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Generating sets of an infinite semigroup of transformations preserving a zig-zag order

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Abstract: A zig-zag order is like a directed path, only with alternating directions. A generating set of minimal size for the semigroup of all full transformations on a finite set preserving the zig-zag order was determined by Fenandes et al. in 2019. This paper deals with generating sets of the semigroup $F_{\mathbb{N}}$ of all full transformations on the set of all natural numbers preserving the zig-zag order. We prove that $F_{\mathbb{N}}$ has no minimal generating sets and present two particular infinite decreasing chains of generating sets of $F_{\mathbb{N}}$.

Key words: Fence, zig-zag order, order-preserving, generating set, transformation

1. Introduction

This paper deals with generating sets of transformation semigroups. A full transformation on a set X is a self-mapping on X. The set of all full transformations on X forms a semigroup T_X under the usual composition of mappings. If X is the n-element set $\{1, 2, \ldots, n\}$, then we write T_n rather than T_X. In particular, T_n is a finite semigroup of full transformations, which is the disjoint union of the symmetric group and the singular part Sing_n. In fact, Sing_n is an ideal of T_n consisting of all full transformations with rank $< n$. The semigroup Sing_n is generated by the idempotents of rank $n-1$ [9]. Ayik et al. found a necessary and sufficient condition for any set of full transformations with rank $n-1$ to be a generating set of Sing_n [1]. The generating sets of the ideals $K(n,r), r \in \{1, 2, \ldots, n-1\}$, of Sing_n were determined by Ayik and Bugay [3].

The set O_n of all order-preserving full transformations on $\{1, 2, \ldots, n\}$ with respect to the usual linear order on the natural numbers forms a semigroup, which is the disjoint union of the identity mapping on $\{1, 2, \ldots, n\}$ and the singular part. The minimal size of a generating set of O_n (i.e. the rank of O_n) is n while the singular part is generated by its idempotents of rank $n-1$ [6]. A necessary and sufficient condition for any set of full transformations in the ideal $O(n,r), r \in \{1, 2, \ldots, n-1\}$, to be a generating set of $O(n,r)$ was provided by Ayik and Bugay [2].

Generating sets for other (finite) semigroups of full transformations have been determined by several authors. Among these semigroups is the semigroup F_n of all full transformations on $\{1, 2, \ldots, n\}$ preserving the zig-zag order. Recall that the zig-zag order is a partial order, which is like a path, only with alternating directions. Full transformations on $\{1, 2, \ldots, n\}$ preserving the zig-zag order were first studied by Currie and

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Visentine [4] and Rutkowski [14] in 1991 and 1992, respectively. In both papers, the authors calculated the cardinality of F_n, depending on the parity of n. In [5], Fernandes, Koppitz, and Musunthia determined a generating set of F_n of minimal size and gave a formula to calculate the rank of F_n. Algebraic properties of F_n were investigated by several authors in the last decade (e.g., [10, 11, 15]).

Recall that uncountable semigroups have only uncountable generating sets. In order to make the situation more comfortable, Ruškuc introduced the concept of a relative generating set (i.e. a relative rank) [13]. For example, in [7, 8], the authors considered the uncountable semigroup $T_{\mathbb{N}}$ and the semigroup $O_{\mathbb{N}}$ of all order-preserving full transformations on the set $\mathbb{N}$ of all natural numbers with respect to the usual linear order on $\mathbb{N}$. One needs only one $\alpha \in T_{\mathbb{N}} \setminus O_{\mathbb{N}}$ such that $O_{\mathbb{N}} \cup \{\alpha\}$ generates $T_{\mathbb{N}}$, i.e. the relative rank of $T_{\mathbb{N}}$ modulo $O_{\mathbb{N}}$ is one, where $\{\alpha\}$ is said to be a relative generating set of $T_{\mathbb{N}}$ modulo $O_{\mathbb{N}}$. On the other hand, in [7], Higgins, Mitchell, and Ruškuc considered the set C of all contractions on $\mathbb{N}$ and obtained that the relative rank of $T_{\mathbb{N}}$ modulo C is uncountable. Also in [7], the authors pointed out that the relative rank of $T_{\mathbb{N}}$ modulo a so-called dominated set is uncountable.

