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MUSTAFA ÇETİN GÜLEÇYÜZ

AHMET ERSOY

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The Milne problem using the P_N method for a nonabsorbing medium with linearly anisotropic scattering

Mustafa Çetin GÜLEÇYÜZ[1,*], **Ahmet ERSOY**[1,2]
[1]Department of Physics, Ankara University, Ankara, Turkey
[2]Department of Physics, Karamanoğlu Mehmetbey University, Karaman, Turkey

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Abstract: The Milne problem, known as one of the classical problems of radiative transfer and neutron transport theory, is solved using the P_N method for a nonabsorbing medium (c = 1) with a linearly anisotropic scattering kernel. Specular and diffuse reflection boundary conditions are taken into account. The numerical results are listed for different selected parameters. Some results are also compared with the literature.

Key words: P_N method, Milne problem, extrapolated endpoint, nonabsorbing medium, specular and diffuse reflectivity, anisotropic scattering

1. Introduction

The Milne problem is a well-known problem in both the radiative transfer field and neutron transport theory [1]. In this problem, the angular distribution of the flux due to monoenergetic neutrons or radiations diffusing from a source at infinity ($x > 0$) can be obtained in a source-free half-space ($x < 0$) [2,3].

In this paper the effect of linearly anisotropic scattering with specular and diffuse reflecting boundaries on the extrapolated endpoint z_0 for the Milne problem is studied in a nonabsorbing medium using Legendre polynomial approximation. Numerical results for z_0 and the emergent angular distribution $\Psi(0,-\mu)$ are given and compared with the literature.

In the plane geometry, the one-speed, time-independent neutron transport equation for linearly anisotropic scattering is [4]

$$\mu\frac{\partial\Psi(x,\mu)}{\partial x}+\Psi\left(x,\mu\right)=\frac{c}{2}\int\limits_{-1}^{+1}(1+3f_1\mu\mu')\Psi\left(x,\mu'\right)d\mu' \tag{1}$$

where $\Psi(x,\mu)$ is the angular distribution of neutrons, c is the number of secondary neutrons per collision, μ is the cosine of the angle between the direction of the neutron velocity and the positive x axis, and $3f_1$ ($-1 \leq 3f_1 \leq +1$) is the coefficient of the linearly anisotropic scattering.

The Milne problem will be solved with the following boundary conditions [5]:

$$\Psi\left(0,\mu\right)=\rho^s\Psi\left(0,-\mu\right)+2\rho^d\int\limits_{-1}^{+1}\mu'\Psi\left(0,\mu'\right)d\mu', \qquad \mu>0 \tag{2}$$

*Correspondence: gulecyuz@science.ankara.edu.tr

and

$$\lim_{x\to\infty} \Psi(x,\mu) = 0, \tag{3}$$

where $\rho^s(0 \leq \rho^s \leq 1)$ and $\rho^d(0 \leq \rho^d \leq 1)$ are the specular and diffuse reflectivity of the boundary, respectively.

2. Method and calculation

In the P_N method, by using the expansion of the angular distribution in terms of Legendre polynomials [6],

$$\Psi(x,\mu) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \Psi_n(x) P_n(\mu) \tag{4}$$

and multiplying both sides of Eq. (1) by $P_m(\mu)$, then integrating over μ in an interval [–1, +1] and using the orthogonality of the Legendre polynomials leads to the following moment equations, which consist of the infinite sets of coupled differential equations with m = 0,1,...,N [7], [8]:

$$m\frac{d\Psi_{m-1}(x)}{dx} + (m+1)\frac{d\Psi_{m+1}(x)}{dx} + (2m+1)\left[1 - c\left(\delta_{m0} - f_1\delta_{m1}\right)\right]\Psi_m(x) = 0 \tag{5}$$

Here

$$\int_{-1}^{+1} P_n(\mu)P_m(\mu)d\mu = \frac{2}{2m+1}\delta_{nm}, \tag{6}$$

$$\mu P_m(\mu) = \frac{1}{2m+1}\left[(2m+1)P_{m+1}(\mu) + mP_{m-1}(\mu)\right]. \tag{7}$$

In order to solve Eq. (5) it is sufficient to get $d\Psi_{m+1}(x)/dx = 0$. As a result of this, N + 1 equations that consist of N + 1 unknowns are obtained.

