

1-1-2020

## Asymptotic theory for a critical class of third-order differential equations

AZIZ S. A. ALHAMMADI

Follow this and additional works at: <https://dctubitak.researchcommons.org/math>



Part of the [Mathematics Commons](#)


---

### Recommended Citation

ALHAMMADI, AZIZ S. A. (2020) "Asymptotic theory for a critical class of third-order differential equations," *Turkish Journal of Mathematics*: Vol. 44: No. 6, Article 13. <https://doi.org/10.3906/mat-1910-5>  
Available at: <https://dctubitak.researchcommons.org/math/vol44/iss6/13>

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals.

## Asymptotic theory for a critical class of third-order differential equations

Aziz S. A. AL-HAMMADI\* 

Department of Mathematics, College of Science, University of Bahrain, Sakhir, Bahrain

Received: 02.10.2019

Accepted/Published Online: 08.09.2020

Final Version: 16.11.2020

**Abstract:** An asymptotic theory is developed for a class of third-order differential equations. We identify a critical case to obtain the asymptotic form of three linearly independent solutions for large  $x$ .

**Key words:** Asymptotic form of solutions, critical case

### 1. Introduction

In this paper, we consider the asymptotic form of three linearly independent solutions of the third-order differential equation

$$(q(qy'))' + py' + ry = 0, \quad (1.1)$$

as  $x \rightarrow \infty$ , where the coefficients  $q$ ,  $p$  and  $r$  are nowhere zero in some interval  $[a, \infty)$ .

$x$  is the independent variable and the prime is  $d/dx$ . We do not need to restrict ourselves to real valued coefficients. We shall consider the case where  $r$  and  $q$  are small compared to  $p$  to identify the critical case which is given by:

$$\frac{p'}{r} \rightarrow \sigma \quad \text{as } x \rightarrow \infty, \quad (1.2)$$

where  $\sigma$  is a nonzero constant.

If  $p = Bx^{\alpha_1}$  and  $r = Cx^{\alpha_2}$ , then (1.2) is given by

$$\alpha_1 - \alpha_2 = 1, \quad (1.3)$$

as we shall see in Section 5. The critical case (1.2) is given in Section 4. Pfeiffer [9] considered (1.1) with  $q = 1$ , subject to the condition  $r' = o(r^{4/3})$  as  $x \rightarrow \infty$ . Similar third-order equations to (1.1) have been also investigated by Al-Hammadi [1–3] and Unsworth [10]. Eastham [4] considered a critical case for a fourth-order differential equation and showed that this case represents a borderline between situations where all solutions have a certain exponential character as  $x \rightarrow \infty$  and where only two solutions have this character.

In this paper we use the asymptotic theorem of Eastham (see Section 2 of [5] and Theorem 1.6.1 in [6]) to obtain the solutions of (1.1).

The general features of our method are given in Sections 2 and 3, the main theorem for the differential equation (1.1) is stated and proved in Section 4. Finally, in Section 5, we give some examples.

For the convenience of the reader we state here the aforementioned asymptotic theorem of Eastham.

\*Correspondence: profmaths\_alhammadi@hotmail.com

2010 AMS Mathematics Subject Classification: 34E05

**Theorem 1.1 (Eastham 1985, [5])** *In the system*

$$Y'(x) = \{\Lambda(x) + R(x)\}Y(x), \quad (1.4)$$

let  $\Lambda(x)$  be an  $n \times n$  diagonal matrix,

$$\Lambda(x) = \text{diag}(\lambda_1(x), \dots, \lambda_n(x)),$$

in which  $\lambda_i(x) - \lambda_j(x)$  is nowhere zero in some interval  $[0, \infty)$  for each pair of unequal integers  $i$  and  $j$  in  $[1, n]$ . Let the  $n \times n$  matrix  $R(x)$  satisfy

$$(i) \quad \{\lambda_i(x) - \lambda_j(x)\}^{-1}R(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty;$$

$$(ii) \quad (\{\lambda_i(x) - \lambda_j(x)\}^{-1}R(x))' \in L(a, \infty),$$

for each unequal  $i$  and  $j$  in  $[1, n]$ . Also for  $1 \leq k \leq n$ , let the eigenvalues  $\mu_k(x)$  of  $\Lambda(x) + R(x)$  satisfy the condition (L) in [5].

