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Mixed boundary value problem for a class of quasi-linear elliptic operators containing p-Laplacian

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Abstract: In this paper, we consider a mixed boundary value problem to a class of quasi-linear elliptic operators containing a p-Laplace operator. We show the existence of a unique weak solution and an estimate. We also demonstrate the continuity of the solution on the given data. Moreover, we consider the dual problem.

Key words: Mixed boundary value problem, p-Laplacian, existence of a weak solution, variational method, continuous dependence of solution, dual problem

1. Introduction

Fundamental nonlinear stationary conservation law reads

$$\begin{cases} \operatorname{div} \boldsymbol{j} = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_1, \\ \boldsymbol{j} \cdot \boldsymbol{n} = h & \text{on } \Gamma_2, \end{cases} \tag{1.1}$$

where $\boldsymbol{j}$ is the current density. Here Ω is a bounded domain in $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and Γ_1 and Γ_2 are disjoint open subsets of Γ such that $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \Gamma$ and $\boldsymbol{n}$ denotes the unit outer normal vector to the boundary Γ. If $\boldsymbol{j}$ is of the form

$$\boldsymbol{j} = -\alpha(|\boldsymbol{\nabla} u|^2)\boldsymbol{\nabla} u, \tag{1.2}$$

problem (1.1) corresponds to many physical problems, for example, hydrodynamics and gas dynamics, electrostatics, heat conduction, elasticity, and plasticity.

For an intuitive picture, let $N = 3$. If we regard $u(x)$ as the temperature of a body Ω at the point $x \in \Omega$, then $\boldsymbol{j}$ in (1.1) is the current density vector of the stationary heat flow in Ω. The function f describes outer heat sources. The boundary conditions prescribe the temperature u on the boundary part Γ_1 and heat flow through the boundary part Γ_2. If $\alpha \equiv 1$, then the problem (1.1) becomes

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial \boldsymbol{n}} = h & \text{on } \Gamma_2. \end{cases} \tag{1.3}$$

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From the mathematical point of view, this is a mixed boundary value problem for the Poisson equation. If $\alpha(|\boldsymbol{\nabla} u|^2) = |\boldsymbol{\nabla} u|^{p-2}$, the problem (1.1) corresponds to the p-Laplacian equation. In the case $\Gamma_2 = \emptyset$, i.e. Dirichlet boundary condition, or in the case $\Gamma_1 = \emptyset$, i.e. Steklov or Neumann boundary condition, there are many articles. For example, see Figueiredo et al. [7], Dong and Xu [8], Zhang and Wang [18], Wang [14], Yuan and Du [15], Torné [13], Zerouali et al. [17], Nastasi [11], Colasuonno and Noris [5]. On the contrary, for mixed boundary problem, there are only a few articles (cf. Ge and Tian [10], Arerna et al. [4]). Equation (1.2) represents a constitutive law which depends on the specific properties of the material. If α is a positive constant, α represents the heat conductivity and (1.2) is called heat conductivity.

In this paper, we consider the following problem.

$$\begin{cases} -\operatorname{div}[S_t(x, |\boldsymbol{\nabla} u|^2)\boldsymbol{\nabla} u] = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_1, \\ -S_t(x, |\boldsymbol{\nabla} u|^2)\frac{\partial u}{\partial \boldsymbol{n}} = h & \text{on } \Gamma_2, \end{cases} \tag{1.4}$$

where a function $S(x,t)$ is a Carathéodory function on $\Omega \times [0,\infty)$. Then

$$\boldsymbol{j} = -S_t(x, |\boldsymbol{\nabla} u|^2)\boldsymbol{\nabla} u \tag{1.5}$$

corresponds to a nonlinear constitutive law, where the heat conductivity depend on x and the gradient of temperature. The negative sign in (1.5) reflects the fact that heat flows from points with higher temperature to points with lower temperature.

An advantage which considers such an operator is in the ability to prove that the problem (1.4) is well-posed. For the existence of a weak solution to (1.4), we consider the following variational problem: to find a minimizer of the functional

$$\int_\Omega \big(S(x, |\boldsymbol{\nabla} u|^2) - fu\big)dx + \int_{\Gamma_2} hud\sigma \tag{1.6}$$

on the function space such that $u = g$ on Γ_1, where $d\sigma$ is the surface measure on Γ. We show the existence of a unique weak solution to (1.5) and an estimate. Moreover, we can obtain the continuous dependence of the solution on the data f, g and h. Furthermore, we consider the dual problem of (1.6). We show that the dual problem also has a unique solution which is related to the unique minimizer of (1.6).

The paper is organized as follows. Section 2 consists of three subsections. In subsection 2.1, we define a function $S(x,t)$ having some structure conditions and its properties. In subsection 2.2, we introduce the spaces of functions used in this paper. In subsection 2.3, we give one of the main theorem (Theorem 2.7) in the paper. Section 3 is devoted to a proof of Theorem 2.7. In section 4, we show the continuous dependence of solution obtained in section 3 on the given data. In section 5, we consider the duality problem for conservation laws.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{0,1}$-boundary Γ. Moreover, we assume that Γ_1 and Γ_2 are disjoint open subset of Γ such that

$$\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \Gamma \text{ and } \Gamma_1 \neq \emptyset. \tag{2.1}$$

Throughout this paper, for $1 < p < \infty$ we denote the Hölder conjugate exponent of p by p', i.e. $(1/p) + (1/p') = 1$, and we only consider vector spaces over $\mathbb{R}$. For any space B, we denote B^N by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a} = (a_1, \dots, a_N)$ and $\boldsymbol{b} = (b_1, \dots, b_N)$ in $\mathbb{R}^N$ by $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^N a_i b_i$ and $|\boldsymbol{a}| = (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2}$. Furthermore, we denote the dual space of B by B^* and the duality bracket by $\langle \cdot, \cdot \rangle_{B^*,B}$.

2.1. Function $S(x,t)$ satisfying some structure conditions

We assume that a function $S(x,t)$ is a Carathéodory function in $\Omega \times [0,\infty)$ and for a.e. in $x \in \Omega$, $S(x,t) \in C^2((0,\infty)) \cap C([0,\infty))$ satisfies the following structure conditions: there exist a constant $1 < p < \infty$ and positive constants $0 < \lambda \le \Lambda < \infty$ such that for a.e. $x \in \Omega$

$$S(x,0) = 0 \text{ and } \lambda t^{(p-2)/2} \le S_t(x,t) \le \Lambda t^{(p-2)/2} \text{ for } t > 0. \tag{2.2a}$$

$$\lambda t^{(p-2)/2} \le S_t(x,t) + 2tS_{tt}(x,t) \le \Lambda t^{(p-2)/2} \text{ for } t > 0. \tag{2.2b}$$

$$\text{If } 1 < p < 2, S_{tt}(x,t) < 0, \text{ and if } p \ge 2, S_{tt}(x,t) \ge 0 \text{ for } t > 0, \tag{2.2c}$$

where $S_t = \partial S/\partial t$ and $S_{tt} = \partial^2 S/\partial t^2$. We note that from (2.2a), we have

$$\frac{2}{p}\lambda t^{p/2} \le S(x,t) \le \frac{2}{p}\Lambda t^{p/2} \text{ for } t \ge 0. \tag{2.3}$$

We introduce two examples. When $S(x,t) = \nu(x)t^{p/2}$, where ν is a measurable function in Ω satisfying $0 < \nu_* \le \nu(x) \le \nu^* < \infty$ for a.e. in Ω, the function $S(x,t)$ satisfies (2.2a)–(2.2c). This example corresponds the p-Laplacian operator. As an another example, we can take

$$g(t) = \begin{cases} ae^{-1/t} + a & \text{for } t > 0, \\ a & \text{for } t = 0, \end{cases}$$

where $a > 0$ is a constant. Then we can see that $S(x,t) = \nu(x)g(t)t^{p/2}$ satisfies (2.2a)–(2.2c) if $p \ge 2$, (cf. Aramaki [2]).

We have the following strict monotonicity of S_t.