P_1 approximation: In P_1 approximation for nonabsorbing medium i.e. c = 1, the moment equations from Eq. (5) are

$$\frac{d\Psi_1(x)}{dx} = 0, \tag{8a}$$

$$\frac{d\Psi_0(x)}{dx} + 3(1-f_1)\Psi_1(x) = 0. \tag{8b}$$

$\Psi_1(x)$ is a constant as seen in Eq. (8a) and so assuming $\Psi_1(x) = -1$ and then substituting it in Eq. (8b) we get

$$\Psi_0(x) = 3(1-f_1)x + A_0 \tag{9}$$

and from Eq. (4)

$$\Psi(0,\mu) = \frac{1}{2}\Psi_0(0)P_0(\mu) + \frac{3}{2}\Psi_1(0)P_1(\mu), \tag{10}$$

where A_0 is constant and can be obtained using the Marshak boundary condition given by [9],

$$\int_{-1}^{+1} \mu^m \Psi(0,\mu)d\mu = 0, \quad m = 1,3,...,N \tag{11}$$

Using Eq. (2), we can rewrite the Marshak boundary condition as

$$\int_0^1 \mu^m \Psi(0,\mu) d\mu - \rho^s \int_0^1 \mu^m \Psi\left(0,-\mu\right) d\mu - \frac{2\rho^d}{m+1} \int_0^1 \mu^{'} \Psi\left(0,\mu^{'}\right) d\mu^{'} = 0,$$

$$m = 1, 3, ..., N \tag{12}$$

Then A_0 can be obtained as

$$A_0 = 2\frac{1+\rho^s+\rho^d}{1-\rho^s-\rho^d}, \tag{13}$$

The extrapolated endpoint z_0 is defined as a distance from a vacuum boundary to where the asymptotic intensity vanishes and it can be written mathematically as $\Psi_0(-z_0) = 0$. From the definitions one can get

$$z_0 = \frac{A_0}{3(1-f_1)} \tag{14}$$

P_3 approximation: In P_3 approximation the moment equations from Eq. (5) can be written as

$$\frac{d\Psi_1(x)}{dx} = 0, \tag{15a}$$

$$2\frac{d\Psi_2(x)}{dx} + \frac{d\Psi_0(x)}{dx} + 3\left(1-f_1\right)\Psi_1\left(x\right) = 0, \tag{15b}$$

$$3\frac{d\Psi_3(x)}{dx} + 2\frac{d\Psi_1(x)}{dx} + 5\Psi_2\left(x\right) = 0, \tag{15c}$$

$$3\frac{d\Psi_2(x)}{dx} + 7\Psi_3\left(x\right) = 0. \tag{15d}$$

From Eq. (15c) and Eq. (15d), a differential equation for $\Psi_2(x)$ is

$$\frac{-9}{7}\frac{d^2\Psi_2(x)}{dx^2} + 5\Psi_2\left(x\right) = 0 \tag{16}$$

and the solution of this equation is given by

$$\Psi_2\left(x\right) = A_1 \exp\left(-\alpha_1 x\right); \quad \alpha_1 = \left(\frac{35}{9}\right)^{\frac{1}{2}} \tag{17}$$

where A_1 is constant. Inserting Eq. (17) into Eq. (15d), $\Psi_3(x)$ can be written as

$$\Psi_3\left(x\right) = \frac{3}{7} A_1 \alpha_1 \exp\left(-\alpha_1 x\right) \tag{18}$$

Using Eqs. (17) and (18) and $\Psi_1(x) = -1$ in Eq. (15b)

$$\Psi_0\left(x\right) = 3\left(1-f_1\right)x + A_0 - 2A_1 \exp\left(-\alpha_1 x\right) \tag{19}$$

is obtained. Here A_0 is constant and from Eq. (4) we get

$$\Psi(0,\mu)=\frac{1}{2}\Psi_0(0)P_0(\mu)+\frac{3}{2}\Psi_1(0)P_1(\mu)+\frac{5}{2}\Psi_2(0)P_2(\mu)+\frac{7}{2}\Psi_3(0)P_3(\mu) \tag{20}$$

The lowest order of Marshak boundary conditions given in Eq. (12) and Eq. (20) are used to determine A_0 and A_1 constants

$$A_0=-2-\frac{4}{\rho^s\rho^d-1}+\frac{42(1+\rho^s)}{175(\rho^s-1)+96\alpha_1(1+\rho^s)}, \tag{21a}$$

$$A_1=\frac{56(1+\rho^s)}{175(\rho^s-1)+96\alpha_1(1+\rho^s)}. \tag{21b}$$

Finally, the extrapolated endpoint z_0 is calculated from Eq. (14) using A_0 in Eq. (21a) for P_3 approximation.

