Turkish Journal of Mathematics

Volume 44 | Number 6 Article 17

1-1-2020

# Almost symmetric Arf partitions

NİHAL GÜMÜŞBAŞ ÖZTÜRK

NESRİN TUTAŞ

NACİ ER



Recommended Citation

ÖZTÜRK, NİHAL GÜMÜŞBAŞ; TUTAŞ, NESRİN; and ER, NACİ (2020) "Almost symmetric Arf partitions," *Turkish Journal of Mathematics*: Vol. 44: No. 6, Article 17. https://doi.org/10.3906/mat-2003-74
Available at: https://dctubitak.researchcommons.org/math/vol44/iss6/17

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 2185 – 2198
© TÜBİTAK
doi:10.3906/mat-2003-74

# Almost symmetric Arf partitions

**Nihal GÜMÜŞBAŞ, Nesrin TUTAŞ*, Naci ER**
Department of Mathematics, Faculty of Science, Akdeniz University, Antalya, Turkey

**Received:** 17.03.2020 • **Accepted/Published Online:** 16.09.2020 • **Final Version:** 16.11.2020

**Abstract:** In this paper, we introduce almost symmetric Arf partitions (for short, ASA-partitions) and using properties of partitions of positive integers, we give the number of almost symmetric Arf semigroups of genus $g$.

**Key words:** Numerical semigroup, Arf semigroup, partition, Arf partition, almost symmetric Arf partition

## 1. Introduction

Partitions of a positive integer have several applications in many branches of mathematics. In this work, we are interested in applications to numerical sets and numerical semigroups, in particular Arf numerical semigroups.

In recent years, the authors gave very interesting relations between partitions, numerical sets and Young diagrams, see [7, 11, 12, 16, 17]. In [7], the authors studied a correspondence between numerical sets and integer partitions that leaded to a bijection between simultaneous core partitions and the integer points of a certain polytope. In [11], the authors gave an associative operation $\oplus$ on Young diagrams which was carried to partitions and numerical sets by means of the correspondences obtained in [17]. This operation is used to introduce a decomposition of a partition into the so called hook partitions and a decomposition of a numerical set into the so called hook numerical sets. We observe that under some conditions, the hook numerical sets appearing in the decomposition of a numerical set become primitive numerical semigroups.

In this paper we introduce almost symmetric Arf partitions and give criterias to determine them. We present a formula for calculating the number of almost symmetric Arf partitions of a positive integer $N$. We tabulate those numbers in Table for $N \leq 80$. In Theorem 2.16 we give the number of almost symmetric Arf semigroups of genus $g$, where $g$ is a given nonnegative integer. Moreover, we describe the Kunz coordinates of an almost symmetric Arf semigroup with a given multiplicity $m$ in Proposition 2.11.

We first briefly introduce some notations necessary to explain our main results. We denote the set of integers by $\mathbb{Z}$ and the set of positive integers by $\mathbb{N}$. We put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The cardinality of any set $K$ will be denoted by $|K|$. For two subsets $U, V$ of $\mathbb{Z}$ and $z \in \mathbb{Z}$, we set

$$U + V = \{u + v : u \in U, v \in V\}, \quad U - V = \{x \in \mathbb{Z} : x + v \in U \text{ for all } v \in V\},$$

and $z + U = \{z\} + U$.

A numerical set $S$ is a subset of $\mathbb{N}_0$ which contains $0$ and has a finite complement $G(S) = \mathbb{N}_0 \setminus S$. $\mathbb{N}_0$ itself is a numerical set with $G(\mathbb{N}_0) = \emptyset$. A numerical set $S$ is said to be proper if $S \neq \mathbb{N}_0$. If $S$ is a proper

*Correspondence: ntutas@akdeniz.edu.tr
2010 *AMS Mathematics Subject Classification:* 20M14, 05A17.

numerical set, the elements of $G(S)$ are called gaps of $S$. The number of gaps is called the genus of $S$ and denoted by $g(S)$. The largest gap of $S$ is called the Frobenius number of $S$ and the Frobenius number of $S$ is denoted by $F(S)$. $C(S) = F(S) + 1$ is called the conductor of $S$. The conductor of $S$ is the smallest element of $S$ such that every subsequent integer is an element of $S$. Note that F($\mathbb{N}_0$)= $-1$, so that $C(\mathbb{N}_0) = 0$. The elements of $S$ which are smaller than $C(S)$ are called the small elements of $S$. If a numerical set has $n = n(S)$ small elements, it is customary to list them as $s_0 = 0 < s_1 < \cdots < s_{n-1}$ and write

$$S = \{s_0 = 0, s_1, \ldots, s_{n-1}, s_n = C(S), \rightarrow\},$$

where the arrow means that all subsequent integers belong to $S$. We note that $C(S) = g(S) + n(S)$ for any numerical set $S$.

For example, the numerical set $S = \{0, 4, 5, 7, \rightarrow\}$ has the complement $G(S) = \{1, 2, 3, 6\}$. Hence, $g(S) = 4$, $F(S) = 6$, $n(S) = 3$ and $C(S) = 7$.

Given a numerical set $S = \{s_0 = 0, s_1, \ldots, s_{n-1}, s_n = C(S), \rightarrow\}$, for each $i \geq 0$ we define

$$S_i = \{x \in S : x \geq s_i\}, \; S(i) = S - S_i.$$

For each $i = 0, \ldots, n-1$, the set $-s_i + S_i$ is a numerical set whose Frobenius number is $F(-s_i + S_i) = F(S) - s_i$ and $G(-s_i + S_i) = -s_i + \{b \in G(S) : b > s_i\}$; the set $S(i)$ is a numerical set for each $i \geq 0$, $S(i) \subseteq -s_i + S_i$, and $S(i) = \mathbb{N}_0$ for all $i \geq n$. We also have $S(0) \subseteq S = S_0$, $S_{i+1} \subset S_i$ and $S(i) \subseteq S(i+1)$ for all $i \geq 0$. For each $i = 1, \ldots, n$, the set $T_i(S) = S(i) \setminus S(i-1)$ is called the $i$-th type set of $S$, and the sequence $\{t_i = |T_i(S)| : 1 \leq i \leq n\}$ is called the type sequence of $S$. $S(1) = S - S \setminus \{0\}$ is defined to be the dual of $S$.

A numerical set $S$ is called a numerical semigroup if $x + y \in S$ for all $x, y \in S$. If $A$ is a subset of $\mathbb{N}_0$, we will denote by $\langle A \rangle$ the submonoid of $\mathbb{N}_0$ generated by $A$. If $S = \langle A \rangle$, $A$ is called a set of generators for $S$. If $A = \{a_1, \ldots, a_r\}$, we write $\langle A \rangle = \langle a_1, \ldots, a_r \rangle$. The monoid $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$. Note that a set of generators of a numerical semigroup is a minimal set of generators if none of its proper subsets generates the numerical semigroup. Let $\{a_1 < \ldots < a_r\}$ be the minimal system of generators of $S$. Then $a_1$ is known as the multiplicity of $S$ and denoted by $m(S)$. For general concepts and notations about numerical semigroups, we refer to [3, 14].