In the present paper, we consider an extension of the zig-zag order on $\{1, 2, \ldots, n\}$ to the set of all natural numbers $\mathbb{N}$. Let

$$\begin{array}{ll} n \prec n+1 & \text{if } n \text{ is odd;} \\ n+1 \prec n & \text{otherwise.} \end{array}$$

The binary relation $\prec$ together with the diagonal on $\mathbb{N}$ is a partial order on $\mathbb{N}$, in fact, $\preceq$ is called the zig-zag order on $\mathbb{N}$. Any element in the partially ordered set $(\mathbb{N}, \preceq)$, which is called a fence, is either minimal or maximal. The set $F_{\mathbb{N}}$ of all full transformations on $\mathbb{N}$ preserving the zig-zag order forms a submonoid of $T_{\mathbb{N}}$ with the identity mapping $\mathrm{id}_{\mathbb{N}}$ on $\mathbb{N}$. Corollary 2.2. in [7] and the fact that $F_{\mathbb{N}}$ is dominated imply that the relative rank of $T_{\mathbb{N}}$ modulo $F_{\mathbb{N}}$ is uncountable infinite. In fact, the study of the semigroup $F_{\mathbb{N}}$ extends the study of F_n on another level (we have now an uncountable semigroup of full transformations). Furthermore, congruences on $F_{\mathbb{N}}$ were already determined in [12]. Hence, a more detailed study of the semigroup $F_{\mathbb{N}}$ seems reasonably enough. An investigation of generating sets of F_n will be provided in this paper.

Besides the zig-zag order $\preceq$ on $\mathbb{N}$, we also deal with the usual liner order $\leq$ on $\mathbb{N}$. Excluding any confusion, we introduce the following agreements. Let A be a nonempty subset of $\mathbb{N}$. We use $\min(A)$ and $\max(A)$ for the smallest and the greatest element (if exists), respectively, in A with respect to $\leq$. Moreover, A is said to be convex if A is an interval with respect to $\leq$. Note that the image of α (in symbols: $\operatorname{im}\alpha$) is a convex set. For $B \subseteq \mathbb{N}$, we write $A < B$ if $a < b$ for all $a \in A$ and all $b \in B$.

In the next section, we show that any transformation in $F_{\mathbb{N}}$ can be expressed as the product of one element from each of the sets

$$\Theta := \{\alpha \in F_{\mathbb{N}} : a\alpha^{-1} \text{ is a convex set for all } a \in \operatorname{im}\alpha\} \text{ and}$$

$$\Lambda_n := \{\alpha \in F_{\mathbb{N}} : |\mathrm{nb}(\alpha)| = 0,\ \mathrm{c}(\alpha) > 0, 1\alpha \geq n,\ \text{and } |\{1, 2, \ldots, n\}\alpha| = n\}$$

for any $n \in \mathbb{N}$, where

$$\mathrm{nb}(\alpha) := \{a \in \mathbb{N} : a\alpha = (a+1)\alpha\} \text{ and}$$

$$c(\alpha) := \left| \bigcup \{a\alpha^{-1} : a \in \operatorname{im}\alpha \text{ and } |a\alpha^{-1}| \geq 2\} \right|.$$

Obviously, $c(\alpha) \leq c(\alpha\beta)$ for all $\alpha, \beta \in F_{\mathbb{N}}$ and $\mathrm{c}(\alpha) = 0$ if and only if α is injective. It is worth mentioning that $F_{\mathbb{N}}$ has no minimal generating sets. The main purpose of paper is to give two particular infinite decreasing chains of generating sets of $F_{\mathbb{N}}$, which will be provided in Section 3.