P_5 approximation: In P_5 approximation the moment equations are obtained from Eq. (5)

$$\frac{d\Psi_1(x)}{dx}=0, \tag{22a}$$

$$2\frac{d\Psi_2(x)}{dx}+\frac{d\Psi_0(x)}{dx}+3(1-f_1)\Psi_1(x)=0, \tag{22b}$$

$$3\frac{d\Psi_3(x)}{dx}+2\frac{d\Psi_1(x)}{dx}+5\Psi_2(x)=0, \tag{22c}$$

$$4\frac{d\Psi_4(x)}{dx}+3\frac{d\Psi_2(x)}{dx}+7\Psi_3(x)=0, \tag{22d}$$

$$5\frac{d\Psi_5(x)}{dx}+4\frac{d\Psi_3(x)}{dx}+9\Psi_4(x)=0, \tag{22e}$$

$$5\frac{d\Psi_4(x)}{dx}+11\Psi_5(x)=0. \tag{22f}$$

After some algebraic manipulations in Eqs. (22), a differential equation is found as follows:

$$\frac{44}{77}\frac{d^4\Psi_4(x)}{dx^4}-\frac{378}{55}\frac{d^2\Psi_4(x)}{dx^2}+9\Psi_4(x)=0, \tag{23}$$

and the solution of this equation is

$$\Psi_4(x)=A_1\exp(-\alpha_1 x)+A_2\exp(-\alpha_2 x) \tag{24}$$

$\Psi_5(x)$ can be derived from Eq. (22f)

$$\Psi_5(x)=\frac{-5}{11}\frac{d\Psi_4(x)}{dx} \tag{25}$$

and the other solutions of moment equations can be found as

$$\Psi_3(x)=-\int\frac{9}{4}\left(\Psi_4(x)+\frac{5}{9}\frac{d\Psi_5(x)}{dx}\right)dx,$$

$$\Psi_2(x) = -\int \frac{7}{3}\left(\Psi_3\left(x\right) + \frac{4}{7}\frac{d\Psi_4\left(x\right)}{dx}\right)dx,$$

$$\Psi_1\left(x\right) = -1,$$

$$\Psi_0\left(x\right) = -3\int\left[\left(1-f_1\right)\Psi_1\left(x\right) + \frac{2}{3}\frac{d\Psi_2\left(x\right)}{dx}\right]dx + A_0. \tag{26}$$

A_0, A_1, A_2 are constants and can be calculated by using Marshak boundary conditions given in Eq. (12) and from Eq.(4) we obtained

$$\Psi\left(0,\mu\right) = \frac{1}{2}\Psi_0\left(0\right)P_0\left(\mu\right) + \frac{3}{2}\Psi_1\left(0\right)P_1\left(\mu\right) + \frac{5}{2}\Psi_2\left(0\right)P_2\left(\mu\right) + \frac{7}{2}\Psi_3\left(0\right)P_3\left(\mu\right)$$

$$+\frac{9}{2}\Psi_4\left(0\right)P_4\left(\mu\right) + \frac{11}{2}\Psi_5\left(0\right)P_5\left(\mu\right) \tag{27}$$

As a result, we can generalize the moment equations for P_N approximation as

$$\frac{d\Psi_1(x)}{dx} = 0,$$

$$2\frac{d\Psi_2(x)}{dx} + \frac{d\Psi_0(x)}{dx} + 3\left(1-f_1\right)\Psi_1\left(x\right) = 0,$$

$$3\frac{d\Psi_3(x)}{dx} + 2\frac{d\Psi_1(x)}{dx} + 5\Psi_2\left(x\right) = 0,$$

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$$N\frac{d\Psi_N(x)}{dx} + \left(N-1\right)\frac{d\Psi_{N-2}\left(x\right)}{dx} + \left(2N-1\right)\Psi_{N-1}\left(x\right) = 0,$$

$$N\frac{d\Psi_{N-1}(x)}{dx} + \left(2N+1\right)\Psi_N\left(x\right) = 0, \tag{28}$$

and the solutions of these equations are

$$\Psi_{N-1}\left(x\right) = \sum_{k=1}^{\frac{N-1}{2}} A_k \exp(-\alpha_k x),$$

$$\Psi_N\left(x\right) = \frac{-N}{2N+1}\frac{d\Psi_{N-1}\left(x\right)}{dx},$$

$$\Psi_{k-2}(x) = -\int \frac{2k-1}{k-1}\left(\Psi_{k-1}\left(x\right) + \frac{k}{2k-1}\frac{d\Psi_k\left(x\right)}{dx}\right)dx, \qquad k = 4, 5, ..., N$$

$$\Psi_1\left(x\right) = -1,$$

$$\Psi_0\left(x\right) = -3\int\left[\left(1-f_1\right)\Psi_1\left(x\right) + \frac{2}{3}\frac{d\Psi_2\left(x\right)}{dx}\right]dx + A_0, \tag{29}$$

The A_0 and A_k constants can be determined by Eq. (12) and the extrapolated endpoint is calculated from Eq. (14).