Given a numerical set S, the set of pseudo-Frobenius numbers of $S$ is defined by

$$PF(S) = \{x \in \mathbb{Z} \setminus S : x + S \setminus \{0\} \subseteq S\}.$$

Equivalently, $PF(S) = S(1) \setminus S = T_1(S)$ and $|PF(S)| = t_1$. The set of gaps of the first type is

$$N(S) = \{x \in \mathbb{N} \setminus S : F(S) - x \in S\}$$

and the set of the second type gaps $L(S)$ consists of the remaining gap numbers, i.e., $L(S) = G(S) \setminus N(S)$. It is well known that if $S$ is symmetric (pseudo-symmetric, resp.), then $L(S) = \emptyset$ ($L(S) = \{F(S)/2\}$, resp.).

A numerical semigroup $S$ is called almost symmetric (for short, $AS$-semigroup) when $L(S) \subseteq PF(S)$.

In [5], two algorithmic procedures were given to compute the whole set of almost symmetric numerical semigroups with fixed Frobenius number and type. For a given numerical semigroup $S$, we are going to use the following equivalent conditions:

$$S \text{ is an AS-semigroup} \Longleftrightarrow PF(S) = L(S) \bigcup \{F(S)\} \Longleftrightarrow g(S) = \frac{F(S) + t_1}{2}.$$

Several equivalent conditions are given for $AS$-semigroups. For details, we refer to [4, 5, 9, 13]. In Proposition 2.1, we explicitly describe the elements of $PF(S)$ with respect to gap numbers of $S$.

Given a positive integer $N$, a partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ of $N$ is a nonincreasing finite sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = N$. For each $i = 1, 2, \ldots, n$, the number $\lambda_i$ is called a part of the partition, and the number $n$ of parts is called the length of the partition. The length of $\lambda$ is denoted by $\mathrm{l}(\lambda)$. If $\lambda_i \neq \lambda_{i+1}$ for each $i = 1, 2, \ldots, n-1$, then $\lambda$ is called a strict dominant partition. If $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ is a partition of $N$, then we write

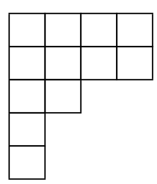
$$\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] \vdash N.$$

If $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ is strict dominant, then $n \leq \lambda_1$.

A Young diagram is a series of top aligned columns of boxes such that the number of boxes in each column is not less than the number of boxes in the column immediately right to it. The number of boxes in a column (or a row) is called the length of that column (or, respectively, that row). Flipping a Young diagram over its main diagonal (from upper left to lower right) gives the conjugate diagram. The conjugate partition of $\lambda$ is the partition corresponding to the conjugate diagram of the Young diagram of $\lambda$. If $\lambda$ and its conjugate are equal, then $\lambda$ is a symmetric partition.

Given a box of a Young diagram, the shape formed by the boxes directly to the right of it, the boxes directly below it and the box itself is called the hook of that box. The number of boxes in the hook of a box is called the hook-length of that box. A Young tableau is a Young diagram with the hook-length of each box is written in that box.

Here, we give an example of a Young diagram and a Young tableau with 4 columns, respectively.



| | | | |
|---|---|---|---|
| 8 | 5 | 3 | 2 |
| 7 | 4 | 2 | 1 |
| 4 | 1 | | |
| 2 | | | |
| 1 | | | |

For general concepts and notations about integer partitions, we refer to [1, 8, 12].

Given a partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] \vdash N$, the Young diagram $Y_\lambda$ corresponding to $\lambda$ consists of $n$ columns of boxes with lengths $\lambda_1, \lambda_2, \ldots, \lambda_n$. Clearly, every Young diagram represents a uniquely determined partition. Therefore, we get a bijection $\alpha : \mathbb{P} \to \mathbb{Y}$, $\alpha(\lambda) = Y_\lambda$, where $\mathbb{P}$ denotes the collection of all partitions and $\mathbb{Y}$ denotes the collection of all Young diagrams.

In a Young diagram, the length of a row is at most the number of columns. Let us assume that there are $n$ columns in a Young diagram $Y$ and there are $u_i$ rows of length $i$ for each $i = 1, 2, \ldots, n$. Then we denote such a Young diagram by $Y = 1^{u_1} 2^{u_2} \cdots n^{u_n}$. If $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ is the partition corresponding to $Y$, then

$$\lambda_j = \sum_{i=j}^{n} u_i, \quad 1 \leq j \leq n.$$

Note also that $\lambda_j - \lambda_{j+1} = u_j$, for each $j = 1, \ldots, n-1$ and $\lambda_n = u_n$.

Numerical sets can be represented by Young diagrams. Given a proper numerical set $S$, a uniquely determined Young diagram and thus a uniquely determined partition can be constructed as follows. We use the first quadrant of the Cartesian $xy-$plane for the construction by drawing a continuous polygonal path which starts from the origin. Starting with $x=0$, if $x \in S$, we draw a line segment of unit length to the right. If $x \notin S$, we draw a line segment of unit length up and we repeat for $x+1$. We continue this until $x = F(S)$. Then a path with $n$ horizontal and $g(S)$ vertical segments will be obtained, the lattice lying above this path and below the horizontal line defines a Young diagram, for details see [6, 7, 11, 16, 17].

It is clear that every Young diagram corresponds to a unique proper numerical set. Thus the correspondence $\beta : \mathbb{S} \to \mathbb{Y},\ \beta(S) = Y_S$ is a bijection between the collection $\mathbb{S}$ of proper numerical sets and the collection $\mathbb{Y}$ of Young diagrams. Let us note that the composition $\alpha^{-1}\beta$ is a bijection from the set $\mathbb{S}$ of proper numerical sets to the set $\mathbb{P}$ of partitions of positive integers: $\alpha^{-1}\beta : \mathbb{S} \to \mathbb{P},\ \alpha^{-1}\beta(S) = \alpha^{-1}(Y_S)$. If $Y_S = 1^{u_1}2^{u_2}\ldots n^{u_n}$, then $\{u_1, u_2, \ldots, u_n\}$ is called the Young sequence of $S$.

Let $S = \{s_0 = 0, s_1, \ldots, s_{n-1}, s_n = C(S), \rightarrow\}$ be a proper numerical set. The construction of $Y_S$ implies that the number of columns of $Y_S$ is $n$ and the number of rows is $g(S)$. We denote the columns of $Y_S$ by $G_0(S)$, $G_1(S), \ldots, G_{n-1}(S)$. It is clear that for each $j = 0, 1, \ldots, n-1$, the $j$-th column $G_j(S)$ corresponds to $s_j$ and the length of $G_j(S)$ is $g(S) - s_j + j$. We identify each column with the set of hook-lengths of boxes in it. The $i$-th row of $Y_S$ from the bottom corresponds to the $i$-th gap of $S$; the hook-length of the box of that row in the first column is the $i$-th gap of $S$. Thus $G_0(S)$ consists of the gaps of $S$, that is, $G_0(S) = G(S)$. Additionally, $G_j(S) = G(-s_j + S_j)$ for each $j = 0, 1, \ldots, n-1$, and $G(S(i)) = \bigcup_{j=i}^{n} G_j(S)$ for each $i = 1, \ldots, n-1$. Moreover, $S$ is a numerical semigroup if and only if $G_j(S) \subseteq G(S)$, for each $j = 1, \ldots, n-1$ (for a detailed proof we refer [17]).

