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Asymptotic theory for a critical class of third-order differential equations

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Abstract: An asymptotic theory is developed for a class of third-order differential equations. We identify a critical case to obtain the asymptotic form of three linearly independent solutions for large x.

Key words: Asymptotic form of solutions, critical case

1. Introduction

In this paper, we consider the asymptotic form of three linearly independent solutions of the third-order differential equation

$$(q(qy')')' + py' + ry = 0, \tag{1.1}$$

as $x \to \infty$, where the coefficients q, p and r are nowhere zero in some interval $[a, \infty)$.
x is the independent variable and the prime is d/dx. We do not need to restrict ourselves to real valued coefficients. We shall consider the case where r and q are small compared to p to identify the critical case which is given by:

$$\frac{p'}{r} \to \sigma \qquad \text{as } x \to \infty, \tag{1.2}$$

where σ is a nonzero constant.
If $p = Bx^{\alpha_1}$ and $r = Cx^{\alpha_2}$, then (1.2) is given by

$$\alpha_1 - \alpha_2 = 1, \tag{1.3}$$

as we shall see in Section 5. The critical case (1.2) is given in Section 4. Pfeiffer [9] considered (1.1) with $q = 1$, subject to the condition $r' = o(r^{4/3})$ as $x \to \infty$. Similar third-order equations to (1.1) have been also investigated by Al-Hammadi [1–3] and Unsworth [10] . Eastham [4] considered a critical case for a fourth-order differential equation and showed that this case represents a borderline between situations where all solutions have a certain exponential character as $x \to \infty$ and where only two solutions have this character.
In this paper we use the asymptotic theorem of Eastham (see Section 2 of [5] and Theorem 1.6.1 in [6]) to obtain the solutions of (1.1).
The general features of our method are given in Sections 2 and 3, the main theorem for the differential equation (1.1) is stated and proved in Section 4. Finally, in Section 5, we give some examples.
For the convenience of the reader we state here the aforementioned asymptotic theorem of Eastham.

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Theorem 1.1 (Eastham 1985, [5]) *In the system*

$$Y'(x) = \{\Lambda(x) + R(x)\}Y(x), \tag{1.4}$$

let $\Lambda(x)$ *be an* $n \times n$ *diagonal matrix,*

$$\Lambda(x) = \text{diag}\left(\lambda_1(x), ..., \lambda_n(x)\right),$$

in which $\lambda_(x) - \lambda_j(x)$ *is nowhere zero in some interval* $[0, \infty)$ *for each pair of unequal integers* i *and* j *in* $[1, n]$. *Let the* $n \times n$ *matrix* $R(x)$ *satisfy*

(i) $\{\lambda_(x) - \lambda_j(x)\}^{-1}R(x) \to 0 \quad \text{as} \quad x \to \infty$*;*

(ii) $\left(\{\lambda_(x) - \lambda_j(x)\}^{-1}R(x)\right)' \in L(a, \infty)$,

for each unequal i *and* j *in* $[1, n]$. *Also for* $1 \leq k \leq n$, *let the eigenvalues* $\mu_k(x)$ *of* $\Lambda(x) + R(x)$ *satisfy the condition (L) in [5].*
Then, as $x \to \infty$, *(1.4) has solution* $Y_k(x)$ *with the asymptotic form*

$$Y_k(x) = \{e_k + 0(1)\}\text{exp}\left(\int_a^x \mu_k(t)dt\right),$$

where e_k *is the coordinate vector whose* k*th component is unity and other components are zero.*

We note that in section 2 of [5] it is shown that the above theorem also applies to the more general system

$$Y'(x) = \{\Lambda(x) + R(x) + S(x)\}Y(x),$$

where S is $L(a, \infty)$ and then we need only Λ and R to satisfy the above theorem.

2. The general method

We write (1.1) in a standard way [8] as a first-order system.

$$Y' = A(x)Y, \tag{2.1}$$

where the vector Y and the matrix $A(x)$ are given by

$$Y = \begin{pmatrix} y \\ qy' \\ q(qy')' \end{pmatrix} \text{ and } A(x) = \begin{pmatrix} 0 & q^{-1} & 0 \\ 0 & 0 & q^{-1} \\ -r & -pq^{-1} & 0 \end{pmatrix} \tag{2.2}$$