3. Results

Numerical values of the extrapolated endpoint are given in isotropic scattering for diffuse and specular reflection in Tables 1 and 2, respectively. They are compared with the exact values in Williams's paper [10]. Table 3 shows the extrapolated endpoints for different values of ρ^s and ρ^d in isotropic scattering. These results are compared with Degheidy [11]. It can be seen from these tables that z_0, the distance at which the flux drops to zero, increases with increasing specular and diffuse components of reflectivity. In Tables 4 and 5, numerical values of z_0 are given for different values of ρ^s and ρ^d in linearly anisotropic scattering compared with Atalay [12,13]. In these tables, the extrapolated endpoint values increase with increasing linearly anisotropic scattering function. Table 6 shows the emergent angular distribution $\Psi(0, -\mu)$ for various values of ρ^s and ρ^d in isotropic scattering ($f_1 = 0$). In all our computations, we used Mathematica programming.

Table 1. The extrapolated endpoint for diffuse reflection ($\rho^s = 0$) and isotropic scattering ($f_1 = 0$).

ρ^d	P_1	P_3	P_5	P_7	P_9
0	0.6667	0.7051	0.7082	0.7092	0.7096
0.1	0.8148	0.8532	0.8564	0.8573	0.8578
0.2	1.0000	1.0338	1.0415	1.0425	1.0430
0.3	1.2381	1.2765	1.2796	1.2806	1.2810
0.4	1.5556	1.5940	1.5971	1.5981	1.5985
0.5	2.0000	2.0384	2.0415	2.0425	2.0430
0.6	2.6667	2.7051	2.7082	2.7092	2.7096
0.7	3.7778	3.8162	3.8193	3.8203	3.8208
0.8	6.0000	6.0384	6.0415	6.0425	6.0430
0.9	12.6667	12.7051	12.7082	12.7092	12.7096
0.99	132.6670	132.7051	132.7082	132.7092	132.7097
0.999	1332.6700	1332.7050	1332.7080	1332.7090	1332.7090
ρ^d	P_{23}	P_{25}	P_{27}	P_{29}	Exact
0	0.7103	0.7103	0.7103	0.7104	0.7104
0.1	0.8585	0.8585	0.8585	0.8585	0.8585
0.2	1.0437	1.0437	1.0437	1.0437	1.0437
0.3	1.2817	1.2818	1.2818	1.2818	1.2818
0.4	1.5992	1.5992	1.5992	1.5992	1.5993
0.5	2.0436	2.0437	2.0437	2.0437	2.0437
0.6	2.7103	2.7103	2.7103	2.7104	2.7104
0.7	3.8214	3.8214	3.8215	3.8215	3.8214
0.8	6.0436	6.0437	6.0437	6.0437	6.0436
0.9	12.7103	12.7103	12.7103	12.7104	12.7104
0.99	132.7103	132.7103	132.7103	132.7104	-
0.999	1332.7100	1332.7100	1332.7100	1332.7100	-

Exact, Williams [10]

4. Discussion

The Milne problem with specular and diffuse reflecting boundary conditions is solved using Legendre polynomial approximation, which is known as the P_N method. Extrapolated endpoint z_0 is the distance from the boundary to where the asymptotic component of density goes to zero. It was calculated for the nonabsorbing medium where the average number of secondary neutrons per collision equals unity. There are only scattering collisions with no neutron loss. Furthermore, the linearly anisotropic scattering function is considered. For Nth order approximation, N + 1 set of coupled differential equations are defined and reduced to one (N − 1)th order differential equation for $\Psi_{N-1}(x)$ in the P_N method. The remaining N moments for $\Psi_N(x)$ are derived after writing the solution of $\Psi_{N-1}(x)$. The constants of these solutions are calculated using Marshak boundary conditions with specular and diffuse reflectivity of the boundary. Then z_0 is obtained using the A_0 constant for each Nth order approximation. Some of our results are compared with the literature and these results are in agreement.

