously, the task here is to solve Eq. (8). Tracking down approximate forms of its solutions has always aroused interest [1–19]. Apart from being useful in the understanding of many physical phenomena, the importance of searching for them also stems from the fact that they very often provide a good starting point for undertaking perturbative calculations of more complex systems. Considering all these earlier works and the experience gained from such calculations, we propose below in Section 3.2 an alternative simple approach, having a physically motivated visualizable framework, which enables explanations of the precise numerical findings or experimental measurements concerning the interaction of interest.

### 3.1. Attempts for the solution of Eq. (8)

In the last three decades, with the rapid development of nonlinear science, there has appeared ever increasing interest of scientists and engineers in the analytical techniques for nonlinear problems. In particular, most of the one-particle potentials that are encountered in such quantum mechanical applications do not allow closed exact solutions either for the energy eigenvalues or for the wave functions, as in the present case. In these circumstances, one resorts to approximate methods, which may be suitable for the particular situation to obtain approximate solutions. Perhaps the most useful solutions in this context are the perturbative solutions. With this consideration, we first show that Eq. (8) may also be transformed to a perturbative scheme. In this case, recalling the full wave function description  $\Psi = F(g)f$  used throughout the article, one needs to rearrange Eq. (8) as  $F_n(g)f_n'' + 2(F_n'g')f_n' = 2m[\Delta V(r) - \Delta E]F_n(g)f_n$ . Using the spirit of perturbation theories,  $f(r)$  and  $\Delta E$  are expanded in terms of the perturbation parameter  $\lambda$  such as

$$f_n(r, \lambda) = 1 + \sum_{j=1}^{\infty} \lambda^j f_{nj}(r), \Delta E_n(\lambda) = \sum_{j=1}^{\infty} \lambda^j \Delta E_{nj}, \Delta V(r) = \lambda^1 (br),$$

where  $j$  denotes the perturbation order. Substitution of these expansions into the new form of Eq. (8) above and by equating terms with the same power of  $\lambda$  on both sides yields, interestingly, the old but well-known alternative perturbation procedure in [24]. However, our exhaustive application results, which are not given here for the sake of clarity, have revealed once more that such approaches inherently work well for only quite small parameters ( $b \ll 1$ ) of the linear potential in Eq. (9).

We then drew our attention to the supersymmetric perturbation theory (SSPT) [25], a lesser-known alternative to the usual perturbative treatments in quantum mechanics. Therefore, in addition to our first attempt mentioned above, the whole formalism discussed in this article has been expressed within the framework of SSPT defining the forms of the superpotentials there as  $W_n^{ES} = -\frac{1}{\sqrt{2m}} \frac{F_n'g'}{F_n(g)}$ , corresponding to the exact (Coulomb-like) potential, and  $\Delta W_n = -\frac{1}{\sqrt{2m}} \frac{f_n'}{f_n}$  concerning the modifying (linear potential) solutions, for the alternative treatment of Eq. (8). Later, as the next step, the familiar expression of the supersymmetric quantum mechanics  $(W_n^{ES} + \Delta W_n)^2 - (W_n^{ES} + \Delta W_n)' / \sqrt{2m} = V - E_n$  was split into two parts as  $(W_n^{ES})^2 -$

$(W_n^{ES})' / \sqrt{2m} = V_{ES} - E_n^{ES}$  and  $2W_n^{ES} \Delta W_n + \Delta W_n^2 - (\Delta W_n)' / \sqrt{2m} = \Delta V - \Delta E_n$ , generating strictly Eqs. (7) and (8), respectively, for individual quantum states. Nevertheless, the expansion of  $\Delta W$  as described in [25], together with the proper expansions of  $\Delta V$  and  $\Delta E$  at each successive perturbation order for the approximate solution of the refinements, makes the calculation scheme considerably cumbersome due to a tedious iterative procedure used for the description of the partner potentials in the model. Besides this inconvenience, which is not illustrated here as it is out of the scope of the present work, we have seen again that this type of calculation works for only rather small confining potential parameters. These two points make the calculation scheme naturally useless and impracticable, as in our previous attempt.

To sum up, with the experience gained from these our earlier calculations and also taking full advantage of the approaches in [24,25], we present here a more economical but instructive prescription for the interpretation of accurate solutions of Eq. (8) in connection with the analysis of Cornell-like funnel potentials.

