Turkish Journal of Mathematics

Volume 44 | Number 6 Article 14

1-1-2020

Entire large positive radial symmetry solutions for combined quasilinear elliptic system

SESHADEV PADHI

SMITA PATI

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 2155 – 2165
© TÜBİTAK
doi:10.3906/mat-1806-14

Entire large positive radial symmetry solutions for combined quasilinear elliptic system

Seshadev PADHI[1,*], **Smita PATI**[2]
[1]Department of Mathematics, Birla Institute of Technology, Mesra, Ranchi, India
[2]Department of Mathematics, Amity University, Ranchi, India

Received: 04.06.2018 • **Accepted/Published Online:** 10.09.2020 • **Final Version:** 16.11.2020

Abstract: We prove the existence of entire large positive solutions to the system

$$\begin{cases} (r^{N-1}\varphi_1(u'))' = r^{N-1}P_1(r)f(u,v),\ 0 \le r < \infty \\ (r^{N-1}\varphi_2(v'))' = r^{N-1}P_2(r)g(u,v),\ 0 \le r < \infty \\ u(0) = a,\ v(0) = b,\ u'(0) = 0,\ v'(0) = b, \end{cases}$$

where the functions $\varphi_i(s) = \alpha_i(s^2)s,\ i = 1,2$ are odd, increasing homeomorphisms, $P_1, P_2 : [0,\infty) \to [0,\infty)$ are continuous, and $f, g : [0,\infty)\times[0,\infty) \to [0,\infty)$ are continuous and increasing functions.

Key words: Elliptic system, positive solutions, radial solutions, large solutions

1. Introduction

In this paper, we study the existence of entire positive radial symmetry solutions for the combined semilinear elliptic system

$$\begin{cases} \operatorname{div}(\alpha_1(|\nabla u|^2)\nabla u) = P_1(|x|)f(u,v) \\ \operatorname{div}(\alpha_2(|\nabla v|^2)\nabla v) = P_2(|x|)g(u,v) \end{cases} \tag{1.1}$$

$x \in \mathbb{R}^N$ and $N \ge 3$. We assume that the functions P_1, P_2, f and g satisfy the following conditions:

(L1) $P_1, P_2 : [0,\infty) \to [0,\infty)$ are spherically symmetric continuous functions, that is, for $r = |x|$, we have $p_1(|x|) = p_1(r)$ and $p_2(|x|) = p_2(r)$;

(L2) $f, g : [0,\infty)\times[0,\infty) \to [0,\infty)$ are continuous and increasing functions with $g(t,s) > 0$ whenever $t, s > 0$;

(L3) For fixed parameters $a, b \in (0,\infty)$, there exist continuous, nondecreasing functions $G_1, G_2, \varphi : [0,\infty) \to [0,\infty)$, and constants C_1 and C_2 such that

$$g(t_1,t_2) \le C_1 G_1(t_1+t_2)$$

*Correspondence: spadhi@bitmesra.ac.in
2010 *AMS Mathematics Subject Classification:* 35J25, 45F10, 35B08, 47N20, 35B09, 35B40.

and

$$f(t_1, MC_1 s G_1(t_1+t_2)) \le C_2 G_2(t_1+t_2)\varphi(s),\ s \ge 1,$$

where $M = max\left\{1, \frac{b}{g(a,b)}\right\}$; and

(L4) The functions $\varphi_i(s) = \alpha_i(s^2)s,\ i = 1,2$ are odd, increasing homeomorphisms on $\mathbb{R}$ and for each $r > 0$, there exists a constant $B_{ir} > 0$ with $B_{ir} \to 0$ as $r \to 0$, and satisfies

$$\varphi_i^{-1}(rz) \le B_{ir}\varphi_i^{-1}(z) \text{ for } z \ge 0 \text{ and } i = 1,2.$$

We say that the functions f and g are increasing if for every pair (u_1, v_1), $(u_2, v_2) \in [0,\infty) \times [0,\infty)$ with $0 \le u_1 \le u_2$ and $0 \le v_1 \le v_2$, we have $f(u_1, v_1) \le f(u_2, v_2)$ and $g(u_1, v_1) \le g(u_2, v_2)$.