As in [1], we express A in its diagonal form

$$T^{-1}AT = \Lambda, \tag{2.3}$$

where

$$\Lambda = dg(\lambda_1, \lambda_2, \lambda_3). \tag{2.4}$$

We therefore require the eigenvalues λ_j and eigenvectors ν_j of A , where$(1 \leq j \leq 3)$. Writing

$$q^2 = s, \tag{2.5}$$

then the characteristic equation of A is given by

$$s\lambda^3 + p\lambda + r = 0. \tag{2.6}$$

The eigenvector u_j that corresponds to λ_j is

$$u_j = \begin{pmatrix} 1 & s^{1/2}\lambda_j & s\lambda_j^2 \end{pmatrix}^t \tag{2.7}$$

where the superscript t denotes the transpose, and at this stage we assume that λ_j are distinct. Thus the matrix T in (2.3) is given by

$$T = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \tag{2.8}$$

Now by (2.7) and (2.8),

$$T^{-1} = \{s(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)\}^{-1} \begin{pmatrix} s\lambda_2\lambda_3(\lambda_3 - \lambda_2) & s^{1/2}(\lambda_2^2 - \lambda_3^2) & \lambda_3 - \lambda_2 \\ s\lambda_1\lambda_3(\lambda_1 - \lambda_3) & s^{1/2}(\lambda_3^2 - \lambda_1^2) & \lambda_1 - \lambda_3 \\ s\lambda_1\lambda_2(\lambda_2 - \lambda_1) & s^{1/2}(\lambda_1^2 - \lambda_2^2) & \lambda_2 - \lambda_1 \end{pmatrix} \tag{2.9}$$

By (2.3) , the transformation

$$Y = TZ, \tag{2.10}$$

takes (2.1) into

$$Z' = (\Lambda - T^{-1}T')Z. \tag{2.11}$$

From (2.7)–(2.9), we obtain $T^{-1}T' = (t_{ik})$, where

$$t_{1k} = \frac{(s^{1/2}\lambda_k)'(2\lambda_k - \lambda_2 - \lambda_3)}{s^{1/2}(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}, \qquad (1 \leq k \leq 3) \tag{2.12}$$

$$t_{2k} = \frac{(s^{1/2}\lambda_k)'(\lambda_3 + \lambda_1 - 2\lambda_2)}{s^{1/2}(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}, \qquad (1 \leq k \leq 3) \tag{2.13}$$

$$t_{3k} = \frac{(s^{1/2}\lambda_k)'(2\lambda_k - \lambda_1 - \lambda_2)}{s^{1/2}(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}. \qquad (1 \leq k \leq 3) \tag{2.14}$$

Now we need to work out (2.12)–(2.14) in some detail in terms of r, p and s in order to determine the form of the system (2.11) and then make progress towards (1.1).

3. The matrices (Λ) and $(T^{-1}T')$

In our analysis, we impose the following conditions in the coefficients r, s and p as follows:
(I) p, r and s are nowhere zero in the interval $[a, \infty)$, and

$$rs^{1/2} = o(p^{3/2}) \qquad (x \to \infty). \tag{3.1}$$

Then we write

$$\frac{rs^{1/2}}{p^{3/2}} = \delta = o(1) \qquad (x \to \infty). \tag{3.2}$$

(II) Let

$$\frac{p'}{p}\delta, \quad \frac{s'}{s}\delta, \quad \frac{r'}{r}\delta \qquad \text{are all} \quad L(a,\infty) \tag{3.3}$$

As in [1], we can solve (2.6) subject to (3.1) to obtain

$$\lambda_j = \omega_j (p/s)^{1/2}(1+\delta_j), \qquad (j=1,2) \text{ and } \omega_1 = i = \overline{\omega}_2 \tag{3.4}$$

$$\lambda_3 = -\frac{r}{p}(1+\delta_3) \tag{3.5}$$

where

$$\delta_j = O(\delta) \qquad (j=1,2) \tag{3.6}$$

and

$$\delta_3 = O(\delta^2). \tag{3.7}$$

Hence , by (3.2)

$$\lambda_3 = o(\lambda_{2-j}), \qquad (j=0,1) \quad (x \to \infty) \tag{3.8}$$

Note that in [2] we considered the case where

$$\lambda_1 \sim (\text{const.})\lambda_2 \sim (\text{const.})\lambda_3 \qquad (x \to \infty) \tag{3.9}$$

Thus

$$\frac{\lambda_1}{\lambda_2} \to \text{const.}, \frac{\lambda_2}{\lambda_3} \to \text{const.}, \frac{\lambda_1}{\lambda_3} \to \text{const.}, \qquad (x \to \infty).$$