### 3.2. Alternative approach to Eq. (8)

Clearly, the total wave function of the system under consideration assumes its asymptotic behavior at a very large distance that is completely determined by the modifying linear term ( $br$ ) in the full potential that appeared in Eq. (9), which is the same for all (ground and excited) states, while at intermediate distances the wave function still decays exponentially but is now governed by the Coulomb term. Hence, the lack of a unique frame for the equal treatments of the main and modifying parts of the interaction potential considered in perturbation theories, unlike the model presented here, does not decisively affect the description of especially the ground-state with small parameters, but it destroys the ability of the potential to describe such interactions beyond small perturbation domains. In this respect, the present consideration would offer the opportunity to improve such calculations due to the clear visualization of the refinements, in an explicit manner, in connection with the confinement potential term.

As the full solution for the Coulomb-like potentials at intermediate distances of the present interaction is known, Eq. (10) and both sides of Eq. (7) are equal to each other in the asymptotic region due to the structure of  $R(g)$  in Eq. (11) when  $r \rightarrow \infty$ , clarifying the noncontribution of the Coulomb potential at large distances, and one needs to focus on just Eq. (8) to obtain the corrections that appeared in the asymptotic domain. In this case, evidently the unnormalized solution in this region is given by the Airy function as in [19],  $f(r) = Ai((2mb)^{1/3}r)$ , if (considering only the ground-state for the present purpose)

$$\Delta E_{n=0} \simeq -\frac{1}{m} (F'_{n=0} g' / F_n) f' / f \Rightarrow f''(r) - (2mbr) f'(r) = 0. \quad (12)$$

Thus, this assumption guides us to define the entire solution as  $\Psi_{n=0}(r) = F_{n=0}(g) f(r) = \Psi_{n=0}^{ES}(r) Ai\left((2mb)^{1/3}r\right)$ , where  $\Psi_{n=0}^{ES}$  is given by Eq. (10).

In this circumstance, the correction term for the energy contribution in arbitrary dimensions takes the following form:

$$\Delta E_{n=0} = \left( \frac{a}{\Lambda + 1} - \frac{\Lambda + 1}{m r} \right) \frac{f'(r)}{f(r)}, \quad (13)$$

which depends on  $r$ . This peculiar behavior is of course due to the present consideration of the nonexactly solvable nature of the Cornell potential. Before proceeding, however, it is worthwhile to pay some attention to

Eq. (13). Recalling our previous second attempt to propose an approximate scheme for the solution of Eq. (8) within the framework of the supersymmetric perturbation theory, which was discussed in the previous section, it is clear that

$$2W_n^{ES}\Delta W + \left(\Delta W_n^2 - \frac{(\Delta W)'}{\sqrt{2m}}\right) = br - \Delta E_n, \quad (14)$$

where  $W_n^{ES} = -F'_n g' / \sqrt{2m} F_n(g)$  and  $\Delta W = -f' / \sqrt{2m} f$  for the present case. The substitution of these superpotentials into Eq. (14) yields explicitly Eq. (8) and subsequently Eqs. (12) and (13) for  $n = 0$ , as  $\Delta V = \Delta W^2 - (\Delta W)' / \sqrt{2m} = f'' / 2m f = br$  and

$$\Delta E_{n=0} = 2W_{n=0}^{ES}\Delta W = \left(\frac{a}{\Lambda + 1} - \frac{\ell + 1}{m r}\right) \frac{f'(r)}{f(r)}. \quad (15)$$

From Eq. (15), one sees that a proper choice of  $\Delta W$  in connection with the pertinent wave function ( $f$ ) yields a constant value for  $\Delta E$  reducing the problem to an analytically solvable case. Otherwise, as in the present nonsolvable potential consideration,  $\Delta W$  and  $\Delta E$  should be expanded as in either the usual perturbation or supersymmetric perturbation theories to solve the problem in an approximate form. Unfortunately, such treatments have some drawbacks, as mentioned in Section 3.1.

Therefore, instead of removing this apparent disadvantage in Eq. (15) using an iterative or perturbative scheme, we adopt here a different strategy concerning the clarification of the physics behind nearly precise numerical calculations in light of Eq. (13). This will reveal an interesting relation between the critical  $r$ -values in the wave function, yielding exact energies, and the choice of an appropriate parent potential in perturbation-like theories. A quick analysis of Eq. (13) at this stage tells us that if  $r = r_0 = \frac{(\Lambda+1)^2}{m a}$  then  $\Delta E = 0$ , in which case all the refinements disappear, reducing the entire system to the purely exactly solvable Coulombic case where  $E \rightarrow E_{n=0}^{ES} = -a/2r_0 = -ma^2/2(\Lambda + 1)^2$ . This location in the corresponding energy expression is in fact the maximum of the ground-state Coulomb wave function,  $\Psi_{n=0}^{ES} = r^{\Lambda+1} \exp\left(-\frac{ma}{\Lambda+1}r\right)$ , which can be easily checked through  $(\partial\Psi_{n=0}^{ES}/\partial r) = 0$ .