Lemma 2.1 *There exists a constant $c > 0$ depending only on p and λ such that for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ $(d \ge 1)$,*

$$\big(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}\big) \cdot (\boldsymbol{a} - \boldsymbol{b}) \ge \begin{cases} c|\boldsymbol{a} - \boldsymbol{b}|^p & \text{if } p \ge 2, \\ c(|\boldsymbol{a}| + |\boldsymbol{b}|)^{p-2}|\boldsymbol{a} - \boldsymbol{b}|^2 & \text{if } 1 < p < 2. \end{cases}$$

In particular,

$$\big(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}\big) \cdot (\boldsymbol{a} - \boldsymbol{b}) > 0 \text{ if } \boldsymbol{a} \ne \boldsymbol{b}.$$

For the proof, see Aramaki [3, Lemma 3.6].

Lemma 2.2 *There exists a constant $C > 0$ depending only on p and Λ such that for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$,*

$$|S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}| \le \begin{cases} C|\boldsymbol{a} - \boldsymbol{b}|^{p-1} & \text{if } 1 < p < 2, \\ C(|\boldsymbol{a}| + |\boldsymbol{b}|)^{p-2}|\boldsymbol{a} - \boldsymbol{b}| & \text{if } p \ge 2. \end{cases}$$

For the proof, see Aramaki [1].

We can see that the following strictly convexity.

Lemma 2.3 *If* $S(x,t)$ *satisfies* (2.2a) *and* (2.2b)*, then for a.e.* $x \in \Omega$*, a function* $\mathbb{R}^d \ni \boldsymbol{a} \mapsto S(x,|\boldsymbol{a}|^2)$ *is strictly convex.*

Proof For fixed a.e. $x \in \Omega$, if we put $\varphi(\boldsymbol{a}) = S(x,|\boldsymbol{a}|^2) : \mathbb{R}^d \mapsto \mathbb{R}$, then φ is Gâteaux differentiable and $\varphi'(\boldsymbol{a}) = S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a}$. By Lemma 2.1, φ' is strictly monotone, so φ is strictly convex. (cf. [2, Lemma 2.3]). □

2.2. Spaces of functions

From now on we use the notations $L^p(\Omega), W^{m,p}(\Omega)$ $(1 < p < \infty$ and $m \geq 0,$ integer$)$, $W^{s,p}(\Gamma)$ $(s \in \mathbb{R})$, and so on, for the standard Lebesgue space and Sobolev spaces of real valued functions.

For $0 < s < 1$ and $1 < p < \infty$, by definition, we say $b \in W^{s,p}(\Gamma)$ iff

$$\int_\Gamma |b|^p d\sigma < \infty \text{ and } \iint_{\Gamma\times\Gamma} \frac{|b(x)-b(y)|^p}{|x-y|^{N-1+ps}} d\sigma d\sigma < \infty,$$

and the norm is defined by

$$\|b\|_{W^{s,p}(\Gamma)} = \left(\int_\Gamma |b|^p d\sigma + \iint_{\Gamma\times\Gamma} \frac{|b(x)-b(y)|^p}{|x-y|^{N-1+ps}} d\sigma d\sigma\right)^{1/p}.$$

Then $W^{s,p}(\Gamma)$ is a real, separable and reflexive Banach space with respect to this norm.

For every $i = 1,2$, by definition, a function $g : \Gamma_i \to \mathbb{R}$ belongs to $W^{1-1/p,p}(\Gamma_i)$ iff there exists $\widetilde{g} \in W^{1-1/p,p}(\Gamma)$ such that $\widetilde{g} = g$ on Γ_1. If we set

$$\|g\|_{W^{1-1/p,p}(\Gamma_i)} = \inf_{\substack{\widetilde{g}\in W^{1-1/p,p}(\Gamma),\\ \widetilde{g}=g \text{ on } \Gamma_i}} \|\widetilde{g}\|_{W^{1-1/p,p}(\Gamma)},$$

then the space $W^{1-1/p,p}(\Gamma_i)$ is a real normed space (we identify two functions whose differ on a subset of Γ_i of surface measure zero). Clearly we can regard $W^{1-1/p,p}(\Gamma) \subset W^{1-1/p,p}(\Gamma_i)$ and the inclusion mapping is linear and continuous.

We give a key lemma which plays an important role in the later.

Lemma 2.4 *For every* $i = 1,2$ *and for any* $g \in W^{1-1/p,p}(\Gamma_i)$*, there exists* $\overline{g} \in W^{1,p}(\Omega)$ *such that* $\overline{g} = g$ *on* Γ_i*, and there exists a constant* $C > 0$ *depending only on* p, Ω *and* Γ_i *such that*

$$\|\overline{g}\|_{W^{1,p}(\Omega)} \leq C\|g\|_{W^{1-1/p,p}(\Gamma_i)}. \tag{2.4}$$

Proof For any $g \in W^{1-1/p,p}(\Gamma_i)$, by the definition, there exists $\widetilde{g} \in W^{1-1/p,p}(\Gamma)$ such that $\widetilde{g} = g$ on Γ_i and

$$\|g\|_{W^{1-1/p,p}(\Gamma_i)} = \inf_{\substack{\widetilde{g}\in W^{1-1/p,p}(\Gamma),\\ \widetilde{g}=g \text{ on } \Gamma_i}} \|\widetilde{g}\|_{W^{1-1/p,p}(\Gamma)}.$$

It is well known that for $\widetilde{g}$, there exists $\widetilde{u} \in W^{1,p}(\Omega)$ such that $\widetilde{u} = \widetilde{g}$ on Γ, in particular, $\widetilde{u} = g$ on Γ_i, and there exists a constant $C > 0$ depending only on p, Ω and Γ_i such that

$$\|\widetilde{u}\|_{W^{1,p}(\Omega)} \leq C\|\widetilde{g}\|_{W^{1-1/p,p}(\Gamma)}.$$

Thus we have

$$\inf_{\substack{\widetilde{u}\in W^{1,p}(\Omega),\\ \widetilde{u}=g \text{ on } \Gamma_i}} \|\widetilde{u}\|_{W^{1,p}(\Omega)} \leq C \inf_{\substack{\widetilde{g}\in W^{1-1/p,p}(\Gamma),\\ \widetilde{g}=g \text{ on } \Gamma_i}} \|\overline{g}\|_{W^{1-1/p,p}(\Gamma)}.$$

It suffices to note that the infimum of the left-hand side is achieved at some $\overline{g} \in W^{1,p}(\Omega)$ such that $\overline{g} = g$ on Γ_i. □

We note that the inclusion mapping

$$W^{1-1/p,p}(\Gamma_i) \hookrightarrow L^p(\Gamma_i) \tag{2.5}$$

is linear and continuous.

The following lemma follows from Zeidler [16, Appendix (53c), p. 1033] (cf. Dautray and Lions [6, Chapter IV, §7, Remark 4]).

Lemma 2.5 *For $1 \leq p < \infty$, if $\Gamma_1 \neq \emptyset$, then for any $v \in W^{1,p}(\Omega)$,*

$$\left(\int_\Omega |\boldsymbol{\nabla} v|^p dx + \int_{\Gamma_1} |v|^p d\sigma\right)^{1/p}$$

is an equivalent norm to $\|v\|_{W^{1,p}(\Omega)}$. In particular, if $v \in W^{1,p}(\Omega)$ satisfies that $v = 0$ on Γ_1, then $\|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^p(\Omega)}$ and $\|v\|_{W^{1,p}(\Omega)}$ are equivalent.

2.3. Main theorem

We consider problem (1.4) where $f \in W^{1,p}(\Omega)^*, g \in W^{1-1/p,p}(\Gamma_1)$ and $h \in W^{1-1/p,p}(\Gamma_2)^*$ are given functions.

Define

$$X_g = \{v \in W^{1,p}(\Omega); v = g \text{ on } \Gamma_1\} \text{ and } X = \{v \in W^{1,p}(\Omega); v = 0 \text{ on } \Gamma_1\}. \tag{2.6}$$

We give the notion of a weak solution to (1.4).