An Arf semigroup $S$ is a numerical set which satisfies

$$x, y, z \in S,\ x \geq y \geq z \implies x + y - z \in S.$$

This condition, given in [2], is known as the Arf condition. $\mathbb{N}_0$ is an Arf numerical semigroup. Since every numerical set contains $0$, the Arf condition implies that every Arf semigroup is a numerical semigroup. It is not difficult to see that a numerical set $S = \{s_0 = 0, s_1, \ldots, s_{n-1}, s_n = C(S), \rightarrow\}$ satisfies the Arf condition if and only if the small elements of $S$ satisfy it. In other words, $S$ is an Arf numerical semigroup if and only if $s_i + s_j - s_k \in S$ for all $1 \leq k \leq j \leq i \leq n-1$. For details, we refer to [2, 3, 10, 15, 18]. We note that Theorem 8 and Proposition 17 in [17] state Arf conditions depending on gap numbers and partitions.

## 2. Almost symmetric Arf semigroups and ASA-partitions

**Proposition 2.1** *Let PF(S) denote the set of pseudo-Frobenius numbers of a numerical semigroup $S$ with n(S)=n. Let $N(S)$ be the set of first type gaps of $S$ and $L(S) = G(S) \setminus N(S)$. Then the following statements hold:*

1. $PF(S) = G(S) \setminus \bigcup_{j=1}^{n-1} G_j(S)$, *where* $G_j(S) = G(-s_j + S_j)$.
2. $L(S) = G(S) \setminus \bigcup_{j=0}^{n-1} \{F(S) - s_j\}$, *where* $s_j$ *is the* $j$*-th element of* $S$.
3. $|N(S)| = n,\ |L(S)| = g(S) - n.$

**Proof**

1. The proof follows from Lemma 5(ii) in [17], $PF(S) = T_1(S) = G(S) \setminus \bigcup_{j=1}^{n-1} G_j(S)$.

2. For any $s_j < C(S)$, we have $F(S)-s_j \notin S$. Otherwise, $F(S)-s_j = s_k$ for some $k$, and $F(S) = s_j+s_k \in S$, but this is impossible. Then $N(S) = \{F(S) - s_j : 0 \leq j \leq n-1\}$ and we get the set of remaining gap numbers as follows:

$$L(S) = \{x \in G(S) : x \notin N(S)\} = G(S) \setminus \bigcup_{j=0}^{n-1} \{F(S) - s_j\}.$$

3. The proof follows from (i) and (ii).

□

Recall that $Ap(S,m) = \{s \in S : s - m \notin S\}$ is the Apéry set of $m$ in $S$, where $m(S) = m$. It is well known that the set $(Ap(S,m) \setminus \{0\}) \cup \{m\}$ contains the minimal generators of $S$.

**Proposition 2.2** *Let $S$ be a numerical semigroup and $m$ be the multiplicity of $S$. Let $G_i(S)$ be the $i$-th column of the Young diagram $Y_S$, $i = 0, \ldots, n-1$. For the Apéry set $Ap(S,m)$, the following statements hold:*

1. $Ap(S,m) \setminus \{0\} = S \bigcap \{b + m : b \in G(S)\} = (m + G(S)) \setminus \bigcup_{i=0}^{n-1} G_i(S)$.

2. *$Ap(S,m) \setminus \{0\}$ is the first type set of the numerical set $(m + S) \bigcup \{0\}$.*

**Proof** Let $S$ be a numerical semigroup with multiplicity $m(S) = m$.

1. For any $s \in Ap(S,m)$, there exists $b = s - m \in G(S)$. By Lemma 4(iii) in [17], $\bigcup_{i=0}^{n-1} G_i = G(S)$ and

$$Ap(S,m) \setminus \{0\} = S \cap \{b + m : b \in G(S)\} = (\mathbb{N} \setminus G(S)) \cap \{b + m : b \in G(S)\} = \{b + m : b \in G(S)\} \setminus G(S).$$

2. Let $K = (m + S) \bigcup \{0\}$. Then, we have

$$\begin{aligned} G_i(K) &= G_{i-1}(S), \quad \text{for } i = 1, \ldots, n, \\ G(K) &= \{1, 2, \ldots, m-1\} \bigcup \{m + G(S)\}, \end{aligned}$$

and $\bigcup_{i=0}^{n-1} G_i(S) = G(S)$. By Lemma 5(ii) in [17], the first type set of K can be calculated as follows:

$$\begin{aligned} T_1(K) = K(1) \setminus K(0) = G(K) \setminus \bigcup_{i=1}^{n} G_i(K) &= G(K) \setminus G(S) \\ &= (\{1, 2, \ldots, m-1\} \bigcup \{m + G(S)\}) \setminus G(S) \\ &= S \bigcap (\{1, 2, \ldots, m-1\} \bigcup \{m + G(S)\}) \\ &= S \bigcap \{m + G(S)\}. \end{aligned}$$

Since $\{1, 2, \ldots, m-1\} \subseteq G(S)$, we have that $T_1(K) = Ap(S,m) \setminus \{0\}$.

□

**Definition 2.3** *If a numerical semigroup $S$ is both AS-semigroup and Arf, then $S$ is called an ASA-semigroup.*

Notions of AS-semigroup and Arf semigroup have been studied extensively. Symmetric and pseudo-symmetric semigroups are $AS$-semigroups. For example, $\mathbb{N}_0$ and $S = \{0, 4, 6, 8, 10, 12, \longrightarrow\}$ are ASA-semigroups.

Now, recall bijections $\beta : \mathbb{S} \to \mathbb{Y}$, $\beta(S) = Y_S$ and $\alpha : \mathbb{P} \to \mathbb{Y}$, $\alpha(Y) = \lambda$. These bijections allow us to define notions of $AS$-partitions and ASA-partitions.

**Definition 2.4** *Let* $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]$ *is a partition.*

1. *If* $\lambda = \alpha^{-1}\beta(S)$, *for some* $AS$*-semigroup* $S$, *then* $\lambda$ *is called an* $AS$*-partition.*
2. *If* $\lambda = \alpha^{-1}\beta(S)$, *for some ASA-semigroup* $S$, *then* $\lambda$ *is called an ASA-partition.*

**Theorem 2.5** *Any ASA-partition* $\lambda$ *is of the form either* $\lambda = [j]$, *or* $\lambda = [j+k, j, j-1, \dots, 2, 1]$, *where* $k \in \{1, 3, 5, \dots, 2j-3, 2j-1, \rightarrow\}$, *for some* $j \geq 1$.