From this short instructive discussion, a good choice is to proceed with another maximum ( $r_0$ ) in connection with the ground-state of the full wave function in the asymptotic domain,  $\Psi \sim r^{\Lambda+1}f(r)$ , for the precise calculations of  $\Delta E$ . In fact, for the Cornell potential, because of the attractive Coulomb term, the potential function is in general not bounded below and therefore the choice of  $r_0$  to be the location of the maximum of the ground-state wave function in the asymptotic region, as an initial guess, seems reasonable. Hence, one should expect the physically reasonable  $r$ -values in Eq. (13), responsible for the exact energies, in the vicinity of this maximum point. To define exactly the positions of these shifted  $r = r_{\Delta E}$  values, we need to consider the exact energies calculated through precise numerical techniques such as the ones in [13,19] and the related references therein. As  $\Delta E = E_{exact} - E_{ES}$ , Eq. (13) can be expressed as:

$$E_{exact}^{n=0} = -\frac{ma}{2(\Lambda + 1)^2} + \left(\frac{a}{\Lambda + 1} - \frac{\Lambda + 1}{m r_{\Delta E}}\right) \frac{f'(r)}{f(r)}, \quad (16)$$

which allows us to find exact locations for  $r$ -values related to the exact energy eigenvalues of the individual  $\ell$ -states. The systematic calculation of the  $r_{\Delta E}$  value via Eq. (16) offers no difficulty if we resort to a computer algebra system like Mathematica, Maple, or Reduce.

The results obtained are tabulated into three tables. For the purpose of consistency, we have calculated each  $r_{\Delta E}$  value to 16 significant figures, unlike the other two values rounded there for clarity. In Tables 1 and 2 we report the eigenvalues for the Schrödinger equation in distinct dimensions with the potential  $V = -1/r + br$  considering the different sizes of the linear potential strength. Table 3 illustrates the  $S$ -wave ground-state eigenenergies for the potential  $V = -a/r + r$  with dissimilar values of the parameter  $a$ . From these results, the following comments apply:

**Table 1.** Ground-state  $r_{\Delta E}$ -values for  $V(r) = \frac{-1}{r} + br$  in  $N = 3$  dimensional space.

$b$	$\ell$	$r_0$	$r_{\Delta E}$ (Eq. (13))	$\Delta E_{n=0}$
0.01	0	4.103	1.7436481087936350	0.029
	1	6.873	4.9460678940048420	0.080
	2	9.195	7.9252834202633755	0.130
	3	11.265	10.555231364572375	0.175
	4	13.163	12.921702562396474	0.216
	5	14.935	15.094068452133266	0.254
1	0	0.884	0.7994448794104599	1.648
	1	1.481	1.5319979780777233	2.888
	2	1.981	2.1271924940550924	3.878
	3	2.427	2.6476825358214430	4.742
	4	2.836	3.120002415656866	5.527
	5	3.218	3.5579114353323114	6.255
100	0	0.190	0.20718831032409812	46.652
	1	0.319	0.3568814759026067	70.079
	2	0.427	0.48025853095063736	89.743
	3	0.523	0.5892382922692437	107.350
	4	0.611	0.6887636263960271	123.572
	5	0.693	0.7814337345649496	138.768

**Table 2.** Ground-state  $r_{\Delta E}$ -values for  $V(r) = \frac{-1}{r} + br$  in  $N = 4$  dimensional space.

$b$	$\ell$	$r_0$	$r_{\Delta E}$	$\Delta E_{n=0}$
0.01	0	5.566	3.3312700304565390	0.534
	1	8.074	6.4834209674432110	0.106
	2	10.255	9.278604516903260	0.153
	3	12.232	11.766427983937469	0.196
	4	14.063	14.028816948623492	0.235
	5	15.783	14.585313802191063	0.293
1	0	1.199	1.1896870585180375	2.314
	1	1.739	1.8415039963613402	3.404
	2	2.209	2.394675709114608	4.322
	3	2.635	2.888823024670750	5.143
	4	3.030	2.4699633437009440	7.094
	5	3.400	2.3228580785446677	8.805

1. All  $r_{\Delta E}$ -values obtained, in  $1/GeV$  units, satisfy the  $r_{\Delta E} \prec \frac{(\Lambda+1)^2}{m a}$  condition, as  $\Delta E$  is always positive, whereas  $f'/f$  has a negative structure in Eq. (13), except for the case of  $a = 0$ .