Definition 2.6 *We say $u \in W^{1,p}(\Omega)$ is a weak solution to (1.4) if $u \in X_g$ and u satisfies*

$$\int_\Omega S_t(x, |\boldsymbol{\nabla} u|^2)\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v dx = \langle f, v\rangle_\Omega - \langle h, v\rangle_{\Gamma_2} \textit{ for all } v \in X, \tag{2.7}$$

where $\langle f, v\rangle_\Omega = \langle f, v\rangle_{W^{1,p}(\Omega)^*, W^{1,p}(\Omega)}$, and $\langle h, v\rangle_{\Gamma_2} = \langle h, v\rangle_{W^{1-1/p,p}(\Gamma_2)^*, W^{1-1/p,p}(\Gamma_2)}$.

We are in a position to state one of the main theorems.

Theorem 2.7 *Assume that Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{0,1}$-boundary Γ satisfying (2.1). Let $f \in W^{1,p}(\Omega)^*, g \in W^{1-1/p,p}(\Gamma_1)$ and $h \in W^{1-1/p,p}(\Gamma_2)^*$. Then the mixed boundary value problem (1.4) has a unique weak solution $u \in W^{1,p}(\Omega)$, and there exists a constant $C > 0$ depending only on $p, \lambda, \Gamma_1, \Gamma_2$ and Ω such that*

$$\|u\|^p_{W^{1,p}(\Omega)} \leq C(\|f\|^{p'}_{W^{1,p}(\Omega)^*} + \|g\|^p_{W^{1-1/p,p}(\Gamma_1)} + \|h\|^{p'}_{W^{1-1/p,p}(\Gamma_2)^*}). \tag{2.8}$$

3. Proof of Theorem 2.7

In this section, we give a proof of Theorem 2.7 by the variational mathod. Assume that Ω is a bounded domain in $\mathbb{R}^N$ with a $C^{0,1}$ boundary Γ satisfying (2.1), and that $f \in W^{1,p}(\Omega)^*, g \in W^{1-1/p,p}(\Gamma_1)$ and $h \in W^{1-1/p,p}(\Gamma_2)^*$. Since the inclusion mappings $W^{1,p}(\Omega) \hookrightarrow W^{1-1/p,p}(\Gamma) \hookrightarrow W^{1-1/p,p}(\Gamma_1)$ are continuous, the set X_g defined in section 2 is a closed convex subset of $W^{1,p}(\Omega)$. We define a functional on X_g by

$$F(v) = \frac{1}{2}\int_\Omega S(x, |\nabla v|^2)dx - \langle f, v\rangle_\Omega + \langle h, v\rangle_{\Gamma_2},$$

and consider the following minimization problem: to find $u \in X_g$ such that

$$F(u) = \inf_{v\in X_g} F(v). \tag{3.1}$$

We call such a u a minimizer of F on X_g.

Proposition 3.1 *The minimization problem (3.1) has a unique minimizer $u \in X_g$.*

Proof First we show that F is weakly coercive on X_g, i.e., $F(v) \to \infty$ if $v \in X_g$ and $\|v\|_{W^{1,p}(\Omega)} \to \infty$. Indeed, from Lemma 2.5, we see that

$$\left(\int_\Omega |\nabla v|^p dx + \int_{\Gamma_1} |g|^p d\sigma\right)^{1/p}$$

is an equivalent norm to $\|v\|_{W^{1,p}(\Omega)}$ on X_g. By (2.3), there exists constants $c, C > 0$ such that

$$\begin{aligned} F(v) &\geq \frac{\lambda}{p}\int_\Omega |\nabla v|^p dx - \|f\|_{W^{1,p}(\Omega)^*}\|v\|_{W^{1,p}(\Omega)} - \|h\|_{W^{1-1/p,p}(\Gamma_2)^*}\|v\|_{W^{1-1/p,p}(\Gamma_2)} \\ &\geq c\|v\|^p_{W^{1,p}(\Omega)} - \int_{\Gamma_1}|g|^p d\sigma - C(\|f\|_{W^{1,p}(\Omega)^*} + \|h\|_{W^{1-1/p,p}(\Gamma_2)^*})\|v\|_{W^{1,p}(\Omega)}. \end{aligned}$$

Since $1 < p < \infty$, we can see that if $v \in X_g$ and $\|v\|_{W^{1,p}(\Omega)} \to \infty$, then $F(v) \to \infty$.

Next, from Lemma 2.3, the functional F is strictly convex. Here we note that if $v, w \in X_g$ and $v \neq w$, then $\nabla v \neq \nabla w$ since $\Gamma_1 \neq \emptyset$.

Clearly F is proper.

Finally we show that F is lower semicontinuous on X_g. Indeed, since $S(x,t)$ is a Carathéodory function on $\Omega \times [0,\infty)$ and satisfies

$$S(x, |\nabla v|^2) \leq \frac{2\Lambda}{p}|\nabla v|^p$$

from (2.3). Define the Nemyskii operator E by

$$(E(\boldsymbol{p}))(x) = S(x, |\boldsymbol{p}(x)|^2) \text{ for } \boldsymbol{p} \in \boldsymbol{L}^p(\Omega).$$

Then it follows from [16, Proposition 26.6] that $E : \boldsymbol{L}^p(\Omega) \to L^1(\Omega)$ is continuous and bounded, that is,

$$\|E(\boldsymbol{p})\|_{L^1(\Omega)} \leq C\|\boldsymbol{p}\|_{\boldsymbol{L}^p(\Omega)},$$

so we have

$$\|S(x,|\boldsymbol{p}|^2)\|_{L^1(\Omega)} \le C\|\boldsymbol{p}\|_{\boldsymbol{L}^p(\Omega)}.$$

This implies that if $v_j \to v$ in X_g, then $\boldsymbol{\nabla} v_j \to \boldsymbol{\nabla} v$ in $\boldsymbol{L}^p(\Omega)$, so $S(x,|\boldsymbol{\nabla} v_j|^2) \to S(x,|\boldsymbol{\nabla} v|^2)$ in $L^1(\Omega)$, that is,

$$\int_\Omega S(x,|\boldsymbol{\nabla} v_j|^2)dx \to \int_\Omega S(x,|\boldsymbol{\nabla} v|^2)dx.$$

Therefore, F is continuous on X_g. Since F is convex and continuous, F is lower semi-continuous on M_g. By the Ekeland and Témam [9, Chapter II, Proposition 1.2], the minimization problem (3.1) has a unique solution $u \in X_g$. □

We continue the proof of Theorem 2.7. Let $u \in X_g$ be the minimizer of (3.1). For any $v \in X$ and any $t \in \mathbb{R}$, since $u + tv \in X_g$, we have $F(u) \le F(u+tv)$. From the Euler-Lagrange equation,

$$\frac{d}{dt}F(u+tv)\Big|_{t=0} = 0 \text{ for all } v \in X.$$

This means that u satisfies (2.7).

We show the uniqueness of a weak solution to (1.4). Let $u_1, u_2 \in X_g$ be two weak solutions to (1.4). Then for every $i = 1, 2$,

$$\int_\Omega S_t(x,|\boldsymbol{\nabla} u_i|^2)\boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} v dx = \langle f, v\rangle_\Omega - \langle h, v\rangle_{\Gamma_2} \text{ for all } v \in X. \tag{3.2}$$

Since $u_1 - u_2 \in X$, we take $v = u_1 - u_2$ as a test function of (3.2). Thus for $i = 1, 2$,

$$\int_\Omega S_t(x,|\boldsymbol{\nabla} u_i|^2)\boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla}(u_1 - u_2)dx = \langle f, u_1 - u_2\rangle_\Omega - \langle h, u_1 - u_2\rangle_{\Gamma_2}.$$

Therefore, we have

$$\int_\Omega \big(S_t(x,|\boldsymbol{\nabla} u_1|^2)\boldsymbol{\nabla} u_1 - S_t(x,|\boldsymbol{\nabla} u_2|^2)\boldsymbol{\nabla} u_2\big) \cdot \boldsymbol{\nabla}(u_1 - u_2)dx = 0.$$

By the strict monotonicity of S_t (Lemma 2.2), $\boldsymbol{\nabla} u_1 = \boldsymbol{\nabla} u_2$ in Ω. Hence $u_1 - u_2$ is a constant. Since $u_1 = u_2 = g$ on Γ_1, the constant is equal to zero, so we have $u_1 = u_2$.