Also, by substituting (3.4) and (3.5) into (2.6) and differentiating, we obtain:

$$\delta_j' = O(\delta'), \qquad (j=1,2) \tag{3.10}$$

and

$$\delta_3' = O(\delta\delta'), \tag{3.11}$$

where

$$\delta' = O(\frac{p'}{p}\delta) + O(\frac{s'}{s}\delta) + O(\frac{r'}{r}\delta) \tag{3.12}$$

Hence, by (3.10)–(3.12), and (3.3)

$$\delta', \quad \delta_j' \quad \text{and} \quad \delta_3' \qquad \text{are} \quad L(a,\infty) \qquad (j=1,2) \tag{3.13}$$

Now by (3.4)

$$(s^{1/2}\lambda_k)' = \frac{1}{2}\omega_k \frac{p'}{p} p^{1/2}(1+\delta_k) + \omega_k p^{1/2}\delta_k' \qquad (k=1,2) \tag{3.14}$$

and by (3.5)

$$(s^{1/2}\lambda_3)' = \left[\frac{p'}{p} - \frac{1}{2}\frac{s'}{s}\right]\frac{rs^{1/2}}{p}(1+\delta_3) - \frac{r's^{1/2}}{p}(1+\delta_3) \tag{3.15}$$

We can now substitute the estimates (3.4)–(3.7) and (3.14)–(3.15) into (2.12)–(2.14), the calculations are similar to those in [1, 2], and we obtain the following expressions for t_{jk}:

$$\begin{aligned} t_{11} &= \frac{3}{4}\frac{p'}{p}(1+O(\delta)) + O(\delta_1'), \\ t_{12} &= \frac{1}{4}\frac{p'}{p}(1+O(\delta)) + O(\delta_2'), \\ t_{13} &= O\left(\frac{p'}{p}\delta\right) + O\left(\frac{r'}{r}\delta\right) + O\left(\frac{s'\delta}{s}\right) + O\left(\delta\delta_3'\right) \end{aligned} \tag{3.16}$$

$$\begin{aligned} t_{21} &= \frac{1}{4}\frac{p'}{p}(1+O(\delta)) + O(\delta_1'), \\ t_{22} &= -\frac{1}{4}\frac{p'}{p}(1+O(\delta)) + O(\delta_2'), \\ t_{23} &= O(\left(\frac{p'}{p}\delta\right) + O\left(\frac{r'}{r}\delta\right) + O\left(\frac{s'\delta}{s}\right) + O(\delta\delta_3'). \end{aligned} \tag{3.17}$$

$$\begin{aligned} t_{31} &= -\frac{p'}{p}(1+O(\delta)) + O(\delta_1'), \\ t_{32} &= -\frac{p'}{p}(1+O(\delta)) + O(\delta_2'), \\ t_{33} &= O\left(\frac{p'}{p}\delta^2\right) + O\left(\frac{r'}{r}\delta^2\right) + O\left(\frac{s'}{s}\delta^2\right) + O(\delta\delta_3'). \end{aligned} \tag{3.18}$$

Now by (3.3), (3.13) and (3.16)–(3.18), we can write the system (2.11) as

$$Z' = (\Lambda + R + S)Z \tag{3.19}$$

where

$$R = \begin{pmatrix} -3\varrho & -\varrho & 0 \\ -\varrho & \varrho & 0 \\ 4\varrho & 4\varrho & 0 \end{pmatrix} \tag{3.20}$$

with

$$\varrho = \frac{1}{4}\frac{p'}{p} \tag{3.21}$$

and S is $L(a,\infty)$

4. The asymptotic form of solutions

Theorem 4.1 *Let the coefficients r, p and s in (1.1) nowhere zero in $[a, \infty)$ with p to be $C^{(2)}[a, \infty)$ while r and s are $C^{(1)}[a, \infty)$. Let (3.1) and (3.3) hold. Let*

$$\varrho(x) = \sigma(r/p)(1 + \psi) \tag{4.1}$$

where σ nonzero constant and $\psi(x) \to 0$ as $x \to \infty$. Also let

$$\left(\frac{p'}{r}\right)' \text{ is } \quad L(a, \infty) \tag{4.2}$$

Let $Re(\nu_i - \nu_j)$ have one sign in $[a, \infty)$ for each unequal pair (i, j) where

$$\nu_1 = \lambda_1 - 3\varrho, \quad \nu_2 = \lambda_2 + \varrho, \quad \nu_3 = \lambda_3 \tag{4.3}$$