In a recent paper [1], Covei obtained sufficient conditions for the existence of bounded and unbounded entire solutions to the semilinear elliptic system

$$\begin{cases} \Delta u = p_1(|x|)f(u,v), x \in \mathbb{R}^N, \\ \Delta v = p_2(|x|)g(u),\ x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

for $N \ge 3$, where the functions p_1, p_2, f and g satisfy the following conditions:

(H1) $p_1, p_2 : [0,\infty) \to [0,\infty)$ are spherically symmetric continuous functions, that is, for $r = |x|$, we have $p_1(|x|) = p_1(r)$ and $p_2(|x|) = p_2(r)$;

(H2) $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ and $g : [0,\infty) \to [0,\infty)$ are continuous increasing functions with $g(s) > 0$ whenever $s > 0$; and

(H3) for fixed parameters $a, b \in (0,\infty)$, there exist continuous functions $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ and $\varphi : [0,\infty) \to [0,\infty)$ such that h is nondecreasing and there exists $C \in (0,\infty)$ such that

$$f(t_1, t_2.s_2) \le Ch(t_1, t_2).\varphi(s_2),\ \forall t_2 \ge M.g(a),\ s_2 \ge 1,$$

where $M = b/g(a)$ if $b > g(a)$ or $M = 1$ if $b \le g(a)$.

Many authors have studied positive entire solutions for equations of the form (1.2) with various nonlinearities. One may see the works in [5–10, 12–14] and the references cited therein.

Equations of the form

$$\begin{cases} \operatorname{div}(\varphi_1(|\nabla u|)\nabla u) + a_1(|x|)\varphi_1(|\nabla u|)\nabla u = b_1(|x|)f_1(u,v), x \in \mathbb{R}^N, \\ \operatorname{div}(\varphi_2(|\nabla v|)\nabla v) + a_2(|x|)\varphi_2(|\nabla v|)\nabla v = b_2(|x|)f_2(u,v), x \in \mathbb{R}^N,\ N \ge 3, \end{cases} \tag{1.3}$$

has been studied by Zhou [16]. The conditions assumed by Zhou [16] are $a_i, b_i : \mathbb{R}^N \to [0,\infty)$ are continuous, $f_i : [0,\infty) \times [0,\infty) \to [0,\infty), i = 1,2$ are continuous and increasing. In another work, Zhang [15] studied the existence of a positive radial solution to the quasilinear elliptic system

$$\begin{cases} \operatorname{div}(\varphi_1(|\nabla u|)\nabla u) = a(|x|)f(v), x \in \mathbb{R}^N, \\ \operatorname{div}(\varphi_2(|\nabla v|)\nabla v) = b(|x|)g(u), x \in \mathbb{R}^N,\ N \ge 3, \end{cases} \tag{1.4}$$

where $a, b : \mathbb{R}^N \to [0, \infty)$ are continuous, $f, g : [0, \infty) \to [0, \infty)$ are continuous and increasing. The case where the function $\varphi_i(t)$, $i = 1, 2$ in (1.3) and (1.4) satisfies the inequality

$$(t\varphi_i(t))' > 0 \quad \text{for} \quad \text{all } t > 0 \text{ and } i = 1, 2. \tag{1.5}$$

has been analysed respectively by the authors [16] and [15].

In a recent paper, Hai and Shivaji [4] obtained sufficient conditions for the existence of a positive radial solution for the combined semilinear system of elliptic equations

$$\begin{cases} -\text{div}(\psi_1(\nabla u)) = \lambda_1 f_1(u), \; a < |x| < b \\ -\text{div}(\psi_2(\nabla v)) = \lambda_2 f_2(v), \; a < |x| < b, \end{cases} \tag{1.6}$$

in the annular region $u = v = 0, |x| \in \{a, b\}$, where $a > 0, \lambda$ is a small positive parameter, $f_i : (0, \infty) \to \mathbb{R}$ is continuous, and the functions $\psi_i(s)$ satisfy the condition

(H4) $\psi_i(s)$ $i = 1, 2$ are odd, increasing homeomorphisms on $\mathbb{R}$ and for each $c > 0$, there exists a constant $A_c > 0$ with $A_c \to \infty$ as $c \to \infty$ such that

$$\psi_i^{-1}(cz) \geq A_c \psi_i^{-1}(z) \text{ for } z \geq 0 \text{ and } i = 1, 2.$$

The motivation of this work has arisen from (1.6) and the condition (**H4**). Unlike the use of (1.5) in (1.3) and (1.4), in this paper, we shall use a condition similar to (**H4**) [see condition (**L4**) given above].

The main result of this paper is Theorem 2.1. The conditions used in our theorem is simple and easy to verify. The conditions we require in our result is the integrability of the coefficient functions and the use of iteration process on the sequence of solutions. We have used the homeomorphism property of the function $\varphi_i(s), i = 1, 2$, which has not been used by any of the authors while dealing with the positive radial solutions of systems of the type (1.1). Our Corollary 2.2 provide simple intrgrable conditions to obtain results on the asymptotic behaviour of solutions. In the process, we provide an example (see Example 2.3), where Theorem 2.1 and Corollary 2.2 are applicable.