Lastly we show the estimate (2.8). Let $u \in X_g$ be the weak solution to (1.4). From Lemma 2.4, for $g \in W^{1-1/p,p}(\Gamma_1)$, there exists $\overline{g} \in W^{1,p}(\Omega)$ such that $\overline{g} = g$ on Γ_1 and

$$\|\overline{g}\|_{W^{1,p}(\Omega)} \le C\|g\|_{W^{1-1/p,p}(\Gamma_1)}, \tag{3.3}$$

where $C > 0$ is a constant depending only on p, Ω and Γ_1. Since $u - \overline{g} \in X$, if we take $v = u - \overline{g}$ as a test function of (2.7), then we have

$$\int_\Omega S_t(x,|\boldsymbol{\nabla} u|^2)\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} u dx = \int_\Omega S_t(x,|\boldsymbol{\nabla} u|^2)\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla}\overline{g} dx + \langle f, u\rangle_\Omega - \langle f, \overline{g}\rangle_\Omega - \langle h, u\rangle_{\Gamma_2} + \langle h, \overline{g}\rangle_{\Gamma_2}. \tag{3.4}$$

From (2.2a),

$$\int_\Omega S_t(x,|\boldsymbol{\nabla} u|^2)\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} u dx \ge \lambda \int_\Omega |\boldsymbol{\nabla} u|^p dx.$$

On the other hand, we estimate the right-hand side of (3.4) from above. By (2.2a), (3.3) and the Hölder and Young inequalities,

$$\begin{aligned}\left|\int_\Omega S_t(x,|\nabla u|^2)\nabla u\cdot\nabla\overline{g}dx\right| &\le \Lambda\int_\Omega|\nabla u|^{p-1}|\nabla\overline{g}|dx\\ &\le \Lambda\|\nabla u\|^{p-1}_{L^p(\Omega)}\|\nabla\overline{g}\|_{L^p(\Omega)}\\ &\le \epsilon\|\nabla u\|^p_{L^p(\Omega)}+C(\epsilon)\|\nabla\overline{g}\|^p_{L^p(\Omega)}\\ &\le \epsilon\|\nabla u\|^p_{L^p(\Omega)}+C'(\epsilon)\|g\|^p_{W^{1-1/p,p}(\Gamma_1)}\end{aligned}$$

for any $\epsilon>0$ and some constants $C(\epsilon)$ and $C'(\epsilon)$. Furthermore,

$$\begin{aligned}|\langle f,u\rangle_\Omega+\langle h,u\rangle_{\Gamma_2}| &\le \|f\|_{W^{1,p}(\Omega)^*}\|u\|_{W^{1,p}(\Omega)}+\|h\|_{W^{1-1/p,p}(\Gamma_2)^*}\|u\|_{W^{1-1/p,p}(\Gamma_2)}\\ &\le C(\|f\|_{W^{1,p}(\Omega)^*}+\|h\|_{W^{1-1/p,p}(\Gamma_2)^*})\|u\|_{W^{1,p}(\Omega)}\\ &\le \epsilon\|u\|^p_{W^{1,p}(\Omega)}+C(\epsilon)(\|f\|^{p'}_{W^{1,p}(\Omega)^*}+\|h\|^{p'}_{W^{1-1/p,p}(\Gamma_2)^*})\end{aligned}$$

for any $\epsilon>0$ and a constant $C(\epsilon)$. Similarly,

$$\begin{aligned}|\langle f,\overline{g}\rangle_\Omega+\langle h,\overline{g}\rangle_{\Gamma_2}| &\le \|f\|_{W^{1,p}(\Omega)^*}\|\overline{g}\|_{W^{1,p}(\Omega)}+\|h\|_{W^{1-1/p,p}(\Gamma_2)^*}\|\overline{g}\|_{W^{1-1/p,p}(\Gamma_2)}\\ &\le C(\|f\|^{p'}_{W^{1,p}(\Omega)^*}+\|h\|^{p'}_{W^{1-1/p,p}(\Gamma_2)^*}+\|g\|^p_{W^{1-1/p,p}(\Gamma_1)})\end{aligned}$$

for a constant C. Since $u\in X_g$,

$$\int_\Omega|\nabla u|^p dx+\int_{\Gamma_1}|g|^p dx$$

is equivalent to $\|u\|^p_{W^{1,p}(\Omega)}$ and the inclusion mapping $W^{1-1/p,p}(\Gamma_1)\hookrightarrow L^p(\Gamma_1)$ is linear and continuous, we choose $\epsilon>0$ small enough, we obtain the estimate (2.8). This completes the proof of Theorem 2.7.

4. Continuity of the weak solution on data

In this section, we derive the continuity of the weak solution to (1.4) on the given data.

We have the following theorem.

Theorem 4.1 *Assume that Ω is a bounded domain in $\mathbb{R}^N$ with a $C^{0,1}$ boundary Γ satisfying (2.1). Moreover, we assume that $f_j,f\in W^{1,p}(\Omega)^*$, $g_j,g\in W^{1-1/p,p}(\Gamma_1)$, $h_j,h\in W^{1-1/p,p}(\Gamma_2)^*$ and $f_j\to f$ strongly in $W^{1,p}(\Omega)^*$, $g_j\to g$ strongly in $W^{1-1/p,p}(\Gamma_1)$, $h_j\to h$ strongly in $W^{1-1/p,p}(\Gamma_2)^*$. Furthermore, let $u_j\in X_{g_j}$ be the weak solution to (1.4) with $f=f_j,g=g_j,h=h_j$ and let $u\in X_g$ be the weak solution to (1.4). Then we can show that $u_j\to u$ strongly in $W^{1,p}(\Omega)$ as $j\to\infty$.*

Proof The weak solution u_j satisfies that

$$\int_\Omega S_t(x,|\nabla u_j|^2)\nabla u_j\cdot\nabla v dx=\langle f_j,v\rangle_\Omega-\langle h_j,v\rangle_{\Gamma_2}\text{ for all }v\in X,\tag{4.1}$$

and

$$\|u_j\|^p_{W^{1,p}(\Omega)} \le C(\|f_j\|^{p'}_{W^{1,p}(\Omega)^*} + \|g_j\|^p_{W^{1-1/p,p}(\Gamma_1)} + \|h_j\|^{p'}_{W^{1-1/p,p}(\Gamma_2)^*}) \le C_0, \tag{4.2}$$

where we can assume that the constant C_0 is independent of j. From (4.1) and (2.7), we have

$$\int_\Omega \big(S_t(x, |\nabla u_j|^2)\nabla u_j - S_t(x, |\nabla u|^2)\nabla u\big) \cdot \nabla v dx = \langle f_j - f, v\rangle_\Omega - \langle h_j - h, v\rangle_{\Gamma_2} \text{ for all } v \in X. \tag{4.3}$$

Since $g_j - g \in W^{1-1/p,p}(\Gamma_1)$, it follows from Lemma 2.4 that there exists $v_j \in W^{1,p}(\Omega)$ such that $v_j = g_j - g$ on Γ_1 and

$$\|v_j\|_{W^{1,p}(\Omega)} \le C\|g_j - g\|_{W^{1-1/p,p}(\Gamma_1)},$$

where C is a constant depending only on p, Ω and Γ_1, but independent of j. Since $u_j - u - v_j \in X$, we take $v = u_j - u - v_j$ as a test function of (4.3). Then we have

$$\begin{aligned}
&\int_\Omega \big(S_t(x, |\nabla u_j|^2)\nabla u_j - S_t(x, |\nabla u|^2)\nabla u\big) \cdot \nabla (u_j - u) dx \\
&\qquad = \int_\Omega \big(S_t(x, |\nabla u_j|^2)\nabla u_j - S_t(x, |\nabla u|^2)\nabla u\big) \cdot \nabla v_j dx \\
&\qquad\qquad + \langle f_j - f, u_j - u - v_j\rangle_\Omega - \langle h_j - h, u_j - u - v_j\rangle_{\Gamma_2}.
\end{aligned}$$

The case $p \ge 2$.