**Table 3.** The lowest level  $r_{\Delta E}$ -values for  $V(r) = \frac{-a}{r} + r$  in  $N = 3$  dimension with  $r_0 = 0.884$ .

$a$	$r_{\Delta E}$	$\Delta E_{n=0}$	$a$	$r_{\Delta E}$	$\Delta E_{n=0}$
0.0	1.0092498710582083	2.338	1.0	0.7994448794104599	1.648
0.1	0.9871174720215355	2.254	1.1	0.7806100091335577	1.593
0.2	0.9650736643159619	2.177	1.2	0.7622476487440810	1.541
0.3	0.9432041027784062	2.101	1.3	0.7443630403947710	1.491
0.4	0.9215784931711197	2.028	1.4	0.7269585604248531	1.443
0.5	0.9002533954125955	1.958	1.5	0.7100341340504672	1.397
0.6	0.8792744720957341	1.891	1.6	0.6935875917754227	1.353
0.7	0.8586783015732216	1.826	1.7	0.6776149777440374	1.311
0.8	0.8384938486996074	1.764	1.8	0.6621108181210410	1.270
0.9	0.8187436656830261	1.705	1.9	0.6470684263448296	1.232

- The first  $r_{\Delta E}$ -values in Tables 1 and 2 for  $b = 0.01$  reveal why one naturally expects a bound only for these situations, which are very close to the critical  $r$ -value,  $r = r_0 = (\Lambda + 1)^2 / ma$ , where the Coulomb-like potential is dominant and the linear part of the Cornell potential may be treated as a perturbation in this case.
- From Table 1, as  $r_0$  and  $r_{\Delta E}$  depend on  $b$ - and  $\ell$ -values due to the form of the related function,  $r^{\ell+1}f(b, r)$ , in the asymptotic region, the decrease in these  $r$ -values can be readily observed through the increase in parameter  $b$ . However, considerable increments in  $r_0$  and  $r_{\Delta E}$  are clearly visible when  $\ell$  increases for each individual  $b$ -value. Since the asymptotic solution is independent of the Coulombic parameter  $a$ , the  $r$ -values of interest are not affected by the changes in  $a$  as in Table 3.
- When  $b$  increases,  $\Delta E$  has larger values. In particular, in the case of  $b = 100$  in Table 1, where the linear confinement is more active, the energy corrections are striking with the rise of  $\ell$ -values. In this circumstance, the Coulombic part may be regarded as a perturbative term. In contrast to this observation, Table 3 shows that  $\Delta E$  decreases with increasing  $a$ -values, which makes the Coulomb potential stronger than the confining part. Consequently, the refinements in eigenvalues due to the linear potential diminish as expected. These results confirm the comments given above.
- Additionally, there is an impressive correlation between the  $r$ -values. In Table 1,  $r_0 \succ r_{\Delta E}$  is valid for  $b = 0.01$  through  $\ell = 0$  to  $\ell = 4$ . The rapid rise in  $r_{\Delta E}$  due to increase in  $\ell$ -values makes them comparable, even larger than  $r_0$  as in the case of  $\ell = 5$ . This is understandable as the angular momentum barrier with large  $\ell$  provides a more positive contribution to the whole potential, which decreases the effects of the Coulombic part while the modifications in  $\Delta E$  get larger. On the other hand, for larger values of  $b$ ,  $r_{\Delta E} \succ r_0$  through all  $\ell$ -values except only  $b = 1$  with  $\ell = 0$ , where both potentials are comparable. From this short discussion, we conclude that  $r_0 \succ r_{\Delta E}$  signalizes the domains where the Coulombic part is dominant while the linear term behaves as a perturbing potential, whereas  $r_{\Delta E} \succ r_0$  indicates the regions where the confining potential is more active and the Coulomb-like potential can be treated as a modifying term. Roughly speaking,  $r_{\Delta E}$  measures the distance at which the potential changes from being dominantly Coulombic ( $r_{\Delta E} \prec r_0$ ) to dominantly linear ( $r_{\Delta E} \succ r_0$ ). In sum, as in a perturbation procedure the values of the model parameters play a crucial role in choosing the parent and perturbative terms, reliable estimations of  $r_{\Delta E}$  would be helpful in providing information on this choice and on the energy corrections as well through Eq. (13). Some such physical insight is usually necessary

in any approach that is based on a judicious choice of either the related potential functions or concerning wave functions. Within this context, the present comments on our findings would be helpful in clarifying the questions concerning the limitations of the traditional perturbation techniques.