We note that solution (u, v) of the system (1.1) is an entire large solution if it is a classical solution of (1.1) and satisfies the property

$$\lim_{|r|\to\infty} u(r) = \infty \text{ and } \lim_{|r|\to\infty} v(r) = \infty. \tag{1.7}$$

If the property (1.7) holds for a positive solution (u, v) of (1.1), then we say that (u, v) is an entire large positive solution of the system (1.1).

Remark 1.1 *Assumptions of the form* (**L4**) *can be found in [4]. We note that* (**L4**) *is implied by* (**H4**)*.*

The present work is split into three sections. Section 1 is introduction. Section 2 presents the notations and the statements pertaining to our results. Finally, Section 3 presents the proof of our main theorem.

2. Statement of the main results and examples

We shall use the following notations throughout the paper.

$$F_{1r} = \int_0^r \varphi_1^{-1}\left(t^{1-N}\int_0^t \theta^{N-1}P_1(\theta)\,d\theta\right)dt$$

$$F_{2r} = \int_0^s \varphi_2^{-1}\left(t^{1-N}\int_0^t \theta^{N-1}P_2(\theta)d\theta\right)dt$$

$$B_{1r} = \int_0^r P_1(s)\varphi\left(1+\int_0^s \varphi_2^{-1}\left(t^{1-N}\int_0^t \theta^{N-1}P_2(\theta)d\theta\right)dt\right)ds$$

$$B_{2r} = \int_0^r P_2(s)\,ds$$

$$P(r) = \int_0^r B_{2\theta}\,d\theta,\ P(\infty) = \lim_{r\to\infty} P(r);$$

$$Q(r) = \int_0^r B_{1\theta}\,d\theta,\ Q(\infty) = \lim_{r\to\infty} Q(r);$$

$$H(r) = \int_{a+b}^r \frac{d\theta}{\varphi_2^{-1}(G_1(\theta)) + \varphi_1^{-1}(G_2(\theta))} \text{ for } r > a+b, \text{ and } H(\infty) = \lim_{r\to\infty} H(r).$$

Our main results of this paper are as follows:

Theorem 2.1 *Assume* $(L1)-(L4)$ *and* $H(\infty)=\infty$. *Then the system (1.1) has a positive radial solution* $(u,v)\in C^2([0,\infty))\times C^2([0,\infty))$.

Corollary 2.2 *The solution* (u,v), *obtained in Theorem 2.1, satisfies the following properties:*

(I) *If*

$$P(\infty)<\infty,\ Q(\infty)<\infty, \tag{2.1}$$

and

$$\int_0^\infty \varphi_2^{-1}\left(G_1\left(H^{-1}\left(C_1P(s)+C_2Q(s)\right)\right)\right)B_{2s}\,ds<\infty, \tag{2.2}$$

then $\lim_{r\to\infty} u(r)<\infty$ *and* $\lim_{r\to\infty} v(r)<\infty$.

(II) *If*

$$\int_0^\infty \varphi_1^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_1(s)f\left(a,b+g(a,b)F_{2s}\right)ds\right)dt=\infty, \tag{2.3}$$

and either

$$\int_0^\infty \varphi_2^{-1}\left(r^{1-N}\int_0^r s^{N-1}P_2(s)\,ds\right)dr=\infty \tag{2.4}$$

or

$$\int_0^\infty \varphi_2^{-1}\left(r^{1-N}\int_0^r s^{N-1}P_2(s)g\left(a+f(a,b)F_{1s},b+g(a,b)F_{2s}\right)ds\right)dr=\infty \tag{2.5}$$

are satisfied, then $\lim_{r\to\infty} u(r)=\infty$ *and* $\lim_{r\to\infty} v(r)=\infty$.

(III) *Let* $P(\infty) < \infty$ *holds. Further, assume that either (2.4) or (2.5) holds. Then* $\lim_{r\to\infty} u(r) < \infty$ *and* $\lim_{r\to\infty} v(r) = \infty$*.*

(IV) *If both (2.2) and (2.3) holds, then* $\lim_{r\to\infty} u(r) = \infty$ *and* $\lim_{r\to\infty} v(r) < \infty$*.*

Below, we give an example of the existence of entire large positive solution for (1.1). Since the existence of an entire large positive solution of the system

$$\begin{cases} (r^{N-1}\varphi_1(u'))' = r^{N-1}P_1(r)f(u,v),\ 0 \le r < \infty \\ (r^{N-1}\varphi_2(v'))' = r^{N-1}P_2(r)g(u,v),\ 0 \le r < \infty \\ u(0) = a,\ v(0) = b,\ u'(0) = 0,\ v'(0) = 0, \end{cases} \tag{2.6}$$

satisfying the property (1.7), is also an entire large positive solution for (1.1), we shall concentrate on the system (2.6).