Let $\alpha \in F_{\mathbb{N}}$. The rank of α, (in symbols: rank α) is the size of the image of α. Then rank α can be finite (in symbols: rank $\alpha < \aleph_0$) or countable infinite (in symbols: rank $\alpha = \aleph_0$). The set of all transformations in $F_{\mathbb{N}}$ with countable infinite rank will be denoted by $F_{\mathbb{N}}^{\text{inf}}$. For $n \in \mathbb{N}$, let $\Theta_n = \Theta \cap \Omega_n$, where

$$\Omega_n := \{\alpha \in F_{\mathbb{N}} : 1\alpha \geq n \text{ and } |\{1, 2, \ldots, n\}\alpha| = n\}.$$

Then we obtain that $\Lambda_n = \Lambda \cap \Omega_n$, where $\Lambda := \{\alpha \in F_{\mathbb{N}} : |\text{nb}(\alpha)| = 0 \text{ and } \text{c}(\alpha) > 0\}$. Just for convenience, for $\alpha \in F_{\mathbb{N}}$, we define the following sets, which will be used subsequently:

$M_\alpha^n := \{X \subseteq \mathbb{N} : |X| = n$ and X is a maximal convex set with respect to $|X\alpha| = 1\}$;
$M_\alpha := \bigcup_{n \in \mathbb{N}} M_\alpha^n$;
$M_\alpha^* := M_\alpha \setminus M_\alpha^1$;
$MS_\alpha^n := \{X \subseteq \bigcup M_\alpha^1 : X$ is a maximal convex set and $|X| = n\}$;
$MS_\alpha := \bigcup_{n \in \mathbb{N}} MS_\alpha^n$.

More in detail, a convex set $X \subseteq \mathbb{N}$ belongs to M_α^n if and only if $|X| = n, |X\alpha| = 1$, and $|Y\alpha| > 1$ for any convex set $Y \subseteq \mathbb{N}$ with $X \subsetneq Y$. Moreover, a convex set $X \subseteq \bigcup M_\alpha^1$ belongs to MS_α^n if and only if $|X| = n$ and $Y \nsubseteq \bigcup M_\alpha^1$ for any convex set $Y \subseteq \mathbb{N}$ with $X \subsetneq Y$. For any $\beta \in F_{\mathbb{N}}$, it is clear that $M_\alpha = M_\beta$ if and only if $M_\alpha^* = M_\beta^*$.
Further, let $C_m := \{X : X \subseteq \{m, m+1, \ldots\}\}$ for all $m \in \mathbb{N}$.

2. On minimal generating sets of $F_{\mathbb{N}}$

First, we describe any transformation α in $F_{\mathbb{N}}$, that is, α preserves the partial order $\preceq$ on $\mathbb{N}$. If $x, y \in \mathbb{N}$ with $x \prec y$, then x is odd and y is even. Moreover, x is the successor of y or conversely y is the successor of x, which implies $|x - y| = 1$. When we apply α to both x and y, their images are related with respect to $\preceq$, that is, $|x\alpha - y\alpha| \leq 1$. This fact will be used subsequently without mentioning. Now, we characterize the elements of $F_{\mathbb{N}}$ by two properties, which are easy to verify.

Proposition 2.1 *Let $\alpha \in T_{\mathbb{N}}$. Then $\alpha \in F_{\mathbb{N}}$ if and only if*

(i) $|x\alpha - (x+1)\alpha| \leq 1$ *for all* $x \in \mathbb{N}$;

(ii) x *and* $x\alpha$ *have the same parity or* $(x-1)\alpha = x\alpha = (x+1)\alpha$ *for all* $x \in \mathbb{N} \setminus \{1\}$.

Proof Suppose $\alpha \in F_{\mathbb{N}}$.
(i) Let $x \in \mathbb{N}$. Then $x \prec x+1$ or $x+1 \prec x$. Since $\alpha \in F_{\mathbb{N}}$, we obtain $x\alpha \preceq (x+1)\alpha$ and $(x+1)\alpha \preceq x\alpha$, respectively. Then $|x\alpha - (x+1)\alpha| \leq 1$.
(ii) Suppose that there exists $x \in \mathbb{N} \setminus \{1\}$ such that x and $x\alpha$ have different parities. Without loss of generality, suppose that x is odd and $x\alpha$ is even. Assume $(x-1)\alpha \neq x\alpha$. Then (i) implies $(x-1)\alpha \in \{x\alpha - 1, x\alpha + 1\}$. It follows that $(x-1)\alpha$ is odd. This shows that $x \prec x-1$ but $(x-1)\alpha \prec x\alpha$, that is, $\alpha \notin F_{\mathbb{N}}$, a contradiction. Hence, $(x-1)\alpha = x\alpha$. Similarly, we can show that $(x+1)\alpha = x\alpha$.