**Proof** $\lambda$ is an ASA-partition if and only if there exists an ASA-semigroup $S$ such that $\lambda = \alpha^{-1}\beta(S)$. By Lemma 2(ii) in [17] and Proposition 2.1, we have $G_i(S) \subset G(S)$, for $i = 1, \dots, n-1$, and $PF(S) = G(S) \setminus \bigcup_{i=1}^{n-1} G_i(S)$. Recall that $N(S) = \{F(S) - s_i : i = 0, \dots, n-1\}$ and

$$S \text{ is an } AS-\text{semigroup} \Longleftrightarrow PF(S) = (G(S) \setminus N(S)) \bigcup \{F(S)\}.$$

If $N(S) = \{F(S)\}$, then $PF(S) = G(S)$, which means that $\mathrm{l}(\lambda) = 1$, $\lambda = [j]$ for some $j \in \mathbb{N}$. Assume that $\mathrm{l}(\lambda) = j + 1 \geq 2$.

$$\begin{aligned} S \text{ is an AS-semigroup } \Longleftrightarrow \quad & PF(S) = G(S) \setminus \{F(S) - s_i : 1 \leq i \leq j\} = G(S) \setminus \bigcup_{i=1}^{j} G_i(S) \\ \Longleftrightarrow \quad & \bigcup_{i=1}^{j} G_i(S) = \{F(S) - s_i : 1 \leq i \leq j\}. \end{aligned}$$

On the other hand, by Theorem 8(v) in [17],

$$S \text{ is an Arf semigroup} \Longleftrightarrow G_i(S) \subset G_{i-1}(S), \quad i = 1, \dots, j+1 \Longleftrightarrow \bigcup_{i=1}^{j} G_i(S) = G_1(S),$$

and we obtain that $\lambda_2 = |\{F(S) - s_i : 1 \leq i \leq j\}| = |G_1(S)| = j$. Since $S$ is Arf, $\lambda$ is strict dominant and $[\lambda_2, \lambda_3, \dots, \lambda_{j+1}]$ is a staircase partition with length $j$. Then $\lambda = [j+k, j, \dots, 1]$, for some $k \geq 1$. Using Proposition 17 in [17] which gives an Arf condition for partitions, we get either $j + k = 2j - (j - 1 - i) + i$, for $i = 0, \dots, j-2$ or $j + k = 2j + b$, for $b \geq j - 1$. □

The only pseudo-symmetric Arf semigroups are $< 3, 4, 5 >$ and $< 3, 5, 7 >$. Hence, $[2]$ and $[3, 1]$ are the only partitions corresponding to pseudo-symmetric Arf semigroups.

Recall that for a given numerical semigroup $S$, the *ratio* is the smallest integer in $S$ that is not a multiple of its multiplicity $m(S)$, we will use $r(S)$ to denote the ratio of $S$.

**Corollary 2.6** *Let $S$ be a proper ASA-semigroup. If $S$ is of the form $\{0, k+1, k+3, \ldots, 2j+k+1, \rightarrow\}$, for some $j \geq 1$ and $k \in \{1, 3, 5, \ldots, 2j-1, \rightarrow\}$, then the following statements hold:*

$$\begin{array}{llll} m(S) = k+1, & F(S) = 2j+k, & t_1 = k, & \\ C(S) = k+2j+1, & g(S) = j+k, & r(S) = \begin{cases} 2j+3, & \text{if } k=1, \\ k+3, & \text{otherwise.} \end{cases} \end{array}$$

*If $S = \{0, k+1, \rightarrow\}$, for some $k \geq 1$, then $m(S) = C(S) = k+1$, $g(S) = F(S) = t_1 = k$ and $r(S) = k+2$.*

**Proof** By Theorem 2.5, if $\lambda = [j+k, j, \ldots, 1]$, then $\beta^{-1}\alpha(\lambda) = S$ and

$$s_1 = j+k-j+1 = k+1, \; s_2 = k+1+(j-(j-1))+1 = k+3, \; \ldots, \; s_{j+1} = k+2j+1.$$

Therefore, $S = \{0, k+1, k+3, \ldots, k+2j+1, \rightarrow\}$. Here, $m(S) = k+1$, $g(S) = k+j$, $F(S) = 2j+k$ and $C(S) = 2j+k+1$. Since $S$ is an ASA-semigroup, we obtain

$$g(S) = \frac{F(S)+t_1}{2} \Longrightarrow k+j = \frac{k+2j+t_1}{2} \Longrightarrow t_1 = k.$$

For $k \neq 1$, we have $(k+1) \nmid (k+3)$ and $r(S) = k+3$. On the other hand, If $k = 1$, then we get $S = \{0, 2, 4, \ldots, 2j+2, \rightarrow\}$ and $r(S) = 2j+3$. If $\lambda = [k]$, $k \geq 1$, then we have $S = \{0, k+1, \rightarrow\}$ and the other assertions follow from definitions. □

**Example 2.7** *Let $\lambda = [7, 4, 3, 2, 1]$. Then the corresponding Young diagram is*

| | | | | |
|---|---|---|---|---|
| 11 | 7 | 5 | 3 | 1 |
| 9 | 5 | 3 | 1 | |
| 7 | 3 | 1 | | |
| 5 | 1 | | | |
| 3 | | | | |
| 2 | | | | |
| 1 | | | | |

*It is obvious that $\lambda$ is an ASA-partition and $k = 3$, $j = 4$. Then the corresponding ASA-semigroup is $S = \{0, 4, 6, 8, 10, 12, \rightarrow\}$ and $G(S) = \{1, 2, 3, 5, 7, 9, 11\}$. Here, $m(S) = 4$, $F(S) = 11$, $C(S) = 12$, $g(S) = 7$, $r(S) = 6$. Moreover, $PF(S) = \{2, 9, 11\}$ and $t_1 = 3$.*

**Corollary 2.8** *Intersection of two ASA-semigroups is again an ASA-semigroup.*

**Proof** Let $S_1$, $S_2$ be ASA-semigroups. Firstly, we observe that $G(S_1 \bigcap S_2) = G(S_1) \bigcup G(S_2)$. Define

$$k_0 = \min\left\{s : 0 \neq s \in S_1 \bigcap S_2\right\}.$$

By Corollary 2.6, it is possible to distinguish the following cases:

*(i)* Let $S_i = \{0, k_i+1, \ldots, k_i+2j_i+1, \rightarrow\}$ for some $k_i \in \{1, 3, \ldots, 2j_i-1, \rightarrow\}$ and $j_i > 0$, $i = 1, 2$. Clearly, $S_1 \bigcap S_2 \neq \emptyset$. For all $l < k_0$, $l \in G(S_1) \bigcup G(S_2)$ and $m(S_1 \bigcap S_2) = k_0$. Since $C(S_i) = k_i + 2j_i + 1$, $i = 1, 2$, we have the following cases:

1) If $k_0 = C(S_1) = C(S_2)$, then $S_1 \bigcap S_2 = \{0, k_0 = C(S_1) = C(S_2), \rightarrow\}$.