Example 2.3 *Consider the system of equations*

$$\begin{cases} (r^{N-1}(u')^3)' = r^{N-1}\frac{1}{3r^{2/3}\left[1+\frac{9s^{10/9}}{10(3N-2)^{1/3}}\right]^{1/2}}u^{1/2}v^{1/2},\ 0 \le r < \infty \\ (r^{N-1}(v')^3)' = r^{N-1}\frac{1}{3r^{2/3}}uv,\ 0 \le r < \infty \\ u(0) = 1,\ v(0) = 1,\ u'(0) = 0,\ v'(0) = 0. \end{cases} \tag{2.7}$$

Here $a = 1, b = 1$, $f(u,v) = u^{1/2}v^{1/2}$, $g(u,v) = uv$, $\varphi_1(r) = \varphi_2(r) = r^3$, $P_1(r) = \frac{1}{3r^{2/3}\left[1+\frac{9s^{10/9}}{10(3N-2)^{1/3}}\right]^{1/2}}$ *and* $P_2(r) = \frac{1}{3r^{2/3}}$. *For the above choice of* f, g, P_1 *and* P_2, *we observe that* (**L1**) *and* (**L2**) *are satisfied. Now, for* $a = b = 1$, *we have* $M = max\left\{1, \frac{b}{g(a,b)}\right\} = 1$. *Since* $g(t_1,t_2) = t_1t_2 < (t_1+t_2)^2$. *Hence, assuming* $C_1 = 1$ *and* $G_1(r) = r^2$, *we have*

$$g(t_1,t_2) \le C_1G_1(t_1+t_2).$$

Now,

$$f\left(t_1, MC_1sG_1(t_1+t_2)\right) = t_1^{1/2}s^{1/2}(t_1+t_2) \le s^{1/2}(t_1+t_2)^{3/2}$$

implies that we can set $C_2 = 1, \varphi(s) = s^{1/2}$ *and* $G_2(r) = r^{3/2}$. *This shows that* (**L3**) *is satisfied. Also, we have*

$$H(r) = \int_2^r \frac{d\theta}{\theta^{2/3}+\theta^{1/2}} = \frac{r^{1/3}-2r^{-2/3}}{1+r^{-1/6}} \to \infty \text{ as } r \to \infty.$$

Now $B_{2r} = r^{1/3} \to 0$ *as* $r \to 0$ *and for any* $r > 0$,

$$\varphi_2^{-1}(rz) = (rz)^{1/3} = r^{1/3}z^{1/3} = B_{2r}\varphi_2^{-1}(z)$$

proves that (**L4**) *is satisfied for* $i = 2$. *Now, we shall verify* (**L4**) *for* $i = 1$. *Clearly,*

$$\varphi\left(1+F_{2s}\right) := \varphi\left(1+\int_0^s \varphi_2^{-1}\left(t^{1-N}\int_0^t \theta^{N-1}P_2(\theta)d\theta\right)dt\right) = \left[1+\frac{9s^{10/9}}{10(3N-2)^{1/3}}\right]^{1/2}$$

yields $B_{1r} = r^{1/3}$, *which tends to zero as* $t \to 0$. *Thus,* (**L4**) *is satisfied for* $i = 1$. *Hence, by Theorem 2.1, (2.7) has a positive radial solution* (u, v).

The following calculations show that this positive radial solution (u, v) *is an entire positive radial solution. Indeed, the condition (2.4) follows from*

$$F_{2s} := \int_0^s \varphi_2^{-1}\left(t^{1-N}\int_0^t \theta^{N-1}P_2(\theta)d\theta\right)dt = \frac{9s^{10/9}}{10(3N-2)^{1/3}},$$

and the condition (2.3) follows from

$$\int_0^n \varphi_1^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_1(s)f\left(a, b + g(a,b)F_{2s}\right)ds\right)dt = \frac{3^{5/3}n^{10/9}}{10(N-2/3)^{1/3}}.$$

Hence, by the condition ***(II)*** *of Corollary 2.2, the solution* (u, v) *is an entire positive radial solution.*