From Lemma 2.1,

$$\int_\Omega \big(S_t(x, |\nabla u_j|^2)\nabla u_j - S_t(x, |\nabla u|^2)\nabla u\big) \cdot \nabla (u_j - u) dx \ge c\int_\Omega |\nabla(u_j - u)|^p dx.$$

On the other hand, from Lemma 2.2 and (4.2),

$$\begin{aligned}
&\left|\int_\Omega \big(S_t(x, |\nabla u_j|^2)\nabla u_j - S_t(x, |\nabla u|^2)\nabla u\big) \cdot \nabla v_j dx\right| \\
&\le C\int_\Omega (|\nabla u_j| + |\nabla u|)^{p-2}|\nabla(u_j - u)||\nabla v_j| dx \\
&\le C\left(\int_\Omega (|\nabla u_j|^p + |\nabla u|^p) dx\right)^{(p-2)/p}\left(\int_\Omega |\nabla(u_j - u)|^p dx\right)^{1/p}\left(\int_\Omega |\nabla v_j|^p dx\right)^{1/p} \\
&\le C_1(2C_0)^{(p-2)/p}\|\nabla(u_j - u)\|_{\boldsymbol{L}^p(\Omega)}\|g_j - g\|_{W^{1-1/p,p}(\Gamma_1)} \\
&\le \epsilon\|\nabla(u_j - u)\|^p_{\boldsymbol{L}^p(\Omega)} + C(\epsilon)\|g_j - g\|^{p'}_{W^{1-1/p,p}(\Gamma_1)}
\end{aligned}$$

for any $\epsilon > 0$ and a constant $C(\epsilon)$. Furthermore, we have

$$\begin{aligned}
|\langle f_j - f, u_j - u - v_j\rangle_\Omega| &\le \|f_j - f\|_{W^{1,p}(\Omega)^*}(\|u_j - u\|_{W^{1,p}(\Omega)} + \|v_j\|_{W^{1,p}(\Omega)}) \\
&\le \|f_j - f\|_{W^{1,p}(\Omega)^*}(\|u_j - u\|_{W^{1,p}(\Omega)} + C\|g_j - g\|_{W^{1-1/p,p}(\Gamma_1)}) \\
&\le \epsilon\|u_j - u\|^p_{W^{1,p}(\Omega)} + C(\epsilon)(\|f_j - f\|^{p'}_{W^{1,p}(\Omega)^*} + \|g_j - g\|^p_{W^{1-1/p,p}(\Gamma_1)}).
\end{aligned}$$

Similarly, we have

$$|\langle h_j - h, u_j - u - v_j\rangle_{\Gamma_2}| \le \epsilon\|u_j - u\|^p_{W^{1,p}(\Omega)} + C(\epsilon)(\|h_j - h\|^{p'}_{W^{1-1/p,p}(\Gamma_2)^*} + \|g_j - g\|^p_{W^{1-1/p,p}(\Gamma_1)}).$$

If we choose $\epsilon > 0$ small enough, we have

$$\|u_j - u\|^p_{W^{1,p}(\Omega)} \le C(\|f_j - f\|^{p'}_{W^{1,p}(\Omega)^*} + \|g_j - g\|^p_{W^{1-1/p,p}(\Gamma_1)} + \|g_j - g\|^{p'}_{W^{1-1/p,p}(\Gamma_1)} + \|h_j - h\|^{p'}_{W^{1-1/p,p}(\Gamma_2)^*}).$$

The case $1 < p < 2$.

In this case we use the reverse Hölder inequality (cf. Sobolev [12, p. 8]) to get

$$\left(\int_\Omega |\nabla(u_j - u)|^p dx\right)^{2/p} \le \int_\Omega |\nabla(u_j - u)|^2(|\nabla u_j| + |\nabla u|^2)^{p-2}dx \left(\int_\Omega (|\nabla u_j| + |\nabla u|)^p dx\right)^{(2-p)/2}.$$

From this inequality, there exists a constant $c > 0$ independent of j such that

$$\int_\Omega (|\nabla u_j| + |\nabla u|)^{p-2}|\nabla(u_j - u)|^2 dx \ge c\|\nabla(u_j - u)\|^2_{\boldsymbol{L}^p(\Omega)}.$$

On the other hand, from Lemma 2.2,

$$\begin{aligned}
&\left|\int_\Omega \big(S_t(x, |\nabla u_j|^2)\nabla u_j - S_t(x, |\nabla u|^2)\nabla u\big) \cdot \nabla v_j dx\right| \\
&\le C\int_\Omega |\nabla(u_j - u)|^{p-1}|\nabla v_j| dx \\
&\le C\left(\int_\Omega |\nabla(u_j - u)|^p dx\right)^{1/p'} \|\nabla v_j\|_{\boldsymbol{L}^p(\Omega)} \\
&= C\|\nabla(u_j - u)\|^{p-1}_{\boldsymbol{L}^p(\Omega)}\|\nabla v_j\|_{\boldsymbol{L}^p(\Omega)} \\
&\le \epsilon\|\nabla(u_j - u)\|^2_{\boldsymbol{L}^p(\Omega)} + C(\epsilon)\|g_j - g\|^{2/(3-p)}_{W^{1-1/p,p}(\Gamma_1)}
\end{aligned}$$

for any $\epsilon > 0$ and a constant $C(\epsilon)$. Furthermore, we have

$$\begin{aligned}
|\langle f_j - f, u_j - u - v_j\rangle_\Omega| &\le \|f_j - f\|_{W^{1,p}(\Omega)^*}(\|u_j - u\|_{W^{1,p}(\Omega)} + C\|g_j - g\|_{W^{1-1/p,p}(\Gamma_1)}) \\
&\le \epsilon\|u_j - u\|^2_{W^{1,p}(\Omega)} + C(\epsilon)(\|f_j - f\|^2_{W^{1,p}(\Omega)^*} + \|g_j - g\|^2_{W^{1-1/p,p}(\Gamma_1)}).
\end{aligned}$$

Similarly, we have

$$|\langle h_j - h, u_j - u - v_j\rangle_{\Gamma_2}| \le \epsilon\|u_j - u\|^2_{W^{1,p}(\Omega)} + C(\epsilon)(\|h_j - h\|^2_{W^{1-1/p,p}(\Gamma_2)^*} + \|g_j - g\|^2_{W^{1-1/p,p}(\Gamma_1)}).$$

If we choose $\epsilon > 0$ small enough, we have

$$\|u_j - u\|^2_{W^{1,p}(\Omega)} \le C(\|f_j - f\|^2_{W^{1,p}(\Omega)^*} + \|g_j - g\|^2_{W^{1-1/p,p}(\Gamma_1)} + \|g_j - g\|^{2/(3-p)}_{W^{1-1/p,p}(\Gamma_1)} + \|h_j - h\|^2_{W^{1-1/p,p}(\Gamma_2)^*}).$$

In each case, if $f_j \to f$ in $W^{1,p}(\Omega)^*, g_j \to g$ in $W^{1-1/p,p}(\Gamma_1)$ and $h_j \to h$ in $W^{1-1/p,p}(\Gamma_2)^*$, then we have $u_j \to u$ in $W^{1,p}(\Omega)$. This completes the proof of Theorem 4.1. □

5. Duality theory for conservation laws

In this section, we consider the duality theory for conservation law (1.1). Let $S(x,t)$ be a Carathéodory function on $\Omega \times [0,\infty)$ satisfying (2.2a)–(2.2c).