2) Otherwise,

$$S_1 \bigcap S_2 = \{0, k_0, k_0+2, \ldots, \max\{C(S_1), C(S_2)\}, \rightarrow\}.$$

*(ii)* If $S_i = \{0, k_i+1, \rightarrow\}$, $k_i \geq 0$, for $i = 1, 2$, then $S_1 \cap S_2 = \{0, \max\{k_1+1, k_2+1\}, \rightarrow\}$.

*(iii)* Let $S_1 = \{0, k_1+1, \ldots, k_1+2j_1+1, \rightarrow\}$, for some $k_1 \in \{1, 3, \ldots, 2j_1-1, \rightarrow\}$, $j_1 > 0$ and $S_2 = \{0, k_2+1, \rightarrow\}$, then

$$S_1 \cap S_2 = \begin{cases} S_1, & \text{if } C(S_2) \leq k_1+1, \\ \{0, k_0, k_0+2, \ldots, C(S_1), \rightarrow\}, & \text{if } k_1+1 < C(S_2) < C(S_1), \\ S_2, & \text{if } C(S_2) \geq C(S_1). \end{cases}$$

For each cases *(i)*-*(iii)*, $S_1 \cap S_2$ is an ASA-semigroup. □

**Corollary 2.9** *Let $S$ be an ASA-semigroup with $n(S) = j+1$, $j \in \mathbb{N}_0$. Then the following statements hold:*

1. *$S(i)$ is also an ASA-semigroup, for each $i = 1, 2, \ldots, j+1$.*
2. *$\alpha^{-1}\beta(S(i)) = [j-i+1, j-i, \ldots, 1]$ is an ASA-partition, for each $i = 1, 2, \ldots, j$.*

**Proof** If $j = 0$, then the corresponding $ASA$-semigroup is $S = \{0, k+1, \rightarrow\}$, for some $k \geq 1$. In this case, we have $\alpha^{-1}\beta(S) = [k]$ and $S(1) = \mathbb{N}_0$. Assertions (i)-(ii) clearly follow from definitions.

If $j > 0$, then by Theorem 2.5 and Corollary 2.6, for some $k \in \{1, 3, \ldots, 2j-1, \rightarrow\}$, we have

$$S = \{0, k+1, k+3, k+5, \ldots, k+2j+1, \rightarrow\}$$

and $\lambda = \alpha^{-1}\beta(S) = [j+k, j, j-1, \ldots, 1]$. Then $S(1) = \{0, 2, 4, \ldots, 2j, \rightarrow\}$ and $\alpha^{-1}\beta(S(1)) = [j, j-1, \ldots, 1]$ is a symmetric ASA-partition. Since S is an Arf semigroup, we get

$$|G(S(i)| = |G_i(S)| = |G(-s_i + S_i)| = g(S) - s_i + 1 = j - i + 1,$$

for $i = 2, \ldots, j$, and $S(j+1) = \mathbb{N}_0$. If we continue iteratively, then we see that $S(i)$ is a symmetric Arf semigroup and $\alpha^{-1}\beta(S(i)) = [j-i+1, j-i, \ldots, 1]$ is an $ASA$-partition by Theorem 2.5. □

**Corollary 2.10** *Let $S$ be an ASA-semigroup and $t_i$ denote the $i$-th term of the type sequence of $S$. If $\lambda = \alpha^{-1}\beta(S) = [j+k, j, j-1, \ldots, 2, 1]$ and $k \in \{1, 3, 5, \ldots, 2j-1, \rightarrow\}$, then $t_1 = k$ and $t_i = 1$, for any $i = 2, \ldots, j+1$.*

**Proof** Since $S$ is Arf, the type sequence of $S$ is identical with Young sequence of $S$ by [17]. Thus, $t_1 = j+k-j = k$, $t_i = \lambda_i - \lambda_{i+1} = j-i+1-(j-i) = 1$, for $i = 2, \ldots, j$ and $t_{j+1} = \lambda_{j+1} = 1$. □

The Apéry set of $m(S) = m$ in $S$ is equivalent to

$$Ap(S, m) = \{0 = w(0), w(1), ..., w(m-1)\},$$

where $w(i)$ is the least element of $S$ congruent with $i$ modulo $m$, for all $i = 1, \ldots, m-1$. For any $i = 1, \ldots, m-1$, there exists uniquely determined $k_i \in \mathbb{N}$ such that $w(i) = k_i m + i$. The positive integer $k_i$ is called the $i$-th Kunz coordinate of $S$ and $K = (k_1, \ldots, k_{m-1})$ is called Kunz vector of $S$.

**Proposition 2.11** *Let $S$ be an ASA-semigroup with $m(S) = m$ and Kunz vector $K = (k_1, k_2, \ldots, k_{m-1})$. Then the following statements hold:*

*i) If n(S)=1, then $k_i = 1$ for $i = 1, \ldots, m-1$.*

*ii) Let $n(S) > 1$. If $m$ is even, then*

$$C(S) \in \left\{ mb + 2s : 0 \leq s \leq \frac{m-2}{2},\ b \in \mathbb{N} \right\} \quad \text{and} \quad k_i = \begin{cases} 1, & \text{if } 2 \mid i, \\ b, & \text{if } 2 \nmid i,\ 2s+1 \leq i \leq m+2s-1, \\ b+1, & \text{if } 2 \nmid i,\ 1 \leq i \leq 2s-1, \end{cases}$$

*where $i = 1, \ldots, m-1$. If $m$ is odd, then*

$$C(S) \in \left\{ m + 2s : 0 \leq s \leq \frac{m-1}{2} \right\} \quad \text{and} \quad k_i = \begin{cases} 1, & \text{if } 2 \mid i, \\ 1, & \text{if } 2 \nmid i,\ 2s+1 \leq i \leq m+2s-1, \\ 2, & \text{if } 2 \nmid i,\ 1 \leq i \leq 2s-1, \end{cases}$$

*where $i = 1, \ldots, m-1$. In the case of $m$ is odd, there exist finitely many ASA-semigroups with multiplicity $m$ and their number is $\frac{m+1}{2}$.*

**Proof** (i) If $n(S) = 1$, then $S =< m, m+1, \ldots, 2m-1 >= \{0, m, \rightarrow\}$ and $k_i = 1$, for $i = 1, \ldots, m-1$.

(ii) If $S$ is an ASA-semigroup with $m(S) = m$, then we obtain that $S = \{0, m, m+2, \ldots, m+2j, \rightarrow\}$, for some $j \geq 0$. We distinguish the following cases:

1) If $m$ is even, then $C(S) = m + 2j$ is also even. By the division algorithm, there exists $b \in \mathbb{N}$ such that $C(S) = mb + 2s$, where $0 \leq s \leq \frac{m-2}{2}$.

When $w(i) \in Ap(S,m)$ with even remainder $i$, we have $m+2 \leq w(i) \leq 2m-2$ and the Kunz coordinate is $k_i = 1$. In the case of $i$ is odd, $w(i)$ must be greater than the conductor of $S$, and

$$F(S) = C(S) - 1 = mb + 2s - 1 < w(i) \leq m(b+1) + 2s - 1.$$

Thus, if $i$ is odd and $2s+1 \leq i \leq m+2s-1$, then the Kunz coordinate is $k_i = b$. If $i$ is odd and $1 \leq i \leq 2s-1$, then we get $k_i = b+1$.