3. Proof of the main results

Proof of Theorem 2.1: Clearly, a positive radial solution of the system of ODE (2.6) is also a positive radial solution of (1.1). Let us define the sequences $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ on $[0, \infty)$, iteratively, by

$$\begin{cases} u(0) = u_0 = a, \ v(0) = v_0 = b, \\ u_n(r) = a + \int_0^r \varphi_1^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_1(s)f\left(u_{n-1}(s), v_{n-1}(s)\right)ds\right)dt, \ r \geq 0, \\ v_n(r) = b + \int_0^r \varphi_2^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_2(s)g\left(u_n(s), v_{n-1}(s)\right)ds\right)dt, \ r \geq 0. \end{cases} \tag{3.1}$$

First, we prove that the sequences $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ are nondecreasing on $[0, \infty)$. For this, we consider

$$\begin{aligned} u_1(r) &= a + \int_0^r \varphi_1^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_1(s)f\left(u_0(s), v_0(s)\right)ds\right)dt \\ &= a + \int_0^r \varphi_1^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_1(s)f(a,b)ds\right)dt. \end{aligned}$$

Since $u_1(r) \geq a$ and $v_1(r) \geq b$ for $n = 1$, then

$$\begin{aligned} u_1(r) \leq& a + \int_0^r \varphi_1^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_1(s)f\left(u_1(s), v_1(s)\right)ds\right)dt \\ &= u_2(r) \end{aligned}$$

implies that

$$\begin{aligned} v_1(r) &= b + \int_0^r \varphi_2^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_2(s)g\left(u_1(s), v_0(s)\right)ds\right)dt \\ &= b + \int_0^r \varphi_2^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_2(s)g\left(u_1(s), b\right)ds\right)dt \\ &\leq b + \int_0^r \varphi_2^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_2(s)g\left(u_2(s), v_1(s)\right)ds\right)dt \end{aligned}$$

$$= v_2(r).$$

Similarly,

$$\begin{aligned} u_2(r) &= a + \int_0^r \varphi_1^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_1(s)f\left(u_1(s), v_1(s)\right)ds\right)dt \\ &\le a + \int_0^r \varphi_1^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_1(s)f\left(u_2(s), v_2(s)\right)ds\right)dt \\ &= u_3(r) \end{aligned}$$

implies

$$\begin{aligned} v_2(r) &= b + \int_0^r \varphi_2^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_2(s)g\left(u_2(s), v_1(s)\right)ds\right)dt \\ &\le b + \int_0^r \varphi_2^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_2(s)g\left(u_3(s), v_2(s)\right)ds\right)dt \\ &= v_3(r). \end{aligned}$$

Hence, by induction, the sequences $\{u_n\}_{n\ge0}$ and $\{v_n\}_{n\ge0}$ are nondecreasing on $[0,\infty)$.

Now we prove that the nondecreasing sequences $\{u_n\}_{n\ge0}$ and $\{v_n\}_{n\ge0}$ are bounded above on any bounded interval. By the monotonicity of $\{u_n\}_{n\ge0}$, $\{v_n\}_{n\ge0}$, f, g, φ_1^{-1} and φ_2^{-1}, we have

$$\begin{aligned} (r^{N-1}\varphi_1(u_n'(r)))' &= r^{N-1}P_1(r)f(u_{n-1}(r), v_{n-1}(r)) \\ &\le r^{N-1}P_1(r)f(u_n(r), v_n(r)) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} (r^{N-1}\varphi_2(v_n'(r)))' &= r^{N-1}P_2(r)g(u_n(r), v_{n-1}(r)) \\ &\le r^{N-1}P_2(r)g(u_n(r), v_n(r)). \end{aligned} \tag{3.3}$$

Integrating (3.2) from 0 to r, we obtain

$$\begin{aligned} u_n'(r) &\le \varphi_1^{-1}\left(r^{1-N}\int_0^r s^{N-1}P_1(s)f\left(u_n(s), v_n(s)\right)ds\right) \\ &= \varphi_1^{-1}\left(r^{1-N}\int_0^r s^{N-1}P_1(s)f\left(u_n(s), b + \int_0^s \varphi_2^{-1}\left(t^{1-N}\int_0^t \theta^{N-1}P_2(\theta)g\left(u_n(\theta), v_{n-1}(\theta)\right)d\theta\right)dt\right)ds\right) \\ &\le \varphi_1^{-1}\left(r^{1-N}\int_0^r s^{N-1}P_1(s)f\left(u_n(s), b + \int_0^s \varphi_2^{-1}\left(t^{1-N}\int_0^t \theta^{N-1}P_2(\theta)g\left(u_n(\theta), v_n(\theta)\right)d\theta\right)dt\right)ds\right) \\ &\le \varphi_1^{-1}\left(r^{1-N}\int_0^r s^{N-1}P_1(s)f\left(u_n(s), b + \int_0^s \varphi_2^{-1}\left(t^{1-N}g\left(u_n(t), v_n(t)\right)\int_0^t \theta^{N-1}P_2(\theta)d\theta\right)dt\right)ds\right) \\ &\le \varphi_1^{-1}\left(r^{1-N}\int_0^r s^{N-1}P_1(s)f\left(u_n(s), b + g\left(u_n(s), v_n(s)\right)F_{2s}\right)ds\right) \end{aligned}$$