Then, as  $x \rightarrow \infty$ , (1.4) has solution  $Y_k(x)$  with the asymptotic form

$$Y_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t) dt\right),$$

where  $e_k$  is the coordinate vector whose  $k$ th component is unity and other components are zero.

We note that in section 2 of [5] it is shown that the above theorem also applies to the more general system

$$Y'(x) = \{\Lambda(x) + R(x) + S(x)\}Y(x),$$

where  $S$  is  $L(a, \infty)$  and then we need only  $\Lambda$  and  $R$  to satisfy the above theorem.

## 2. The general method

We write (1.1) in a standard way [8] as a first-order system.

$$Y' = A(x)Y, \quad (2.1)$$

where the vector  $Y$  and the matrix  $A(x)$  are given by

$$Y = \begin{pmatrix} y \\ qy' \\ q(qy')' \end{pmatrix} \quad \text{and} \quad A(x) = \begin{pmatrix} 0 & q^{-1} & 0 \\ 0 & 0 & q^{-1} \\ -r & -pq^{-1} & 0 \end{pmatrix} \quad (2.2)$$

As in [1], we express  $A$  in its diagonal form

$$T^{-1}AT = \Lambda, \quad (2.3)$$

where

$$\Lambda = dg(\lambda_1, \lambda_2, \lambda_3). \quad (2.4)$$

We therefore require the eigenvalues  $\lambda_j$  and eigenvectors  $\nu_j$  of  $A$ , where  $(1 \leq j \leq 3)$ . Writing

$$q^2 = s, \tag{2.5}$$

then the characteristic equation of  $A$  is given by

$$s\lambda^3 + p\lambda + r = 0. \tag{2.6}$$

The eigenvector  $u_j$  that corresponds to  $\lambda_j$  is

$$u_j = ( 1 \quad s^{1/2}\lambda_j \quad s\lambda_j^2 )^t \tag{2.7}$$

where the superscript  $t$  denotes the transpose, and at this stage we assume that  $\lambda_j$  are distinct. Thus the matrix  $T$  in (2.3) is given by

$$T = ( u_1 \quad u_2 \quad u_3 ) \tag{2.8}$$

Now by (2.7) and (2.8),

$$T^{-1} = \{s(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)\}^{-1} \begin{pmatrix} s\lambda_2\lambda_3(\lambda_3 - \lambda_2) & s^{1/2}(\lambda_2^2 - \lambda_3^2) & \lambda_3 - \lambda_2 \\ s\lambda_1\lambda_3(\lambda_1 - \lambda_3) & s^{1/2}(\lambda_3^2 - \lambda_1^2) & \lambda_1 - \lambda_3 \\ s\lambda_1\lambda_2(\lambda_2 - \lambda_1) & s^{1/2}(\lambda_1^2 - \lambda_2^2) & \lambda_2 - \lambda_1 \end{pmatrix} \tag{2.9}$$

By (2.3), the transformation

$$Y = TZ, \tag{2.10}$$

takes (2.1) into

$$Z' = (\Lambda - T^{-1}T')Z. \tag{2.11}$$

From (2.7)–(2.9), we obtain  $T^{-1}T' = (t_{ik})$ , where

$$t_{1k} = \frac{(s^{1/2}\lambda_k)'(2\lambda_k - \lambda_2 - \lambda_3)}{s^{1/2}(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}, \quad (1 \leq k \leq 3) \tag{2.12}$$

$$t_{2k} = \frac{(s^{1/2}\lambda_k)'(\lambda_3 + \lambda_1 - 2\lambda_2)}{s^{1/2}(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)}, \quad (1 \leq k \leq 3) \tag{2.13}$$

$$t_{3k} = \frac{(s^{1/2}\lambda_k)'(2\lambda_k - \lambda_1 - \lambda_2)}{s^{1/2}(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}. \quad (1 \leq k \leq 3) \tag{2.14}$$

Now we need to work out (2.12)–(2.14) in some detail in terms of  $r, p$  and  $s$  in order to determine the form of the system (2.11) and then make progress towards (1.1).