2) If $m$ is odd, then $C(S) = m + 2j$ is also odd. Take $j = s$. It must be $C(S) = m + 2s < 2m$, since $S$ is a semigroup. Hence, we deduce $0 \leq s \leq \frac{m-1}{2}$.

When $i$ is even, we have $w(i) \leq m + 2s$ or $w(i) \geq m + 2s$. In both cases, $w(i) < 2m$ and $k_i = 1$. Any element of Apéry set with odd remainder is greater than the conductor, and

$$C(S) + 1 = m + 2s + 1 \leq w(i) \leq 2m + 2s - 1.$$

Therefore, if $i$ is odd and $2s+1 \leq i \leq m+2s-1$, then $k_i = 1$. Otherwise, if $1 \leq i \leq 2s-1$ and $i$ is odd, then $k_i = 2$.

Given an odd $m$, there exist finitely many ASA-semigroups with multiplicity $m$, since $0 \leq s \leq \frac{m-1}{2}$, the number of these semigroups is $\frac{m+1}{2}$. □

**Example 2.12** *Let $\lambda = [7, 4, 3, 2, 1]$. Then the corresponding ASA-semigroup is $S = \{0, 4, 6, 8, 10, 12, \rightarrow\}$, and*

$$Ap(S, 4) = \{0 = w(0), w(1) = 13, w(2) = 6, w(3) = 15\}.$$

*Since $C(S) = 3.4$, we get $b = 3$ and $K = (3, 1, 3)$.*

**Theorem 2.13** *Let $n_{ASA}(N)$ denote the number of ASA-partitions of a positive integer N. Let $j \in \mathbb{N}_0$ such that $\frac{(j+1)(j+2)}{2} \leq N < \frac{(j+2)(j+3)}{2}$ and define $t := (N - \frac{(j+1)(j+2)}{2})$. Then the following statement holds:*

$$n_{ASA}(N) = \begin{cases} j+1, & \text{if } (t \geq 2j-2) \vee ((2|t) \wedge ((2 \nmid j) \vee (t \geq j-5))), \\ j-1, & \text{if } (2 \nmid t) \wedge (2 \nmid j) \wedge (t < j-5), \\ j, & \text{otherwise.} \end{cases}$$

**Proof** Given a positive integer $N$, there exists $j \in \mathbb{N}_0$ such that

$$\frac{(j+1)(j+2)}{2} \leq N < \frac{(j+2)(j+3)}{2}.$$

Clearly, $[j+1, j, j-1, \ldots, 2, 1]$ is an $ASA$-partition of $N_0 = \frac{(j+1)(j+2)}{2}$. This partition has the maximum length among all Arf partitions of $N_0$. Let $\lambda$ be an $ASA$-partition of $N$. Then $l(\lambda) \leq j+1$. By Theorem 2.5, $\lambda = [N]$ or $\lambda$ is of the form $\lambda^n := [n-1+k, n-1, \ldots, 1,]$ with $k \in \{1, 3, 5, \ldots, 2n-3, \rightarrow\}$, $2 \leq n \leq j+1$. Let $\lambda_i^n$ denote the $i$-th part of $\lambda^n$. By Proposition 17 in [17], we have

either $\lambda_1^n = 2(n-1) - (n-2-i) + i$, $0 \leq i \leq n-3$ or $\lambda_1^n = 2(n-1) + k_{n-2}$, $k_{n-2} \geq n-2$.

If $\lambda = \lambda^{j+1} = [j+k, j, j-1, \ldots, 2, 1]$ with $k \in \{1, 3, 5, \ldots, 2j-1, \rightarrow\}$, for some $j > 0$, then $\lambda$ can be written as follows:

$$[j+1, j, j-1, \ldots, 2, 1] + [k-1, 0 \ldots, 0],$$

where $t = N - N_0 = k-1$. The second summand $[k-1, 0, \ldots, 0]$ is considered as an extension of the partition $[k-1]$ to length $j+1$.

$n = j+1 \quad \Rightarrow$ either $\lambda_1^{j+1} = j+1+2i$ or $\lambda_1^{j+1} \geq 3j-1$.
$\Rightarrow$ either $t = 2i$, $0 \leq i \leq j-2$, or $t \geq 2j-2$.

Let $\lambda^j = [j+1+t+j,\ j-1, \ldots, 2, 1]$ be a partition of $N$ with length $j$.

$n = j \quad \Rightarrow$ either $\lambda_1^j = 2j+1+t = j+2i$, $0 \leq i \leq j-3$ or $\lambda_1^j \geq 3j-4$.
$\Rightarrow$ either $j+t$ is odd or $t \geq j-5$.

Therefore, ASA-partitions with length both $j$ and $j+1$ are obtained if the following conditions hold:

$$t \geq 2j-2 \text{ or } ((2|t \text{ and } 2 \nmid j) \text{ or } (2|t \text{ and } t \geq j-5)).$$

There are ASA-partitions with length neither $j$ nor $j+1$ if the following conditions hold:

$$t < j-5,\ 2 \nmid t \text{ and } \ 2 \nmid j.$$

Let $\lambda^{j-1} = [j-1+j+j+1+t,\ j-2, \ldots, 2, 1]$ be a partition of $N$ with length $j-1$.

$n = j-1 \quad \Rightarrow$ either $\lambda_1^{j-1} = 3j+t = j-1+2i$, $0 \leq i \leq j-4$ or $\lambda_1^{j-1} \geq 3j-7$.
$\Rightarrow$ either $2j+1+t = 2i$, $0 \leq i \leq j-4$ or $3j+t \geq 3j-7$.
$\Rightarrow$ either $2 \nmid t$ or $t \geq -7$.

Here, $0 \leq 2j+1+t \leq 2j-8$ contradicts with $t \geq 0$. Define

$$t_{j-(z+1)} := (j-z) + (j-(z-1)) + \cdots + j + (j+1) + t,\ 1 \leq z \leq j-2,$$

and consider the partition

$$\left[t_{j-(z+1)}, (j-(z+1)), (j-(z+2)), \ldots, 2, 1\right].$$

By Arf condition, for $k_{j-(z+2)} \geq j-(z+2)$, we have

$$2(j-(z+1))+k_{j-(z+2)} = t_{j-(z+1)} = (z+2)j - \frac{z(z+1)}{2} + 1 + t,$$

and

$$\frac{(z+1)(z-4)}{2} - (z-1)(j+1) - 4 \leq t \Rightarrow \frac{(z-1)}{2}(z-2j-4) - 7 < 0 < t.$$

Hence, we deduce that $N$ has an $ASA$-partition of length $j-z$, since $1 \leq z \leq j-2$. The partition $[N]$ is also an $ASA$-partition with length 1.