Conversely, suppose that (i) and (ii) hold. Let $x, y \in \mathbb{N}$ be such that $x \prec y$. Then x is odd and y is even with $x \in \{y-1, y+1\}$. By (i), we obtain $|x\alpha - y\alpha| \leq 1$. It is enough to consider the case $|x\alpha - y\alpha| = 1$. Since $x \in \{y-1, y+1\}$ and $|x\alpha - y\alpha| = 1$, we obtain that y and $y\alpha$ are even by (ii) and so $x\alpha \prec y\alpha$. Altogether, we conclude $x\alpha \preceq y\alpha$. Therefore, $\alpha \in F_{\mathbb{N}}$. □

An immediate consequence of Proposition 2.1 is that $|A|$ is odd for all $A \in M_\alpha^*$ with $1 \notin A$. In the following, we will use this fact as well as Proposition 2.1 without further mentioning. Any element in $F_{\mathbb{N}}$ can be described as the product of one element from each of the sets Θ and Λ_n for any $n \in \mathbb{N}$.

Proposition 2.2 $F_{\mathbb{N}} = \Theta\Lambda_n = \{\gamma_1\gamma_2 : \gamma_1 \in \Theta, \gamma_2 \in \Lambda_n\}$ *for all* $n \in \mathbb{N}$.

Proof Let $n \in \mathbb{N}$ and $\alpha \in F_{\mathbb{N}}$. Then we consider the following two cases.
Case 1: $|M_\alpha| = \aleph_0$. Suppose $M_\alpha = \{A_i : i \in \mathbb{N}\}$ with $A_i < A_{i+1}$ for all $i \in \mathbb{N}$. Then $|A_i| < \aleph_0$ for all $i \in \mathbb{N}$. For all $i \in \mathbb{N}$, let $m_i = \max(A_i)$. This means $A_i\alpha = \{m_i\alpha\}$ for all $i \in \mathbb{N}$. Obviously, $\alpha \in F_{\mathbb{N}}$ and $|A_i\alpha| = 1$ for all $i \in \mathbb{N}$ imply that for all $i \in \mathbb{N}$,

$$m_i \text{ and } m_i\alpha \text{ have the same parity and } |m_i\alpha - m_{i+1}\alpha| = 1. \tag{2.1}$$

Let $k \in \mathbb{N} \setminus \{1, 2, \ldots, n\}$ be such that k and $m_1\alpha$ have the same parity. We define $\gamma_1 : \mathbb{N} \to \mathbb{N}$ by

$$x\gamma_1 := k + i - 1 \text{ for all } x \in A_i, i \in \mathbb{N}.$$

The transformation γ_1 is well defined since $\bigcup_{i\in\mathbb{N}} A_i = \mathbb{N}$. Moreover, $A_i\gamma_1 = \{k + i - 1\}$ for all $i \in \mathbb{N}$ and thus, $M_{\gamma_1} = M_\alpha$. It is clear that $|x\gamma_1 - (x+1)\gamma_1| \leq 1$ for all $x \in \mathbb{N}$. Since k and $m_1\alpha$ have the same parity and $M_{\gamma_1} = M_\alpha$, we obtain that x and $x\gamma_1$ have the same parity or $(x-1)\gamma_1 = x\gamma_1 = (x+1)\gamma_1$ for all $x \in \mathbb{N} \setminus \{1\}$. Since $y\gamma_1^{-1}$ is a convex set for all $y \in \operatorname{im} \gamma_1$, we obtain $\gamma_1 \in \Theta$. Further, we define $\gamma_2 : \mathbb{N} \to \mathbb{N}$ by

$$x\gamma_2 := \begin{cases} m_1\alpha + k - x & \text{if } x \in \{1, 2, \ldots, k-1\}; \\ m_{x-k+1}\alpha & \text{if } x \in \{k, k+1, \ldots\}. \end{cases}$$