then (1.1) has the asymptotic form of solutions

$$y_1(x) \sim p^{-3/4}(x) \exp\left(\int_a^x \lambda_1(t)dt\right) \tag{4.4}$$

$$y_2(x) \sim p^{1/4}(x) \exp\left(\int_a^x \lambda_2(t)dt\right) \tag{4.5}$$

$$y_3(x) \sim \exp\left(\int_a^x \lambda_3(t)dt\right) \tag{4.6}$$

Proof As in [1] we apply Eastham theorem ([5], Section 2) to the system (3.19) provided only that Λ and R satisfy the conditions and we shall use (3.20),(3.21), (4.1), and (4.2). We first require that

$$\varrho = o\{(\lambda_i - \lambda_j)\} \qquad (1 \le i, j \le 3, i \ne j) \tag{4.7}$$

this being Eq. 2.1 of [5] for our system. Now

$$\lambda_i - \lambda_j \sim const(p/s)^{1/2} \qquad (x \to \infty) \quad \text{for} \quad (1 \le i, j \le 3, i \ne j)$$

thus (4.7) true by (3.1) and (4.1). Second we need

$$\left\{\varrho(\lambda_i - \lambda_j)^{-1}\right\}' \qquad \text{are} \quad L(a, \infty) \quad \text{for} \quad (1 \le i, j \le 3, i \ne j) \tag{4.8}$$

this being Eq. 2.1 of [5] for our system. This requirement is implied by (3.3), (3.13), and (4.2). We also note that by (1.6.36) from [6] the eigenvalue $\mu_j (1 \le j \le 3)$ of $\bigwedge + R$ are given by

$$\mu_j = \nu_j + O(\max_{j \ne l} |R|^2 |\lambda_l - \lambda_j|^{-1}) \qquad (1 \le j \le 3) \tag{4.9}$$

Hence the simplifying condition (2.13) from [5] is satisfied by (**??**), (4.3), and (2.13).
Since (3.19) satisfies all the conditions for the asymptotic theorem in Section 2 of [5], it follows that, as $x \to \infty$ (3.19) has three linearly independent solutions $Z_k(x)$ such that

$$Z_k(x) = \{e_k + o(1)\} \exp(\int_a^x \nu_k(t)dt) \tag{4.10}$$

where ν_k are given by (4.3). Now we transform back to Y by means of (2.10), (2.7), and (2.8). By taking the first component on each side of (4.10), and carrying out the integration of

$$-\frac{3}{4}\frac{p'}{p} \quad \text{and} \quad \frac{1}{4}\frac{p'}{p}$$

we obtain (4.4)–(4.6) after an adjustment of a constant multiple in y_1 and y_2. □

5. Discussion

(i) In the familiar case the coefficients which are covered by theorem 4.1 are

$$p(x) = c_1 x^{\alpha_1}, \quad r(x) = c_2 x^{\alpha_2} \quad \text{and} \quad s(x) = c_3 x^{\alpha_3},$$

with real constants c_i and α_i $(1 \le i \le 3)$, such that c_i are not equal to zero.
Then (3.1) and (3.2) hold if

$$-\alpha_3 + 3\alpha_1 - 2\alpha_2 > 0 \tag{5.1}$$

The critical case (4.1) is given by

$$\alpha_1 - \alpha_2 = 1 \tag{5.2}$$

The nonzero σ in (4.1) is given by

$$\sigma = \frac{c_1 \alpha_1}{4c_2} \qquad (\alpha_1 \neq 0) \tag{5.3}$$

where $\psi(x) = 0$ in (4.1).
(ii) More general coefficients are

$$p(x) = c_1 x^{\alpha_1} \exp(2x^b), \quad r(x) = c_2 x^{\alpha_2} exp(2x^b) \quad \text{and} \quad s(x) = c_3 x^{\alpha_3} exp(2x^a),$$

with real constants c_i , α_i, a and $b > a \ge 0$, such that c_i are not equal to zero.
Conditions (3.1) and (3.2) are all satisfied.
The critical case (4.1) is given by

$$\alpha_2 - \alpha_1 = b - 1 \tag{5.4}$$

and the nonzero σ in (4.1) is given by

$$\sigma = \frac{c_1 b}{2c_2} \tag{5.5}$$

where

$$\psi(x) = \frac{\alpha_1}{2b} x^{-b} \tag{5.6}$$

Then $\psi(x) \to 0$ as $x \to \infty$ and $\psi'(x) \; \epsilon \, L(a, \infty)$ which implies that (4.2) holds.

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