5.1. Preliminary remarks

Define

$$\beta(x,t) = \frac{1}{2}S(x,t^2) \text{ for } (x,t) \in \Omega \times \mathbb{R}.$$

By Lemma 2.3 with $d = 1$, for a.e. $x \in \Omega$, $\beta(x,t)$ is strictly convex and satisfies

$$\frac{\lambda}{p}|t|^p \le \beta(x,t) \le \frac{\Lambda}{p}|t|^p \text{ for all } t \in \mathbb{R}.$$

Therefore, $\beta_t(x,t) = S_t(x,t^2)t$ is a strictly monotone increasing and continuous function in $\mathbb{R}$ (we regard $S_t(x,t^2)t$ as zero at $t = 0$) and satisfies

$$-\Lambda|t|^{p-1} \le \beta_t(x,t) \le \Lambda|t|^{p-1} \text{ a.e. } x \in \Omega \text{ and for all } t \in \mathbb{R}.$$

For a.e. $x \in \Omega$ and $t^* \in \mathbb{R}^* = \mathbb{R}$, if we define

$$\beta^*(x,t^*) = \sup_{t\in\mathbb{R}}(t^*t - \beta(x,t)),$$

then it is clearly achieved at t such that $t^* = \beta_t(x,t)$, i.e. $t = \beta_t^{-1}(x,t^*)$, where β_t^{-1} is the inverse function of $\beta_t(x,t)$.

For a.e. $x \in \Omega$ and $\boldsymbol{p} \in \mathbb{R}^N$, define

$$\gamma(x,\boldsymbol{p}) = \beta(x,|\boldsymbol{p}|) = \frac{1}{2}S(x,|\boldsymbol{p}|^2),$$

and for $\boldsymbol{p}^* \in (\mathbb{R}^N)^* = \mathbb{R}^N$, define

$$\gamma^*(x,\boldsymbol{p}^*) = \sup_{\boldsymbol{p}\in\mathbb{R}^N}(\boldsymbol{p}^* \cdot \boldsymbol{p} - \gamma(x,\boldsymbol{p})).$$

Since $\gamma(x,\boldsymbol{p}) = \beta(x,|\boldsymbol{p}|) \ge \frac{\lambda}{p}|\boldsymbol{p}|^p$ $(p > 1)$, the function $\mathbb{R}^N \ni \boldsymbol{p} \mapsto \gamma(x,\boldsymbol{p}) - \boldsymbol{p}^* \cdot \boldsymbol{p}$ is strictly convex, weakly coercive and C^1. Thus $\inf_{\boldsymbol{p}\in\mathbb{R}^N}(\gamma(x,\boldsymbol{p}) - \boldsymbol{p}^* \cdot \boldsymbol{p})$ is achieved at a unique point $\boldsymbol{p}$ such that

$$\boldsymbol{p}^* = \gamma'(x,\boldsymbol{p}) = S_t(x,|\boldsymbol{p}|^2)\boldsymbol{p},$$

where γ' denotes the Gâteaux differential with respect to $\boldsymbol{p}$, so $|\boldsymbol{p}| = \beta_t^{-1}(x,|\boldsymbol{p}^*|)$. Hence

$$\gamma^*(x,\boldsymbol{p}^*) = \beta_t(x,|\boldsymbol{p}|)|\boldsymbol{p}| - \beta(x,|\boldsymbol{p}|) = \beta^*(x,|\boldsymbol{p}|^*) = \beta^*(x,|\boldsymbol{p}^*|).$$

For a.e. $x \in \Omega$ and a fixed $\boldsymbol{c} \in \mathbb{R}^N$, if we define $\delta(x,\boldsymbol{p}) = \gamma(x,\boldsymbol{p}+\boldsymbol{c})$, then we have

$$\delta^*(x,\boldsymbol{p}^*) = \sup_{\boldsymbol{p}\in\mathbb{R}^N}(\boldsymbol{p}^* \cdot \boldsymbol{p} - \delta(x,\boldsymbol{p})) = \sup_{\boldsymbol{q}\in\mathbb{R}^N}(\boldsymbol{p}^* \cdot \boldsymbol{q} - \gamma(x,\boldsymbol{q}) - \boldsymbol{p}^* \cdot \boldsymbol{c}) = \gamma^*(x,\boldsymbol{p}^*) - \boldsymbol{p}^* \cdot \boldsymbol{c}.$$

5.2. Theorem on the duality theory for conservation laws

Let $f \in W^{1,p}(\Omega)^*$, $g \in W^{1-1/p,p}(\Gamma_1)$ and $h \in W^{1-1/p,p}(\Gamma_2)^*$. Let X_g and X be defined by (2.6) and define $Y = \boldsymbol{L}^p(\Omega)$, we identify Y^* with $\boldsymbol{L}^{p'}(\Omega)$ by the relation

$$\langle \boldsymbol{p}^*, \boldsymbol{p}\rangle_{Y^*,Y} = \int_\Omega \boldsymbol{p}^* \cdot \boldsymbol{p} dx.$$

Since X is a closed subspace of $W^{1,p}(\Omega)$, it is a separable and reflexive Banach space with respect to the norm $\|u\|_X = \|\boldsymbol{\nabla} u\|_{\boldsymbol{L}^p(\Omega)}$ which is equivalent to $\|u\|_{W^{1,p}(\Omega)}$ from Lemma 2.5. From Lemma 2.4, there exists $\overline{g} \in W^{1,p}(\Omega)$ such that $\overline{g} = g$ on Γ_1 and there exists a constant C depending only on p, Ω and Γ_1 such that

$$\|\overline{g}\|_{W^{1,p}(\Omega)} \le C\|g\|_{W^{1-1/p,p}(\Gamma_1)}. \tag{5.1}$$

We consider the following problem: to find $u \in X_g$ such that

$$F(u) = \alpha := \inf_{v \in X_g} F(v), \tag{5.2}$$

together with the so-called dual problem: to find $\boldsymbol{p}^* \in \boldsymbol{L}^{p'}(\Omega)$ such that

$$F_*(\boldsymbol{p}^*) = \beta := \sup_{\boldsymbol{q}^* \in K} F_*(\boldsymbol{q}^*), \tag{5.3}$$

where

$$\begin{aligned} F(v) &= \frac{1}{2}\int_\Omega S(x, |\boldsymbol{\nabla} v|^2)dx - b(v) \text{ for } v \in X_g, \\ b(v) &= \int_\Omega f v dx - \int_{\Gamma_2} h v d\sigma \text{ for } v \in X_g, \\ K &= \{\boldsymbol{q}^* \in Y^* = \boldsymbol{L}^{p'}(\Omega); \langle \boldsymbol{q}^*, \boldsymbol{\nabla} v\rangle_{Y^*,Y} = b(v) \text{ for all } v \in X\}, \\ F_*(\boldsymbol{q}^*) &= -\int_\Omega \beta^*(x, |\boldsymbol{q}^*|)dx + \int_\Omega \boldsymbol{q}^* \cdot \boldsymbol{\nabla}\overline{g} dx - b(\overline{g}) \text{ for } \boldsymbol{q}^* \in K. \end{aligned}$$

We give a theorem on the duality theory for conservation laws.

Theorem 5.1 *Assume that Ω is a bounded domain in $\mathbb{R}^N$ $(N \ge 2)$ with $C^{0,1}$-boundary Γ which satisfies (2.1). Then the problems (5.2) and (5.3) have a unique solution $\overline{u} \in X_g$ and $\overline{\boldsymbol{p}}^* \in K$, respectively, and $\alpha = \beta$,*

$$\overline{\boldsymbol{p}}^* = S_t(x, |\boldsymbol{\nabla}\overline{u}|^2)\boldsymbol{\nabla}\overline{u} \textit{ a.e. in } \Omega, \tag{5.4}$$

and there exists a constant $C > 0$ depending only on p, Γ_1, Γ_2 and Ω such that

$$\|\overline{u}\|^p_{W^{1,p}(\Omega)} + \|\overline{\boldsymbol{p}}^*\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} \le C(\|f\|^{p'}_{W^{1,p}(\Omega)^*} + \|g\|^p_{W^{1-1/p,p}(\Gamma_1)} + \|h\|^{p'}_{W^{1-1/p,p}(\Gamma_2)^*}). \tag{5.5}$$

Moreover,

$$F_*(\boldsymbol{p}^*) \le F_*(\overline{\boldsymbol{p}}^*) = F(\overline{u}) \le F(u) \textit{ for all } u \in X_g \textit{ and } \boldsymbol{p}^* \in K. \tag{5.6}$$