By (2.1) and the fact that k and $m_1\alpha$ have the same parity, we can conclude that (i) and (ii) in Proposition 2.1 are satisfied for γ_2, that is, $\gamma_2 \in F_{\mathbb{N}}$. If rank $\alpha = \aleph_0$, then there exists $y \in \{m_2\alpha, m_3\alpha, \ldots\}$ with $y = m_1\alpha + 1$, that is, γ_2 is not injective. If rank $\alpha < \aleph_0$, then it is clear that γ_2 is not injective. Moreover, we have $|\text{nb}(\gamma_2)| = 0, |\{1, 2, \ldots, n\}\gamma_2| = n$, and $1\gamma_2 = m_1\alpha + k - 1 \geq k > n$. Thus, $\gamma_2 \in \Lambda_n$. By straightforward calculations, we obtain $A_i\gamma_1\gamma_2 = \{m_i\alpha\}$ for all $i \in \mathbb{N}$. This shows $\gamma_1\gamma_2 = \alpha$.

Case 2: $|M_\alpha| < \aleph_0$. Suppose $M_\alpha = \{A_i : 1 \leq i \leq l\}$ for some $l \in \mathbb{N}$ with $A_i < A_j$ for all $1 \leq i < j \leq l$. Then $|A_i| < \aleph_0$ for all $i \in \mathbb{N} \setminus \{l, l+1, \ldots\}$ and $|A_l| = \aleph_0$. Let $m_i = \max(A_i)$ for all $i \in \mathbb{N} \setminus \{l, l+1, \ldots\}$ and $m_l = \min(A_l)$. Then $A_i\alpha = \{m_i\alpha\}$ for all $i \in \{1, 2, \ldots, l\}$. Since $\alpha \in F_{\mathbb{N}}$ and $|A_i\alpha| = 1$ for all $1 \leq i \leq l$, the following properties hold:

(a1) $|m_i\alpha - m_{i+1}\alpha| = 1$ for all $i \in \mathbb{N} \setminus \{l, l+1, \ldots\}$;

(a2) m_i and $m_i\alpha$ have the same parity for all $1 \leq i \leq l$, whenever $l > 1$.

Let $k \in \mathbb{N} \setminus \{1, 2, \dots, n\}$ be such that k and $m_1\alpha$ have the same parity. Then we define $\gamma_1 : \mathbb{N} \to \mathbb{N}$ by

$$x\gamma_1 := k + i - 1 \text{ for all } x \in A_i, 1 \leq i \leq l.$$

The transformation γ_1 is well defined since $\bigcup_{i\in\mathbb{N}} A_i = \mathbb{N}$. Moreover, $A_i\gamma_1 = \{k+i-1\}$ for all $1 \leq i \leq l$. Using the same arguments as in Case 1, we get $\gamma_1 \in F_\mathbb{N}$. Since $y\gamma_1^{-1}$ is a convex set for all $y \in \text{im } \gamma_1$, we have $\gamma_1 \in \Theta$. Further, let $\gamma_2 : \mathbb{N} \to \mathbb{N}$ by

$$x\gamma_2 := \begin{cases} m_1\alpha + k - x & \text{if } x \in \{1, 2, \dots, k-1\}; \\ m_{x-k+1}\alpha & \text{if } x \in \{k, k+1, \dots, k+l-1\}; \\ m_l\alpha + x - k - l + 1 & \text{if } x \in \{k+l, k+l+1, \dots\}. \end{cases}$$