$$\leq \varphi_1^{-1}\Bigg(r^{1-N}\int_0^r s^{N-1}P_1(s)f\Bigg(u_n(s), g\left(u_n(s),v_n(s)\right)\left(\frac{b}{g\left(u_n(s),v_n(s)\right)}+F_{2s}\right)\Bigg)ds\Bigg)$$
$$\leq \varphi_1^{-1}\Bigg(r^{1-N}\int_0^r s^{N-1}P_1(s)f\Bigg(u_n(s), g\left(u_n(s),v_n(s)\right)\left(\frac{b}{g(a,b)}+F_{2s}\right)\Bigg)ds\Bigg)$$
$$\leq \varphi_1^{-1}\Bigg(r^{1-N}\int_0^r s^{N-1}P_1(s)f\Bigg(u_n(s), Mg\left(u_n(s),v_n(s)\right)(1+F_{2s})\Bigg)ds\Bigg)$$
$$\leq \varphi_1^{-1}\Bigg(r^{1-N}\int_0^r s^{N-1}P_1(s)C_2G_2\left(u_n(s)+v_n(s)\right)\varphi\left(1+F_{2s}\right)ds\Bigg)$$
$$\leq C_2\varphi_1^{-1}\Bigg(r^{1-N}G_2\left(u_n(r)+v_n(r)\right)\int_0^r s^{N-1}P_1(s)\varphi\left(1+F_{2s}\right)ds\Bigg).$$

Since G_2 is nondecreasing and $r^{1-N} < s^{1-N}$ for $0 < s < r$, using (**L4**) we have

$$\begin{aligned} u_n'(r) &\leq C_2\varphi_1^{-1}\left(G_2(u_n(r)+v_n(r))\right)r^{1-N}\int_0^r s^{N-1}P_1(s)\varphi\left(1+F_{2s}\right)ds \\ &\leq C_2\left[\varphi_2^{-1}\left(G_1(u_n(r)+v_n(r))\right)+\varphi_1^{-1}\left(G_2(u_n(r)+v_n(r))\right)\right]B_{1r} \end{aligned} \tag{3.4}$$

Integrating (3.3) from 0 to r, we get

$$\begin{aligned} v_n'(r) &\leq \varphi_2^{-1}\left(r^{1-N}\int_0^r s^{N-1}P_2(s)g\left(u_n(s),v_n(s)\right)ds\right) \\ &\leq \varphi_2^{-1}\left(r^{1-N}\int_0^r s^{N-1}C_1P_2(s)G_1(u_n(s)+v_n(s))ds\right) \end{aligned}$$

Since $r^{1-N} < s^{1-N}$ for $0 < s < r$, then by (**L4**), the above inequality gives

$$v_n'(r) \leq C_1\varphi_2^{-1}\left(G_1(u_n(r)+v_n(r))\right)r^{1-N}\int_0^r s^{N-1}P_2(s)ds \tag{3.5}$$
$$\leq C_1\left[\varphi_2^{-1}\left(G_1(u_n(r)+v_n(r))\right)+\varphi_1^{-1}\left(G_2(u_n(r)+v_n(r))\right)\right]B_{2r}. \tag{3.6}$$

Hence from (3.4) and (3.6), we have

$$\frac{u_n'(r)+v_n'(r)}{\varphi_2^{-1}(G_1(u_n(r)+v_n(r)))+\varphi_1^{-1}(G_2(u_n(r)+v_n(r)))} \leq C_1B_{2r}+C_2B_{1r} \tag{3.7}$$

Integrating (3.7) from 0 to r, we get

$$\int_{a+b}^{u_n(r)+v_n(r)}\frac{d\theta}{\varphi_2^{-1}(G_1(\theta))+\varphi_1^{-1}(G_2(\theta))} \leq C_1P(r)+C_2Q(r),$$

that is,

$$H(u_n(r)+v_n(r)) \leq C_1P(r)+C_2Q(r).$$

Since H is a bijection map with H^{-1} increasing, it follows that

$$u_n(r)+v_n(r) \leq H^{-1}(C_1P(r)+C_2Q(r)). \tag{3.8}$$

Then from (3.5), we have

$$v_n'(r) \leq C_1\varphi_2^{-1}\left(G_1(H^{-1}(C_1P(r)+C_2Q(r)))\right)B_{2r}. \tag{3.9}$$