We have the following error estimates. When $p \geq 2$*, there exist constants* $c, c_1 > 0$ *depending only on* p, Γ_1, Γ_2 *and* Ω *such that*

$$c\|u - \overline{u}\|_{L^p(\Omega)}^p \leq c_1\|\boldsymbol{\nabla} u - \boldsymbol{\nabla}\overline{u}\|_{\boldsymbol{L}^p(\Omega)}^p \leq F(u) - F_*(\boldsymbol{p}^*) \text{ for all } u \in X_g, \boldsymbol{p}^* \in K. \tag{5.7}$$

When $1 < p < 2$*, there exists a constant* $c_2 > 0$ *depending only on* p, Γ_1, Γ_2 *and* Ω *such that*

$$c_2(\|\boldsymbol{\nabla} u\|_{\boldsymbol{L}^p(\Omega)}^p + \|\boldsymbol{\nabla}\overline{u}\|_{\boldsymbol{L}^p(\Omega)}^p)^{(p-2)/2}\|u - \overline{u}\|_{L^p(\Omega)}^2 \leq F(u) - F_*(\boldsymbol{p}^*) \text{ for all } u \in X_g, \boldsymbol{p}^* \in K. \tag{5.8}$$

Proof

For any $\boldsymbol{p} \in Y$, define a functional

$$H(\boldsymbol{p}) = \frac{1}{2}\int_\Omega S(x, |\boldsymbol{p} + \boldsymbol{\nabla}\overline{g}|^2)dx - b(\overline{g}).$$

For $v \in X_g$, if we put $w = v - \overline{g}$, then $w \in X$ and $\boldsymbol{\nabla} w = \boldsymbol{\nabla} v - \boldsymbol{\nabla}\overline{g}$. Thus we have

$$H(\boldsymbol{\nabla} w) - b(w) = \frac{1}{2}\int_\Omega S(x, |\boldsymbol{\nabla} v|^2)dx - b(v) = F(v).$$

Therefore, we see that the problem (5.2) is equivalent to the problem: to find $w \in X$ such that

$$H(\boldsymbol{\nabla} w) - b(w) = \inf_{v\in X}(H(\boldsymbol{\nabla} v) - b(v)), \tag{5.9}$$

and

$$\alpha = \inf_{v\in X}(H(\boldsymbol{\nabla} v) - b(v)). \tag{5.10}$$

This plays an important role in many physical problems of elasticity and plasticity theory. From [16, Theorem 51.B], the dual problem of (5.10) reads

$$\gamma = \sup_{\boldsymbol{p}^*\in K}[-H^*(\boldsymbol{p}^*)], \tag{5.11}$$

where

$$H^*(\boldsymbol{p}^*) = \sup_{\boldsymbol{p}\in Y}(\langle \boldsymbol{p}^*, \boldsymbol{p}\rangle_{Y^*,Y} - H(\boldsymbol{p})).$$

We compute H^*. Since

$$H(\boldsymbol{p}) = \int_\Omega \gamma(x, \boldsymbol{p} + \boldsymbol{\nabla}\overline{g})dx - b(\overline{g}) = \int_\Omega \delta(x, \boldsymbol{p})dx - b(\overline{g}),$$

we have

$$\begin{aligned}
H^*(\boldsymbol{p}^*) &= \sup_{\boldsymbol{p}\in Y}(\langle \boldsymbol{p}^*, \boldsymbol{p}\rangle_{Y^*,Y} - H(\boldsymbol{p})) \\
&= \sup_{\boldsymbol{p}\in Y}\left\{\langle \boldsymbol{p}^*, \boldsymbol{p}\rangle_{Y^*,Y} - \int_\Omega \delta(x,\boldsymbol{p})dx\right\} + b(\overline{g}) \\
&= \left(\int_\Omega \delta(x,\boldsymbol{p})dx\right)^*(\boldsymbol{p}^*) + b(\overline{g}) \\
&= \int_\Omega \delta^*(x,\boldsymbol{p}^*)dx + b(\overline{g}) \quad (\text{ from [16, Problem 51.7] }) \\
&= \int_\Omega (\gamma^*(x,\boldsymbol{p}^*) - \boldsymbol{p}^*\cdot \boldsymbol{\nabla}\overline{g})dx + b(\overline{g}) \\
&= \int_\Omega (\beta^*(x,|\boldsymbol{p}^*|) - \boldsymbol{p}^*\cdot \boldsymbol{\nabla}\overline{g})dx + b(\overline{g}).
\end{aligned}$$

Thus we have $-H^*(\boldsymbol{p}^*) = F_*(\boldsymbol{p}^*)$. Hence we see that the problems (5.11) and (5.3) are identical and $\gamma = \beta$.

Now we compute H'. Since

$$H'(\boldsymbol{p}) = \gamma'(x, \boldsymbol{p}+\boldsymbol{\nabla}\overline{g}) = S_t(x, |\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}|^2)(\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}) \text{ for } \boldsymbol{p}\in Y,$$

it follows from Lemma 2.1 that

$$\begin{aligned}
\langle H'(\boldsymbol{p}) - H'(\boldsymbol{q}), \boldsymbol{p}-\boldsymbol{q}\rangle_{Y^*,Y} &= \langle S_t(x,|\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}|^2)(\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}) - S_t(x,|\boldsymbol{q}+\boldsymbol{\nabla}\overline{g}|^2)(\boldsymbol{q}+\boldsymbol{\nabla}\overline{g}), \boldsymbol{p}-\boldsymbol{q}\rangle_{Y^*,Y} \\
&\geq \begin{cases} c\int_\Omega |\boldsymbol{p}-\boldsymbol{q}|^p dx & \text{if } p\geq 2, \\ c\int_\Omega (|\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}| + |\boldsymbol{q}+\boldsymbol{\nabla}\overline{g}|)^{p-2}|\boldsymbol{p}-\boldsymbol{q}|^2 dx & \text{if } 1<p<2, \end{cases}
\end{aligned}$$

where c is a positive constant depending only on p and Ω. When $1 < p < 2$, we use the reverse Hölder inequality (cf. Sobolev [12, p. 8] to get

$$\int_\Omega (|\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}| + |\boldsymbol{q}+\boldsymbol{\nabla}\overline{g}|)^{p-2}|\boldsymbol{p}-\boldsymbol{q}|^2 dx \geq c_1\left(\int_\Omega (|\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}|^p + |\boldsymbol{q}+\boldsymbol{\nabla}\overline{g}|^p)dx\right)^{(p-2)/2}\|\boldsymbol{p}-\boldsymbol{q}\|^2_{\boldsymbol{L}^p(\Omega)},$$

where c_1 ia a positive constant depending only on p and Ω. Thus there exists a constant $c_2 > 0$ depending only on p and Ω such that

$$\begin{aligned}
&\langle H'(\boldsymbol{p}) - H'(\boldsymbol{q}), \boldsymbol{p}-\boldsymbol{q}\rangle_{Y^*,Y} \\
&\qquad \geq \begin{cases} c_2\|\boldsymbol{p}-\boldsymbol{q}\|^p_{\boldsymbol{L}^p(\Omega)} & \text{if } p\geq 2, \\ c_2\left(\int_\Omega (|\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}|^p + |\boldsymbol{q}+\boldsymbol{\nabla}\overline{g}|^p)dx\right)^{(p-2)/2}\|\boldsymbol{p}-\boldsymbol{q}\|^2_{\boldsymbol{L}^p(\Omega)} & \text{if } 1<p<2. \end{cases}
\end{aligned} \tag{5.12}$$