Table 2. The extrapolated endpoint for specular reflection ($\rho^d = 0$) and anisotropic scattering ($f_1 = 0$).

ρ^d	P_1	P_3	P_5	P_7	P_9
0	0.6667	0.7051	0.7082	0.7092	0.7096
0.1	0.8148	0.8569	0.8605	0.8617	0.8623
0.2	1.0000	1.0458	1.0499	1.0513	1.0519
0.3	1.2381	1.2875	1.2922	1.2938	1.2945
0.4	1.5556	1.6085	1.6139	1.6157	1.6166
0.5	2.0000	2.0565	2.0625	2.0646	2.0656
0.6	2.6667	2.7267	2.7334	2.7358	2.7369
0.7	3.7778	3.8414	3.8488	3.8514	3.8526
0.8	6.0000	6.0671	6.0752	6.0781	6.0795
0.9	12.6667	12.7372	12.7461	12.7494	12.7509
0.99	132.6667	132.7403	132.7500	132.7535	132.7552
0.999	1332.6667	1332.7410	1332.7500	1332.7540	1332.7560
ρ^d	P_{23}	P_{25}	P_{27}	P_{29}	Exact
0	0.7103	0.7103	0.7103	0.7104	0.7104
0.1	0.8630	0.8631	0.8631	0.8631	0.8632
0.2	1.0529	1.0529	1.0529	1.0529	1.0531
0.3	1.2957	1.2957	1.2957	1.2957	1.2959
0.4	1.6178	1.6179	1.6179	1.6179	1.6181
0.5	2.0671	2.0671	2.0671	2.0672	2.0674
0.6	2.7386	2.7386	2.7387	2.7387	2.7389
0.7	3.8545	3.8546	3.8546	3.8547	3.8550
0.8	6.0817	6.0817	6.0818	6.0819	6.0822
0.9	12.7533	12.7534	12.7535	12.7535	12.7539
0.99	132.7578	132.7579	132.7580	132.7580	132.7580
0.999	1332.7580	1332.7580	1332.7580	1332.7590	-

Exact, Williams [10]

Table 3. The extrapolated endpoint for different values of ρ^s, ρ^d, and isotropic scattering case ($f_1 = 0$) in P_{29} approximation.

ρ^s / ρ^d	0.1		0.2		0.3		0.4	
	A	B	A	B	A	B	A	B
0.1	1.0483	1.0484	1.2910	1.2912	1.6132	1.6133	2.0624	2.0626
0.2	1.2864	1.2865	1.6085	1.6086	2.0576	2.0578	2.7291	2.7292
0.3	1.6039	1.6039	2.0529	2.0531	2.7243	2.7245	3.8402	3.8453
0.4	2.0483	2.0484	2.7196	2.7197	3.8354	3.8356	6.0624	6.0626

A; our result, B; A. R. Degheidy [11]

Table 4. The extrapolated endpoint for specular and diffuse reflection in linearly anisotropic scattering for P_{29} approximation.

ρ^d or ρ^s	$f_1 = 0.1$			
	Diffuse		Specular	
	A	B	A	B
0	0.7893	0.7894	0.7893	0.7894
0.1	0.9539	0.9568	0.9590	0.9592
0.2	1.1597	1.1660	1.1699	1.1702
0.3	1.4242	-	1.4397	-
0.4	1.7769	-	1.7977	-
0.5	2.2708	2.2959	2.2969	2.2989
0.6	3.0115	-	3.0430	-
0.7	4.2461	-	4.2830	-
0.8	6.7152	-	6.7576	-
0.9	14.1226	14.3479	14.1706	14.2089
0.99	147.4560	149.9330	147.5089	147.9930
0.999	1480.7893	-	1480.8430	-
ρ^d or ρ^s	$f_1 = 0.2$			
	Diffuse		Specular	
	A	B	A	B
0	0.8879	0.8881	0.8879	0.8881
0.1	1.0731	1.0764	1.0789	1.0791
0.2	1.3046	1.3118	1.3162	1.3165
0.3	1.6022	-	1.6197	-
0.4	1.9991	-	2.0224	-
0.5	2.5546	2.5283	2.5840	2.5863
0.6	3.3879	-	3.4234	-
0.7	4.7768	-	4.8184	-
0.8	7.5546	-	7.6023	-
0.9	15.8879	16.1414	15.9419	15.9850
0.99	165.8879	168.6740	165.9476	166.4920
0.999	1665.8879	-	1665.9480	-
ρ^d or ρ^s	$f_1 = 0.3$			
	Diffuse		Specular	
	A	B	A	B
0	1.0148	1.0149	1.0148	1.0149
0.1	1.2264	1.2301	1.2330	1.2332
0.2	1.4910	1.4992	1.5042	1.5046
0.3	1.8311	-	1.8511	-
0.4	2.2846	-	2.3113	-
0.5	2.9196	2.9519	2.9531	2.9557
0.6	3.8719	-	3.9124	-
0.7	5.4592	-	5.5067	-
0.8	8.6338	-	8.6884	-
0.9	18.1577	18.4473	18.2193	18.2686
0.99	189.5862	192.7710	189.6543	190.2770
0.999	1903.8719	-	1903.9410	-