By (a1), we have $|x\gamma_2 - (x+1)\gamma_2| \leq 1$ for all $x \in \mathbb{N}$. Moreover, x and $x\gamma_2$ have the same parity for all $x \in \mathbb{N}$ by (a2) and the property of k. Hence, $\gamma_2 \in F_\mathbb{N}$. Since $\text{im } \gamma_2 = \{m_1\alpha, \dots, m_l\alpha, m_l\alpha + 1, m_l\alpha + 2, \dots\}$ is a convex set, $\text{rank } \gamma_2 = \aleph_0$, and $k\gamma_2 = m_1\alpha$, there exists $y \in \{k+1, k+2, \dots\}$ such that $y\gamma_2 = m_1\alpha + 1$. Since $(k-1)\gamma_2 = m_1\alpha + 1 = y\gamma_2$ and $k - 1 \neq y$, the transformation γ_2 is not injective. Moreover, $|\text{nb}(\gamma_2)| = 0, |\{1, 2, \dots, n\}\gamma_2| = n$, and $1\gamma_2 = m_1\alpha + k - 1 \geq k > n$. Hence, $\gamma_2 \in \Lambda_n$. By straightforward calculations, we obtain $A_i\gamma_1\gamma_2 = \{m_i\alpha\}$ for all $1 \leq i \leq l$. Therefore, $\gamma_1\gamma_2 = \alpha$.

Altogether, we have shown $F_\mathbb{N} \subseteq \Theta\Lambda_n$. Since the converse inclusion is clear, we have $\Theta\Lambda_n = F_\mathbb{N}$. □

By the construction of γ_1 in Proposition 2.2, we observe that the only conditions for γ_1 are $M_\alpha = M_{\gamma_1}$ and $\min(\text{im } \gamma_1) \geq n$. This gives us the following corollary.

Corollary 2.3 *Let $n \in \mathbb{N}$ and $\alpha \in F_\mathbb{N}$. For $\gamma_1 \in \Theta$ with $M_\alpha = M_{\gamma_1}$ and $\min(\text{im } \gamma_1) \geq n$, there exists $\gamma_2 \in \Lambda_n$ such that $\alpha = \gamma_1\gamma_2$.*

As one can see, $F_\mathbb{N}$ is uncountable and thus, any generating set of $F_\mathbb{N}$ is uncountable. It appears the question whether a minimal generating set of $F_\mathbb{N}$ exists. The following constructions clarify that there are no minimal generating sets of $F_\mathbb{N}$, that is to say, we can get a smaller generating set (under the set inclusion) by excluding suitable elements from a given generating set.

Let $\alpha \in F_\mathbb{N}^{\text{inf}}$, $R_\alpha := \{x \in \text{im } \alpha : x\alpha^{-1} \text{ is not a convex set}\}$, and $Q_\alpha := \{x \in \text{im } \alpha : \left|x\alpha^{-1}\right|, \left|(x+1)\alpha^{-1}\right| \geq 3\}$. Further, let $P := \{\alpha \in F_\mathbb{N}^{\text{inf}} : \left|\bigcup_{n>3} M_\alpha^n\right|, |R_\alpha|, |Q_\alpha| < \aleph_0\}$. For $l \in \mathbb{N}$, let

$$K_l := \{\alpha \in P : \left|MS_\alpha^l\right| = \aleph_0 \text{ and } |MS_\alpha^n| < \aleph_0 \text{ for all } n < l\}.$$

Note that $|M_\alpha^*| = \aleph_0$ for all $\alpha \in K_l$. Further, let $K_{\aleph_0} := P \setminus \bigcup_{n\in\mathbb{N}} K_n$.

Lemma 2.4 *Let $\alpha \in F_\mathbb{N}^{\text{inf}}$ with $|R_\alpha| < \aleph_0$. Then there is $k \in \mathbb{N}$ such that $a\alpha \leq b\alpha$ for all $k \leq a < b$.*