This implies that H' is strictly monotone. Furthermore,

$$\begin{aligned}
\langle H'(\boldsymbol{p}), \boldsymbol{p}\rangle_{Y^*,Y} &= \langle S_t(x,|\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}|^2)(\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}), \boldsymbol{p}\rangle_{Y^*,Y} \\
&= S_t(x,|\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}|^2)(\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}), \boldsymbol{p}+\boldsymbol{\nabla}\overline{g}\rangle_{Y^*,Y} - S_t(x,|\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}|^2)(\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}), \boldsymbol{\nabla}\overline{g}\rangle_{Y^*,Y} \\
&\geq \lambda\|\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}\|^p_{\boldsymbol{L}^p(\Omega)} - C\int_\Omega |\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}|^{p-1}|\boldsymbol{\nabla}\overline{g}|dx \\
&\geq \lambda\|\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}\|^p_{\boldsymbol{L}^p(\Omega)} - \epsilon\|\boldsymbol{p}+\boldsymbol{\nabla}\overline{g}\|^p_{\boldsymbol{L}^p(\Omega)} - C(\epsilon)\|\boldsymbol{\nabla}\overline{g}\|^p_{\boldsymbol{L}^p(\Omega)}
\end{aligned}$$

for any $\epsilon > 0$. Here we used the Hölder and the Young inequalities. If we choose ϵ so that $\epsilon < \lambda$, we can see that H' is coercive. Since it is clearly seen that $H' : Y \to Y^*$ is hemicontinuous, the inverse $(H')^{-1} : Y^* \to Y$ exists and $(H')^{-1} = (H^*)'$. Furthermore, we can apply [16, Theorem 53B]. Therefore, the problems (5.10) and (5.11) have a unique solution $\overline{w} \in X$ and $\overline{p}^* \in K$, respectively, $\alpha = \beta$, and

$$\overline{\boldsymbol{p}}^* = H'(\boldsymbol{\nabla}\overline{w}) = S_t(x, |\boldsymbol{\nabla}\overline{w} + \boldsymbol{\nabla}\overline{g}|^2)(\boldsymbol{\nabla}\overline{w} + \boldsymbol{\nabla}\overline{g}).$$

If we put $\overline{u} = \overline{w} + \overline{g}$, then (5.4) holds. Since $\overline{u}$ is a unique solution of (1.4) and

$$\|\overline{\boldsymbol{p}}^*\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} = \|S_t(x, |\boldsymbol{\nabla}\overline{u}|^2)\boldsymbol{\nabla}\overline{u}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} \le C\|\boldsymbol{\nabla}\overline{u}\|^p_{\boldsymbol{L}^p(\Omega)},$$

we have the estimate (5.5) by (2.8). From (5.12) and [16, Theorem 51B], we get (5.7) and (5.8). □

Remark 5.2 *Here the Euler equation to* (5.10) *becomes:*

$$\langle H'(\boldsymbol{\nabla}\overline{w}), \boldsymbol{\nabla}v\rangle_{Y^*,Y} = b(v) \text{ for all } v \in X, \tag{5.13}$$

This means that

$$\int_\Omega S_t(x, |\boldsymbol{\nabla}\overline{u}|^2)\boldsymbol{\nabla}\overline{u}\cdot\boldsymbol{\nabla}v dx = \int_\Omega f v dx + \int_{\Gamma_2} hv d\sigma \text{ for all } v \in X,$$

that is, $\overline{u}$ is a weak solution of (1.4).

The Euler equation to (5.11) *becomes:*

$$\langle \boldsymbol{q}^* - \overline{\boldsymbol{p}}^*, (H')^{-1}(\overline{\boldsymbol{p}}^*)\rangle_{Y^*,Y} \ge 0 \text{ for all } \boldsymbol{q}^* \in K. \tag{5.14}$$

Since $\overline{\boldsymbol{p}}^* = S_t(x, |\boldsymbol{\nabla}\overline{u}|^2)\boldsymbol{\nabla}\overline{u} = H'(\boldsymbol{\nabla}\overline{w})$, we have $(H')^{-1}(\overline{\boldsymbol{p}}^*) = \boldsymbol{\nabla}\overline{w} = \boldsymbol{\nabla}\overline{u} - \boldsymbol{\nabla}\overline{g}$. Since $\boldsymbol{q}^* \in K$, we have $\langle \boldsymbol{q}^*, \boldsymbol{\nabla}\overline{w}\rangle_{Y^*,Y} = b(\overline{w}) = b(\overline{u}) - b(\overline{g})$. Hence (5.14) *becomes*

$$\int_\Omega S_t(x, |\boldsymbol{\nabla}\overline{u}|^2)\boldsymbol{\nabla}\overline{u}\cdot\boldsymbol{\nabla}\overline{u} dx - b(\overline{u}) \le \int_\Omega S_t(x, |\boldsymbol{\nabla}\overline{u}|^2)\boldsymbol{\nabla}\overline{u}\cdot\boldsymbol{\nabla}\overline{g} dx - b(\overline{g}).$$

We can derive the continuity of the solutions $\overline{u}$ and $\overline{\boldsymbol{p}}^*$ on the data.

Corollary 5.3 *Assume that $f_j, f \in W^{1,p}(\Omega)^*$, $g_j, g \in W^{1-1/p,p}(\Gamma_1)$, $h_j, h \in W^{1-1/p,p}(\Gamma_2)^*$ satisfy $f_j \to f$ strongly in $W^{1,p}(\Omega)^*$, $g_j \to g$ strongly in $W^{1-1/p,p}(\Gamma_1)$, $h_j \to h$ strongly in $W^{1-1/p,p}(\Gamma_2)^*$. Let $\overline{u}_j$ and $\overline{\boldsymbol{p}}^*_j$ be solutions of the problems* (5.2) *and* (5.3) *with $f = f_j, g = g_j, h = h_j$. Then we can show that $\overline{u}_j \to \overline{u}$ strongly in $W^{1,p}(\Omega)$ and $\overline{\boldsymbol{p}}^*_j \to \overline{\boldsymbol{p}}^*$ strongly in $\boldsymbol{L}^{p'}(\Omega)$.*

Proof The strong convergence of $\{\overline{u}_j\}$ to $\overline{u}$ follows from Theorem 4.1. Since $\overline{\boldsymbol{p}}^*_j = S_t(x, |\boldsymbol{\nabla}\overline{u}_j|^2)\boldsymbol{\nabla}\overline{u}_j$ by Theorem 5.1, it follows from Lemma 2.2 that

$$|\overline{\boldsymbol{p}}^*_j - \overline{\boldsymbol{p}}^*| \le \begin{cases} C|\boldsymbol{\nabla}\overline{u}_j - \boldsymbol{\nabla}\overline{u}|^{p-1} & \text{if } 1 < p \le 2, \\ C(|\boldsymbol{\nabla}\overline{u}_j| + |\boldsymbol{\nabla}\overline{u}|)^{p-2}|\boldsymbol{\nabla}\overline{u}_j - \boldsymbol{\nabla}\overline{u}| & \text{if } p > 2. \end{cases}$$

When $1 < p \le 2$, we have

$$\int_\Omega |\overline{\boldsymbol{p}}_j^* - \overline{\boldsymbol{p}}^*|^{p'} dx \le C\|\overline{u}_j - \overline{u}\|_{W^{1,p}(\Omega)}^p \to 0 \text{ as } j \to \infty.$$

When $p > 2$, it follows from the Hölder inequality that

$$\begin{aligned} \int_\Omega |\overline{\boldsymbol{p}}_j^* - \overline{\boldsymbol{p}}^*|^{p'} dx &\le C \int_\Omega (|\boldsymbol{\nabla}\overline{u}_j| + |\boldsymbol{\nabla}\overline{u}|)^{(p-2)p'} |\boldsymbol{\nabla}\overline{u}_j - \boldsymbol{\nabla}\overline{u}|^{p'} dx \\ &\le C \left(\int_\Omega (|\boldsymbol{\nabla}\overline{u}_j| + |\boldsymbol{\nabla}\overline{u}|)^p dx \right)^{(p-2)p'/p} \left(\int_\Omega |\boldsymbol{\nabla}\overline{u}_j - \boldsymbol{\nabla}\overline{u}|^p dx \right)^{p'/p} \le C_1 \|\overline{u}_j - \overline{u}\|_{W^{1,p}(\Omega)}^{p'} \to 0 \text{ as } j \to \infty. \end{aligned}$$

□

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