### 3. The matrices $(\Lambda)$ and $(T^{-1}T')$

In our analysis, we impose the following conditions in the coefficients  $r, s$  and  $p$  as follows:

(I)  $p, r$  and  $s$  are nowhere zero in the interval  $[a, \infty)$ , and

$$rs^{1/2} = o(p^{3/2}) \quad (x \rightarrow \infty). \tag{3.1}$$

Then we write

$$\frac{rs^{1/2}}{p^{3/2}} = \delta = o(1) \quad (x \rightarrow \infty). \tag{3.2}$$

(II) Let

$$\frac{p'}{p}\delta, \quad \frac{s'}{s}\delta, \quad \frac{r'}{r}\delta \quad \text{are all} \quad L(a, \infty) \tag{3.3}$$

As in [1], we can solve (2.6) subject to (3.1) to obtain

$$\lambda_j = \omega_j(p/s)^{1/2}(1 + \delta_j), \quad (j = 1, 2) \text{ and } \omega_1 = i = \bar{\omega}_2 \tag{3.4}$$

$$\lambda_3 = -\frac{r}{p}(1 + \delta_3) \tag{3.5}$$

where

$$\delta_j = O(\delta) \quad (j = 1, 2) \tag{3.6}$$

and

$$\delta_3 = O(\delta^2). \tag{3.7}$$

Hence , by (3.2)

$$\lambda_3 = o(\lambda_{2-j}), \quad (j = 0, 1) \quad (x \rightarrow \infty) \tag{3.8}$$

Note that in [2] we considered the case where

$$\lambda_1 \sim (\text{const.})\lambda_2 \sim (\text{const.})\lambda_3 \quad (x \rightarrow \infty) \tag{3.9}$$

Thus

$$\frac{\lambda_1}{\lambda_2} \rightarrow \text{const.}, \quad \frac{\lambda_2}{\lambda_3} \rightarrow \text{const.}, \quad \frac{\lambda_1}{\lambda_3} \rightarrow \text{const.}, \quad (x \rightarrow \infty).$$

Also, by substituting (3.4) and (3.5) into (2.6) and differentiating, we obtain:

$$\delta'_j = O(\delta'), \quad (j = 1, 2) \tag{3.10}$$

and

$$\delta'_3 = O(\delta\delta'), \tag{3.11}$$

where

$$\delta' = O\left(\frac{p'}{p}\delta\right) + O\left(\frac{s'}{s}\delta\right) + O\left(\frac{r'}{r}\delta\right) \tag{3.12}$$

Hence, by (3.10)–(3.12), and (3.3)

$$\delta', \quad \delta'_j \quad \text{and} \quad \delta'_3 \quad \text{are} \quad L(a, \infty) \quad (j = 1, 2) \tag{3.13}$$

Now by (3.4)

$$(s^{1/2}\lambda_k)' = \frac{1}{2}\omega_k \frac{p'}{p} p^{1/2}(1 + \delta_k) + \omega_k p^{1/2} \delta'_k \quad (k = 1, 2) \tag{3.14}$$

and by (3.5)

$$(s^{1/2}\lambda_3)' = \left[ \frac{p'}{p} - \frac{1}{2} \frac{s'}{s} \right] \frac{rs^{1/2}}{p} (1 + \delta_3) - \frac{r's^{1/2}}{p} (1 + \delta_3) \tag{3.15}$$

We can now substitute the estimates (3.4)–(3.7) and (3.14)–(3.15) into (2.12)–(2.14), the calculations are similar to those in [1, 2], and we obtain the following expressions for  $t_{jk}$ :

$$\begin{aligned} t_{11} &= \frac{3}{4} \frac{p'}{p} (1 + O(\delta)) + O(\delta'_1), \\ t_{12} &= \frac{1}{4} \frac{p'}{p} (1 + O(\delta)) + O(\delta'_2), \\ t_{13} &= O\left(\frac{p'}{p} \delta\right) + O\left(\frac{r'}{r} \delta\right) + O\left(\frac{s'}{s} \delta\right) + O(\delta\delta'_3) \end{aligned} \tag{3.16}$$

$$\begin{aligned} t_{21} &= \frac{1}{4} \frac{p'}{p} (1 + O(\delta)) + O(\delta'_1), \\ t_{22} &= -\frac{1}{4} \frac{p'}{p} (1 + O(\delta)) + O(\delta'_2), \\ t_{23} &= O\left(\frac{p'}{p} \delta\right) + O\left(\frac{r'}{r} \delta\right) + O\left(\frac{s'}{s} \delta\right) + O(\delta\delta'_3). \end{aligned} \tag{3.17}$$

$$\begin{aligned} t_{31} &= -\frac{p'}{p} (1 + O(\delta)) + O(\delta'_1), \\ t_{32} &= -\frac{p'}{p} (1 + O(\delta)) + O(\delta'_2), \\ t_{33} &= O\left(\frac{p'}{p} \delta^2\right) + O\left(\frac{r'}{r} \delta^2\right) + O\left(\frac{s'}{s} \delta^2\right) + O(\delta\delta'_3). \end{aligned} \tag{3.18}$$