Collecting what we obtain, we deduce that

$$n_{ASA}(N) = \begin{cases} j+1, & \text{if } (t \geq 2j-2) \vee ((2|t) \wedge ((2 \nmid j) \vee (t \geq j-5))), \\ j-1, & \text{if } (2 \nmid t) \wedge (2 \nmid j) \wedge (t < j-5), \\ j, & \text{otherwise.} \end{cases}$$

□

**Example 2.14** *For $N=16$ and $N=17$, let us compute $n_{ASA}(16)$ and $n_{ASA}(17)$. Since*

$$\frac{5.6}{2} = 15 < 16 < 17 < \frac{6.7}{2} = 21,$$

*the maximum length of an ASA-partition of 16 ( and 17) is less than or equal to $j+1=5$, where $N_0=15$. Since $t=1 \geq j-5$, $n_{ASA}(16)=4$. Thus the set of ASA-partitions of 16 is $\{[16],[15,1],[13,2,1],[10,3,2,1]\}$. Since $17-N_0=2>j-5$, we have $n_{ASA}(17)=5$. The set of ASA-partitions of 17 is*

$$\{[17],[16,1],[14,2,1],[11,3,2,1],[7,4,3,2,1]\}.$$

**Example 2.15** *Take $N=58$. Then $\frac{10.11}{2} < 58 < \frac{11.12}{2}$ and $n_{ASA}(58)=8$. Here, we list all ASA-partitions of 58:* $[58],[57,1],[55,2,1],[52,3,2,1],[48,4,3,2,1],[43,5,4,3,2,1],[37,6,5,4,3,2,1],[30,7,6,5,4,3,2,1]$.

**Theorem 2.16** *For a given $g \in \mathbb{N}_0$, the number of ASA-semigroups of genus $g$ is $\left\lfloor \frac{2g+3}{3} \right\rfloor$.*

**Proof** Let $n_{ASA}(S,g)$ denote the number of $ASA$-semigroups of genus $g$. Then we obtain

$$\begin{aligned} n_{ASA}(S,g) &= |\{S : S \text{ is an } ASA\text{-semigroup of genus g}\}| \\ &= |\{\lambda : \lambda = [g,j,j-1,\ldots,1] \text{ is an Arf partition or } \lambda = [g]\}| \\ &= 1 + |\{j : \lambda = [g,j,j-1,\ldots,1] \text{ is an Arf partition }\}|. \end{aligned}$$

Define $T = \{j : g = 2j + k_{j-1},\ k_{j-1} \geq j-1\}$ and $K = \{j : g = 2i+j+1,\ 0 \leq i \leq j-2\}$. We have $T \cap K = \emptyset$. In fact,

$$\begin{aligned} z \in T \bigcap K &\Longrightarrow g \geq 3z-1 \text{ and } g = 2i+z+1,\ i+2 \leq z \Longrightarrow g \geq 3z-1 \text{ and } g \leq 3z-3 \\ &\Longrightarrow 3z-1 \leq g \leq 3z-3, \end{aligned}$$

which is a contradiction. Thus, we get

$$\begin{aligned} n_{ASA}(S,g) &= 1+|T|+|K| \\ &= 1+\left|\{j: g\geq 3j-1\}\right|+\left|\left\{k: k \text{ is odd } g-1\geq g-k\geq \lceil\tfrac{g+3}{3}\rceil=\lceil\tfrac{g}{3}\rceil+1\right\}\right| \\ &= 1+\left|\left\{j: 1\leq j\leq \lfloor\tfrac{g+1}{3}\rfloor\right\}\right|+\left|\left\{k: k \text{ is odd } 1\leq k\leq g-\lceil\tfrac{g}{3}\rceil-1\right\}\right|. \end{aligned}$$

Here,

$$|T| = \left\lfloor \frac{g+1}{3} \right\rfloor = \begin{cases} u, & \text{if } g=3u+i,\ u\in\mathbb{N},\ i\in\{0,1\}, \\ u+1, & \text{if } g=3u+2,\ u\in\mathbb{N}, \end{cases}$$

$$g-\left\lceil \frac{g}{3} \right\rceil - 1 = \begin{cases} 2u-1, & \text{if } g=3u+i,\ u\in\mathbb{N},\ i\in\{0,1\}, \\ 2u, & \text{if } g=3u+2,\ u\in\mathbb{N}. \end{cases}$$

Hence, we obtain $|K|=u$ and

$$n_{ASA}(S,g) = \begin{cases} 2u+1, & \text{if } g=3u+i,\ u\in\mathbb{N},\ i\in\{0,1\}, \\ 2u+2, & \text{if } g=3u+2,\ u\in\mathbb{N}, \end{cases}$$

which is equivalent to $n_{ASA}(S,g) = \lfloor \frac{2g+3}{3} \rfloor$. □

**Example 2.17** $n_{ASA}(S,7)=5$. *All* $ASA$*-semigroups of genus* $7$ *can be listed as follows:*

$\{0,8,\rightarrow\},\ \{0,7,9,\rightarrow\},\ \{0,6,8,10,\rightarrow\},\ \{0,4,6,8,10,12\rightarrow\},\ \{0,2,4,6,8,10,12,14\rightarrow\}.$

If the Young diagram of a proper numerical set $S$ consists of one single hook, then $S$ is a hook set. A hook numerical set which is a numerical semigroup is called a hook semigroup. In [11], the authors explained hook set decomposition of a given numerical set $S$, and they proved that $S$ has hook semigroup decomposition when $\alpha^{-1}\beta(S)$ is a strict dominant partition. Recall that a numerical semigroup $S$ is primitive when $F(S) < 2m(S)$. Under Arf condition, the hook numerical sets appearing in the decomposition of a numerical set become primitive numerical semigroups. Therefore, any $ASA$-semigroup has primitive semigroup decomposition.

The number of $ASA$-partitions of a positive integer $N \leq 80$ is tabulated in Table, where $n_d$ is the number of strict dominant partitions of $N$, $n_A$ is the number of Arf partitions of $N$, $n_{AS}$ is the number of $AS$-partitions of $N$ and $n_{ASA}$ is the number of $ASA$-partitions of $N$.