Integrating (3.9) from 0 to r, we have

$$v_n(r) \leq b + C_1\int_0^r \varphi_2^{-1}\left(G_1(H^{-1}(C_1P(s)+C_2Q(s)))\right)B_{2s}ds. \tag{3.10}$$

Now, from (3.1), we have

$$v_n(r) \geq b + g(a,b)F_{2r} \geq g(a,b)F_{2r}. \tag{3.11}$$

Consequently, from (3.8) and (3.11), we have

$$u_n(r) \leq H^{-1}(C_1P(r)+C_2Q(r)) - g(a,b)F_{2r}. \tag{3.12}$$

Hence, from (3.10) and (3.12), we see that the upper bounds on $\{u_n(r)\}_{n\geq 0}$ and $\{v_n(r)\}_{n\geq 0}$ depends on r.

Now, we claim that the sequences $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ are bounded and equicontinuous on $[0, C_0]$ for arbitrary $C_0 > 0$. Indeed, from (3.1), we have

$$\begin{aligned} u_n'(r) &= \varphi_1^{-1}\left(r^{1-N}\int_0^r s^{N-1}P_1(s)f\left(u_{n-1}(s), v_{n-1}(s)\right)ds\right) \\ &\geq f(a,b)\varphi_1^{-1}\left(r^{1-N}\int_0^r s^{N-1}P_1(s)ds\right) \geq 0 \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} v_n'(r) &= \varphi_2^{-1}\left(r^{1-N}\int_0^r s^{N-1}P_2(s)g\left(u_n(s), v_{n-1}(s)\right)ds\right) \\ &\geq g(a,b)\varphi_2^{-1}\left(r^{1-N}\int_0^r s^{N-1}P_2(s)ds\right) \geq 0, \end{aligned}$$

which gives

$$u_n(r) \leq u_n(C_0) \leq C_3 \text{ and } v_n(r) \leq v_n(C_0) \leq C_4, \tag{3.14}$$

where,

$$C_3 = H^{-1}\left(C_1P(C_0)+C_2Q(C_0)\right) - g(a,b)\int_0^{C_0}\varphi_2^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_2(s)\,ds\right)dt$$

and

$$C_4 = b + C_1\int_0^{C_0}\varphi_2^{-1}\left(G_1\left(H^{-1}(C_1P(r)+C_2Q(r))\right)\right)B_{2r}\,dr$$

are positive constants. Hence $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ are bounded on $[0, C_0]$ for arbitrary $C_0 > 0$.

Next, we show that $u_n'(r)$ and $v_n'(r)$ are bounded on $[0, C_0]$. Indeed using (3.14), we obtain from (3.13) that

$$u_n'(r) \leq f(C_3, C_4)\varphi_1^{-1}\left(r^{1-N}\int_0^r s^{N-1}P_1(s)ds\right)$$

$$\leq f(C_3, C_4)\|P_1\|_\infty.\varphi_1^{-1}\left(r^{1-N}r^{N-1}\int_0^r ds\right)$$

$$\leq f(C_3, C_4)\|P_1\|_\infty.\varphi_1^{-1}(r).$$

Hence, for $r \in [0, C_0]$, we have

$$u_n'(r) \leq f(C_3, C_4)\|P\|_\infty.\varphi_1^{-1}(C_0). \tag{3.15}$$

Similarly, using (3.14) from (3.3) we have that

$$v_n'(r) = \varphi_2^{-1}\left(r^{1-N}\int_0^r s^{N-1}P_2(s)g(u_n(s), v_{n-1}(s))\,ds\right)$$

$$\leq g(C_3, C_4)\varphi_2^{-1}\left(r^{1-N}r^{N-1}\int_0^r P_2(s)ds\right)$$

$$\leq g(C_3, C_4)\|P_2\|_\infty.\varphi_2^{-1}(r).$$

Hence, for $r \in [0, C_0]$, we have

$$v_n'(r) \leq g(C_3, C_4)\|P_2\|_\infty.\varphi_2^{-1}(C_0). \tag{3.16}$$

Finally, we prove that $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ are equicontinuous functions on $[0, C_0]$ for arbitrary $C_0 > 0$. Let $\epsilon_1 > 0, \epsilon_2 > 0$ are arbitrary constants. Applying mean value theorem on u_n, v_n and using (3.15),(3.16) we obtain that

$$|u_n(x) - u_n(y)| \leq |u_n'(\xi_1)||x - y|$$
$$\leq f(C_3, C_4)\|P_1\|_\infty.\varphi_1^{-1}(C_0)|x - y|$$

and

$$|v_n(x) - v_n(y)| \leq |v_n'(\xi_2)||x - y|$$
$$\leq g(C_3, C_4)\|P_2\|_\infty.\varphi_2^{-1}(C_0)|x - y|.$$