Now by (3.3), (3.13) and (3.16)–(3.18), we can write the system (2.11) as

$$Z' = (\Lambda + R + S)Z \tag{3.19}$$

where

$$R = \begin{pmatrix} -3\varrho & -\varrho & 0 \\ -\varrho & \varrho & 0 \\ 4\varrho & 4\varrho & 0 \end{pmatrix} \tag{3.20}$$

with

$$\varrho = \frac{1}{4} \frac{p'}{p} \tag{3.21}$$

and  $S$  is  $L(a, \infty)$

**4. The asymptotic form of solutions**

**Theorem 4.1** *Let the coefficients  $r, p$  and  $s$  in (1.1) nowhere zero in  $[a, \infty)$  with  $p$  to be  $C^{(2)}[a, \infty)$  while  $r$  and  $s$  are  $C^{(1)}[a, \infty)$ . Let (3.1) and (3.3) hold. Let*

$$\varrho(x) = \sigma(r/p)(1 + \psi) \tag{4.1}$$

where  $\sigma$  nonzero constant and  $\psi(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Also let

$$\left(\frac{p'}{r}\right)' \text{ is } L(a, \infty) \tag{4.2}$$

Let  $Re(\nu_i - \nu_j)$  have one sign in  $[a, \infty)$  for each unequal pair  $(i, j)$  where

$$\nu_1 = \lambda_1 - 3\varrho, \quad \nu_2 = \lambda_2 + \varrho, \quad \nu_3 = \lambda_3 \tag{4.3}$$

then (1.1) has the asymptotic form of solutions

$$y_1(x) \sim p^{-3/4}(x) \exp\left(\int_a^x \lambda_1(t)dt\right) \tag{4.4}$$

$$y_2(x) \sim p^{1/4}(x) \exp\left(\int_a^x \lambda_2(t)dt\right) \tag{4.5}$$

$$y_3(x) \sim \exp\left(\int_a^x \lambda_3(t)dt\right) \tag{4.6}$$

**Proof** As in [1] we apply Eastham theorem ([5], Section 2) to the system (3.19) provided only that  $\Lambda$  and  $R$  satisfy the conditions and we shall use (3.20), (3.21), (4.1), and (4.2). We first require that

$$\varrho = o\{(\lambda_i - \lambda_j)\} \quad (1 \leq i, j \leq 3, i \neq j) \tag{4.7}$$

this being Eq. 2.1 of [5] for our system. Now

$$\lambda_i - \lambda_j \sim \text{const}(p/s)^{1/2} \quad (x \rightarrow \infty) \quad \text{for } (1 \leq i, j \leq 3, i \neq j)$$

thus (4.7) true by (3.1) and (4.1). Second we need

$$\{\varrho(\lambda_i - \lambda_j)^{-1}\}' \quad \text{are } L(a, \infty) \quad \text{for } (1 \leq i, j \leq 3, i \neq j) \tag{4.8}$$

this being Eq. 2.1 of [5] for our system. This requirement is implied by (3.3), (3.13), and (4.2). We also note that by (1.6.36) from [6] the eigenvalue  $\mu_j (1 \leq j \leq 3)$  of  $\Lambda + R$  are given by

$$\mu_j = \nu_j + O(\max_{j \neq l} |R|^2 |\lambda_l - \lambda_j|^{-1}) \quad (1 \leq j \leq 3) \tag{4.9}$$

Hence the simplifying condition (2.13) from [5] is satisfied by (??), (4.3), and (2.13).