6. The situation is nonetheless not similar as the dimension increases. In particular, considering  $N = 4$  for  $b = 1$  with  $\ell = 4$  and  $\ell = 5$  in Table 2, the trend in  $r$ -values is opposite that of  $N = 3$ . The change in the definition of the angular momentum concept in this dimension where  $\ell \rightarrow \ell + 1/2 = \Lambda$ , which has been debated in Section 2, causes this conflicting behavior due to the new shifted larger shapes of the Coulomb wave function,  $r^{\ell+3/2}e^{-mar/(\ell+3/2)}$ , and of the concerning asymptotic wave function,  $r^{\ell+3/2}f(r)$ , for the ground-state solutions in this larger dimension. In addition, unlike the case in Table 1, Columbic interaction seems more powerful than the linear end of the potential for even  $\ell = 5$  in the case of  $b = 1$  shown in Table 2, as still  $r_0 \succ r_{\Delta E}$  for the same reason regarding the other forms of the wave functions.

In sum, the comments above clarify why the usual perturbation treatments, such as the ones discussed in Section 3.1, do not work for large  $b$ - and  $\ell$ -values through the analysis of the Cornell potential, in which the Coulomb potential is admitted as always dominant to the linear part of the entire potential. Moreover, Aitchison and Dudek in [26] put forward an argument about the significance of some critical  $r$ -values in the analysis of heavy quarkonium spectra and showed that with the Coulombic part, as the parent in the Cornell potential, bottomonium spectra are better explained than charmonium, whereas charmonium states are well elucidated with the linear part as the main term of the same potential, which makes the whole analysis above noteworthy in this context.

We finally note that the discussion of excited states within the framework of the present scenario is unnecessary as the main points of the work presented in this article, which have been clarified through the tables and comments above, can be safely given only by the consideration of the ground-state calculations. However for the treatments of  $n \succ 0$  levels within the framework of a similar procedure, the reader is referred to [21].

#### 4. Concluding remarks

We have presented an alternative technique for the solution of the Schrödinger equation with nonsolvable potentials having an analytically solvable part, together with a modifying term. Although we have limited ourselves to one illustrative example, the range of applications of the method is rather large and appears to be straightforward. The formalism has a structure for the analysis of related experimental results and of precise numerical solutions, which can be accomplished with ease. Through the formalism used we have shown how estimates of  $r_{\Delta E}$  could help to provide intuition on whether a given state is likely to be modeled better by the Coulomb or the linear part of the Cornell potential, together with reasonable predictions for the energy refinements with the help of Eq. (13), which are significant in the analysis of heavy quarkonium spectra. In view of the importance in analyzing such corrections in a simple manner, we believe that the present model would serve as a useful toolbox to interpret even properly chosen more realistic situations, which now occur in experimental observations with the advent of quantum technology. Additionally, when the structure of a critically stable quantum system is analyzed, understanding the analytic behavior of the solution as a function of different physical parameters is often of decisive importance. This justifies the significance of analytical treatments, such as the one discussed here, in different disciplines of science.

Aside from the problem posed and discussed in the present article having its own inner mathematical beauty and ability for providing a good starting point for doing calculations perturbatively/nonperturbatively

for more complex systems, the present work once more reveals that a proper perturbation method, requiring no small parameters in the equations, which can readily eliminate the limitations of the traditional perturbation techniques, which often fail miserably for systems on the border of stability, is urgently required as most nonlinear equations have no small parameter at all. Within this context, nonlinear analysis methods involving homotopy perturbation techniques [27–29] and the quasilinearization method [30–35], which approximates the solution of nonlinear differential equations by treating the nonlinear terms as a perturbation about the linear ones, are promising. These models are iterative but not perturbative and give generally stable solutions to nonlinear problems without depending on the existence of a smallness parameter. The use of such powerful treatments for the solution of Eq. (8) within the present framework and its extension to other similar potentials is underway.

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