A; our result, B; M. A. Atalay [12]

Table 5. The extrapolated endpoint for different values of ρ^s and ρ^d in linearly anisotropic scattering ($f_1 \neq 0$) for P_{29} approximation.

$f_1 = 0.1$					
ρ^s / ρ^d	0.1	0.2	0.3	0.4	0.5
0.1	1.16475	1.43446	1.79241	2.29151	3.03757
0.2	1.42930	1.78720	2.28624	3.03225	4.27213
0.3	1.78203	2.28102	3.02698	4.26681	6.74127
0.4	2.27586	3.02176	4.26155	6.73595	14.14871
0.5	3.01662	4.25636	6.73071	14.14339	-
$f_1 = 0.2$					
ρ^s / ρ^d	0.1	0.2	0.3	0.4	0.5
0.1	1.31034	1.61377	2.01646	2.57794	3.41726
0.2	1.60796	2.01060	2.57202	3.41128	4.80615
0.3	2.00479	2.56615	3.40535	4.80017	7.58393
0.4	2.56034	3.39949	4.79424	7.57794	15.91730
0.5	3.39370	4.78840	7.57205	15.91131	-
$f_1 = 0.3$					
ρ^s / ρ^d	0.1	0.2	0.3	0.4	0.5
0.1	1.49754	1.84431	2.30453	2.94622	3.90544
0.2	1.83767	2.29782	2.93945	3.89860	5.49274
0.3	2.29119	2.93275	3.89183	5.48590	8.66735
0.4	2.92611	3.88513	5.47913	8.66051	18.19120
0.5	3.87851	5.47246	8.65377	18.18436	-

Table 6. The emergent angular distribution $\Psi(0, -\mu)$ for selected values of ρ^s and ρ^d in isotropic scattering ($f_1 = 0$).

	$-\mu$	$\rho^d = 0.0$	$\rho^d = 0.2$	$\rho^d = 0.4$	$\rho^d = 0.6$	$\rho^d = 0.8$
$\rho^s = 0$	0.1	1.18027	1.68027	2.51360	4.18027	9.18027
	0.2	1.19913	1.69913	2.53246	4.19913	9.19913
	0.3	1.46110	1.96110	2.79444	4.46110	9.46110
	0.4	1.55339	2.05339	2.88673	4.55339	9.55339
	0.5	1.76704	2.26704	3.10037	4.76704	9.76704
	0.6	1.88438	2.38438	3.21771	4.88438	9.88438
	0.7	2.05383	2.55383	3.38717	5.05383	10.05380
	0.8	2.22933	2.72933	3.56266	5.22933	10.22930
	0.9	2.38267	2.88267	3.71600	5.38267	10.38270
	1.0	2.59895	3.09895	3.93228	5.59895	10.59890

	- μ	$\rho^s = 0.0$	$\rho^s = 0.2$	$\rho^s = 0.4$	$\rho^s = 0.6$	$\rho^s = 0.8$
$\rho^d = 0$	0.1	1.18027	1.68504	2.50759	4.19368	9.38526
	0.2	1.19913	1.67880	2.57907	4.13763	9.07603
	0.3	1.46110	1.96261	2.78528	4.46511	9.59919
	0.4	1.55339	2.04421	2.93737	4.52577	9.48316
	0.5	1.76704	2.27007	3.09607	4.77591	9.88266
	0.6	1.88438	2.38146	3.26017	4.87573	9.86080
	0.7	2.05383	2.55470	3.41334	5.05653	10.08230
	0.8	2.22933	2.73506	3.55956	5.24654	10.34169
	0.9	2.38267	2.88912	3.71281	5.40209	10.49565
	1.0	2.59895	3.11776	3.82831	5.65495	10.97911

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