**Table** . The number of $ASA$-partitions of a positive integer $N \leq 80$.

| $N$ | $n_d$ | $n_A$ | $n_{AS}$ | $n_{ASA}$ | $N$ | $n_d$ | $n_A$ | $n_{AS}$ | $n_{ASA}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 41 | 44583 | 100 | 901 | 7 |
| 2 | 2 | 1 | 1 | 1 | 42 | 53174 | 122 | 1092 | 8 |
| 3 | 3 | 2 | 2 | 2 | 43 | 63261 | 117 | 1169 | 7 |
| 4 | 5 | 2 | 2 | 2 | 44 | 75175 | 124 | 1379 | 8 |
| 5 | 7 | 2 | 3 | 2 | 45 | 89134 | 150 | 1521 | 8 |
| 6 | 11 | 4 | 4 | 3 | 46 | 105558 | 146 | 1784 | 8 |
| 7 | 15 | 3 | 4 | 2 | 47 | 124754 | 144 | 1935 | 8 |
| 8 | 22 | 4 | 6 | 3 | 48 | 147273 | 174 | 2280 | 8 |
| 9 | 30 | 6 | 6 | 3 | 49 | 173525 | 171 | 2473 | 9 |
| 10 | 42 | 6 | 10 | 4 | 50 | 204226 | 177 | 2917 | 8 |
| 11 | 56 | 6 | 9 | 3 | 51 | 239943 | 209 | 3181 | 9 |
| 12 | 77 | 10 | 12 | 4 | 52 | 281589 | 210 | 3671 | 8 |
| 13 | 101 | 7 | 13 | 3 | 53 | 329931 | 208 | 4025 | 9 |
| 14 | 135 | 9 | 19 | 4 | 54 | 386155 | 240 | 4675 | 8 |
| 15 | 176 | 14 | 21 | 5 | 55 | 451276 | 239 | 5117 | 10 |
| 16 | 231 | 13 | 26 | 4 | 56 | 526823 | 249 | 5885 | 8 |
| 17 | 297 | 13 | 27 | 5 | 57 | 614154 | 288 | 6465 | 10 |
| 18 | 385 | 18 | 35 | 4 | 58 | 715220 | 287 | 7424 | 8 |
| 19 | 490 | 17 | 38 | 5 | 59 | 831820 | 286 | 8133 | 10 |
| 20 | 627 | 17 | 50 | 4 | 60 | 966467 | 339 | 9385 | 9 |
| 21 | 792 | 26 | 53 | 6 | 61 | 1121505 | 326 | 10240 | 10 |
| 22 | 1002 | 24 | 67 | 5 | 62 | 1300156 | 325 | 11726 | 9 |
| 23 | 1255 | 26 | 69 | 6 | 63 | 1505499 | 391 | 12849 | 10 |
| 24 | 1575 | 30 | 92 | 5 | 64 | 1741630 | 383 | 14626 | 9 |
| 25 | 1958 | 32 | 102 | 6 | 65 | 2012558 | 398 | 16073 | 10 |
| 26 | 2436 | 31 | 122 | 5 | 66 | 2323520 | 448 | 18346 | 10 |
| 27 | 3010 | 42 | 133 | 6 | 67 | 2679689 | 440 | 20083 | 10 |
| 28 | 3718 | 42 | 161 | 6 | 68 | 3087735 | 442 | 22764 | 10 |
| 29 | 4565 | 42 | 171 | 6 | 69 | 3554345 | 510 | 24999 | 10 |
| 30 | 5604 | 53 | 226 | 7 | 70 | 4087968 | 515 | 28366 | 10 |
| 31 | 6842 | 51 | 233 | 6 | 71 | 4697205 | 518 | 31038 | 10 |
| 32 | 8349 | 52 | 286 | 7 | 72 | 5392783 | 593 | 35184 | 11 |
| 33 | 10143 | 67 | 315 | 6 | 73 | 6185689 | 575 | 38540 | 10 |
| 34 | 12310 | 65 | 374 | 7 | 74 | 7089500 | 593 | 43498 | 11 |
| 35 | 14883 | 68 | 412 | 6 | 75 | 8118264 | 668 | 47742 | 10 |
| 36 | 17977 | 80 | 494 | 8 | 76 | 9289091 | 659 | 53736 | 11 |
| 37 | 21637 | 80 | 534 | 6 | 77 | 10619863 | 680 | 58922 | 10 |
| 38 | 26015 | 83 | 634 | 8 | 78 | 12132164 | 764 | 66326 | 12 |
| 39 | 31185 | 101 | 702 | 7 | 79 | 13848650 | 746 | 72712 | 10 |
| 40 | 37338 | 101 | 839 | 8 | 80 | 15796476 | 763 | 81652 | 12 |

**Acknowledgment**

The authors would like to thank the anonymous referees for giving many helpful suggestions. This work was supported by Research Fund of the Akdeniz University. Project No: FBA-2020-5393.

## References

[1] Andrews GE, Eriksson K. Integer Partitions. Cambridge, UK: Cambridge University Press, 2004.

[2] Arf C. Une interprétation algébrique de la suite des ordres de multiplicité d'une branche algébrique. Proceedings of the London Mathematical Society 1948; 20: 256-287 (in French).

[3] Barucci V, Dobbs DE, Fontana M. Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains. Memoirs of the American Mathematical Society 1997; 125 (598): 1-77.

[4] Barucci V, Fröberg R. One-dimensional almost Gorenstein rings. Journal of Algebra 1997; 188 (2): 418-442. doi: 10.1006/jabr.1996.6837

[5] Branco MB, Ojeda I, Rosales JC. Almost symmetric numerical semigroups with given Frobenius number and type. Journal of Algebra and Its Applications 2019; 18 (11): 1950217. doi:10.1142/S0219498819502177

[6] Bras-Amoros M, De Mier A. Representation of numerical semigroups by dyck paths. Semigroup Forum 2007; 75 (3): 676-681. doi: 10.1007/s00233-007-0717-7

[7] Constantin H, Houston-Edwards B, Kaplan N. Numerical ets, core partitions, and integer points in polytopes. In: Proceedings of Combinatorial and Additive Number Theory II; New York, NY, USA; 2017. pp. 99-127. doi: 10.1007/978-3-319-68032-3_7

[8] Fulton W. Young Tableaux: With Applications to Representation Theory and Geometry. Cambridge, UK: Cambridge University Press, 1997.

[9] Garcia-Sanchez PA, Ojeda I. Almost symmetric numerical semigroups with high type. Turkish Journal of Mathematics 2019; 43 (5): 2499-2510. doi: 10.3906/mat-1906-10

[10] Karakaş Hİ. Parametrizing numerical semigroups with multiplicity up to 5. International Journal of Algebra and Computation 2018; 28 (01): 69-95. doi: 10.1142/S0218196718500042

[11] Karakaş Hİ, Tutaş N. A decomposition of partitions and numerical sets. Semigroup Forum 2020. doi: 10.1007/s00233-019-10080-7

[12] Keith WJ, Nath R. Partitions with prescribed hooksets. Journal of Combinatorics and Number Theory 2011; 3: 39-50.

[13] Nari H. Symmetries on almost symmetric numerical semigroups. Semigroup Forum 2013; 86 (1): 140-154. doi: 10.1007/s00233-012-9397-z

[14] Rosales JC, García-Sánchez PA. Numerical Semigroups. New York, NY, USA: Springer, 2009.

[15] Rosales JC, García-Sánchez PA, García-García JI, Branco MB. Arf numerical semigroups. Journal of Algebra 2004; 276 (1): 3-12. doi: 10.1016/j.jalgebra.2004.03.007

[16] Tutaş N. On partitions and Arf semigroups. Open Mathematics 2019; 17 (1): 343-355. doi: 10.1515/math-2019-0025

[17] Tutaş N, Karakaş Hİ, Gümüşbaş N. Young tableaux and Arf partitions. Turkish Journal of Mathematics 2019; 43 (1): 448-459. doi: 10.3906/mat-1807-181

[18] İlhan S, Karakaş Hİ. Arf numerical semigroups. Turkish Journal of Mathematics 2017; 41 (6): 1448-1457. doi: 10.3906/mat-1512-46