Since (3.19) satisfies all the conditions for the asymptotic theorem in Section 2 of [5], it follows that, as  $x \rightarrow \infty$  (3.19) has three linearly independent solutions  $Z_k(x)$  such that

$$Z_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \nu_k(t)dt\right) \tag{4.10}$$

where  $\nu_k$  are given by (4.3). Now we transform back to  $Y$  by means of (2.10), (2.7), and (2.8). By taking the first component on each side of (4.10), and carrying out the integration of

$$-\frac{3p'}{4p} \quad \text{and} \quad \frac{1p'}{4p}$$

we obtain (4.4)–(4.6) after an adjustment of a constant multiple in  $y_1$  and  $y_2$ .  $\square$

## 5. Discussion

(i) In the familiar case the coefficients which are covered by theorem 4.1 are

$$p(x) = c_1x^{\alpha_1}, \quad r(x) = c_2x^{\alpha_2} \quad \text{and} \quad s(x) = c_3x^{\alpha_3},$$

with real constants  $c_i$  and  $\alpha_i$  ( $1 \leq i \leq 3$ ), such that  $c_i$  are not equal to zero.

Then (3.1) and (3.2) hold if

$$-\alpha_3 + 3\alpha_1 - 2\alpha_2 > 0 \tag{5.1}$$

The critical case (4.1) is given by

$$\alpha_1 - \alpha_2 = 1 \tag{5.2}$$

The nonzero  $\sigma$  in (4.1) is given by

$$\sigma = \frac{c_1\alpha_1}{4c_2} \quad (\alpha_1 \neq 0) \tag{5.3}$$

where  $\psi(x) = 0$  in (4.1).

(ii) More general coefficients are

$$p(x) = c_1x^{\alpha_1} \exp(2x^b), \quad r(x) = c_2x^{\alpha_2} \exp(2x^b) \quad \text{and} \quad s(x) = c_3x^{\alpha_3} \exp(2x^a),$$

with real constants  $c_i$ ,  $\alpha_i$ ,  $a$  and  $b > a \geq 0$ , such that  $c_i$  are not equal to zero.

Conditions (3.1) and (3.2) are all satisfied.

The critical case (4.1) is given by

$$\alpha_2 - \alpha_1 = b - 1 \tag{5.4}$$

and the nonzero  $\sigma$  in (4.1) is given by

$$\sigma = \frac{c_1b}{2c_2} \tag{5.5}$$

where

$$\psi(x) = \frac{\alpha_1}{2b} x^{-b} \tag{5.6}$$

Then  $\psi(x) \rightarrow 0$  as  $x \rightarrow \infty$  and  $\psi'(x) \in L(a, \infty)$  which implies that (4.2) holds.

## Acknowledgment

I would like to thank the anonymous referee for his careful reading and suggestions on the first version of this paper.



**References**

- [1] Al-Hammadi ASA. Asymptotic theory for third-order differential equations. *Mathematika* 1988; 35: 225-232.
- [2] Al-Hammadi ASA. Asymptotic theory for third-order differential equation of Euler type. *Results in Mathematics* 1990; 17: 1-14.
- [3] Al-Hammadi ASA. Asymptotic theory for third-order differential equations with extension to higher odd-order equations. *Proceedings of the Royal Society of Edinburgh* 1991; 117A: 215-223.
- [4] Eastham MSP. Asymptotic theory for a critical class of fourth-order differential equations. *Proceedings of the Royal Society London* 1982; A383: 173-188.
- [5] Eastham MSP. The asymptotic solution of linear differential systems. *Mathematika* 1985; 32: 131-138.
- [6] Eastham MSP. *The Asymptotic Solution of Linear Differential Systems: Applications of Levinson Theorem*. Oxford, UK: Clarendon Press, 1989.
- [7] Eastham MSP. A repeated transformation in the asymptotic solution of linear differential systems. *Proceedings of the Royal Society of Edinburgh* 1986; 102A: 173-397.
- [8] Everitt WN, Zettl A. Generalized symmetric ordinary differential expressions I: the general theory. *Nieuw Archief voor Wiskunde* 1979; 27: 363-397.
- [9] Pfeiffer GW. Asymptotic solutions of  $y''' + qy' + ry = 0$ . *Journal of Differential Equations* 1972; 11: 145-155.
- [10] Unsworth K. Asymptotic expansions and deficiency indices associated with third-order self-adjoint differential operators. *The Quarterly Journal of Mathematics* 1973; 24 (1): 177-188.