The above inequality holds for all $n \in N$ and for $x, y \in [0, C_0]$. Now setting

$$\delta_1 = \frac{\epsilon_1}{f(C_3, C_4)\|P_1\|_\infty\varphi_1^{-1}(C_0)} \text{ and } \delta_2 = \frac{\epsilon_2}{g(C_3, C_4)\|P_2\|_\infty\varphi_2^{-1}(C_0)},$$

we see that $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ are equicontinuous family of functions on $[0, C_0]$. Since, both $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ are bounded and equicontinuous for arbitrary $C_0 > 0$, by Arzela-Ascoli theorem, $\{u_n\}$ and $\{v_n\}$ have subsequences converging uniformly to u and v respectively on $[0, C_0]$. By the arbitrariness of C_0, we see that u and v are positive entire solutions of (2.6), and thus (u, v) is a radially symmetric positive solution of the system (1.1) with $u(0) = a$ and $v(0) = b$. The $u(r)$ and $v(r)$ are given by

$$u(r) = a + \int_0^r \varphi_1^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_1(s)f(u(s), v(s))ds\right)dt,\ r \geq 0$$

and

$$v(r) = b + \int_0^r \varphi_2^{-1}\left(t^{1-N}\int_0^t s^{N-1}P_2(s)g(u(s), v(s))ds\right)dt,\ r \geq 0.$$

This completes the proof of the theorem.

Acknowledgment

The authors are thankful to the anonymous referees and the editors for their constructive remarks in improving the paper to the present form. Also, the authors are thankful to Dr. B.S.R.V. Prasad for careful reading of this manuscript and constructive suggestions.

References

[1] Covei D-P. Solutions with radial symmetry for a semilinear elliptic system with weights. Applied Mathematics Letters 2018; 76: 187-194.

[2] García-Melián J. A remark on uniqueness of large solutions for elliptic systems of competitive type. Journal of Mathematical Analysis and Applications 2007; 331: 608-616.

[3] García-Melián J. Large solutions for an elliptic system of quasilinear equations. Journal of Differential Equations 2008; 245: 3735-3752.

[4] Hai DD, Shivaji R. On radial solutions for combined superlinear elliptic systems on annular domains. Journal of Mathematical Analysis and Applications 2017; 446: 335-344.

[5] Kusano T, Swanson CA. Positive entire solutions of semilinear biharmonic equations. Hiroshima Mathematical Journal 1987; 17: 13-28.

[6] Lair AV. A necessary and sufficient condition for existence of large solutions to semilinear elliptic equations. Journal of Mathematical Analysis and Applications 1999; 240: 205-218.

[7] Lair AV. Large solutions of mixed sublinear/superlinear elliptic equations. Journal of Mathematical Analysis and Applications 2008; 346: 99-106.

[8] Lair AV. A necessary and sufficient condition for the existence of large solutions to sublinear elliptic systems. Journal of Mathematical Analysis and Applications 2010; 365: 103-108.

[9] Lair AV. Entire large solutions to semilinear elliptic systems. Journal of Mathematical Analysis and Applications 2001; 382: 324-333.

[10] Lair AV. Entire large solutions to elliptic equations of power non-linearities with variable exponents. Advanced Nonlinear Studies 2013; 13: 699-719.

[11] Lair AV, Mohammed A. Large solutions to semilinear elliptic systems with variable exponents. Journal of Mathematical Analysis and Applications 2014; 420: 1478-1499.

[12] Lair AV, Wood AW. Large solutions of semilinear ellipric equations with noinlinear gradient terms. International Journal of Mathematics and Mathematical Sciences 1999; 22: 869-883.

[13] Lair AV, Wood AW. Existence of entire large positive solutions of semilinear elliptic systems. Journal of Differential Equations 2000; 164: 380-394.

[14] Li H, Zhang P, Zhang Z. A remark on the existence of entire positive solutions for a class of semilinear elliptic systems. Journal of Mathematical Analysis and Applications 2010; 365: 338-341.

[15] Zhang Z. Existence of positive radial solutions for quasilinear elliptic equations and systems. Electronic Journal of Differential Equations 2016; 50: 1-9.

[16] Zhou S, Existence of entire radial solutions to a class of quasilinear elliptic equations and systems. Electronic Journal of Qualitative Theory of Differential Equations 2016; 38: 1-10.