Proof Since $|R_\alpha| < \aleph_0$, there is $k' \in \mathbb{N}$ such that $x\alpha^{-1}$ is a convex set for all $x \geq k'$. Let $k = \min(k'\alpha^{-1})$ and let $a, b \in \mathbb{N}$ with $k \leq a < b$. Assume that $a\alpha < k'$, i.e. $k < a$. Then $\text{rank } \alpha = \aleph_0$ implies that $\{a, a+1, \dots\}\alpha$ is an infinite convex set containing k', that is, there is $s > a$ with $s\alpha = k'$. Thus, $k'\alpha^{-1}$ is not a convex set because $k < a < s$, where $s, k \in k'\alpha^{-1}$ and $a \notin k'\alpha^{-1}$, a contradiction. Hence, $k' \leq a\alpha$. Assume $b\alpha < a\alpha$. Then $\text{rank } \alpha = \aleph_0$ implies that $\{b, b+1, \dots\}\alpha$ is an infinite convex set containing $a\alpha$, that is, there exists $t \in \mathbb{N}$

with $b < t$ and $t\alpha = a\alpha$. This means that $(a\alpha)\alpha^{-1}$ is not a convex set since $a < b < t$, where $a, t \in (a\alpha)\alpha^{-1}$ and $b \notin (a\alpha)\alpha^{-1}$, a contradiction to $k' \leq a\alpha$. Therefore, $a\alpha \leq b\alpha$. □

As a consequence of Lemma 2.4, we obtain that $\alpha|_B$ is injective for all $B \in MS_\alpha \cap C_k$.

Lemma 2.5 *Let $\alpha, \beta \in F_{\mathbb{N}}^{\text{inf}}$ and let $x \in R_\beta$ be such that $x\beta^{-1} \cap \text{im } \alpha$ is not a convex set. Then $x \in R_{\alpha\beta}$.*

Proof Assume $x \notin R_{\alpha\beta}$. This means that $x(\alpha\beta)^{-1} = x\beta^{-1}\alpha^{-1}$ is a convex set. Then $x\beta^{-1}\alpha^{-1}\alpha$ is a convex set. But $x\beta^{-1}\alpha^{-1}\alpha = x\beta^{-1} \cap \text{im } \alpha$, a contradiction. Hence, $x \in R_{\alpha\beta}$. □

Lemma 2.6 *Let $\beta \in F_{\mathbb{N}}^{\text{inf}}$ and let $X \subseteq \mathbb{N}$ be such that $|X| = \aleph_0$ and $|X\beta| < \aleph_0$. Then $|R_\beta| = \aleph_0$. Moreover, $|R_{\alpha\beta}| = \aleph_0$ for all $\alpha \in F_{\mathbb{N}}^{\text{inf}}$.*

Proof Assume $|R_\beta| < \aleph_0$. By Lemma 2.4, there is $k \in \mathbb{N}$ with $a\alpha \leq b\alpha$ for all $k \leq a < b$. Let $B = \{x \in X : x \geq k\}$ and $c = \max(B\beta)$. Then $|B| = \aleph_0$. Let $t \in \mathbb{N}$ with $t \geq k$. Since $|B| = \aleph_0$, there is $s \in B$ such that $t < s$. Then $t\beta \leq s\beta \leq c$. This implies that rank $\beta \leq k + c < \aleph_0$, a contradiction. Hence, $|R_\beta| = \aleph_0$ and so $\left|\{x \in R_\beta : x\beta^{-1} \subseteq \text{im } \alpha\}\right| = \aleph_0$. Therefore, $|R_{\alpha\beta}| = \aleph_0$ by Lemma 2.5. □

Proposition 2.7 *$F_{\mathbb{N}} \setminus P$ is an ideal of $F_{\mathbb{N}}$.*

Proof Let $\alpha \in F_{\mathbb{N}} \setminus P$ and $\beta \in F_{\mathbb{N}}$. If rank $\alpha < \aleph_0$ or rank $\beta < \aleph_0$, then we obtain that rank $\alpha\beta$, rank $\beta\alpha < \aleph_0$, that is, $\alpha\beta, \beta\alpha \in F_{\mathbb{N}} \setminus P$. Suppose now rank $\alpha =$ rank $\beta = \aleph_0$. Since im α and im β are convex sets, we have that rank $\alpha\beta = \aleph_0$ and rank $\beta\alpha = \aleph_0$, respectively. Let $M_\beta = \{B_i : i \in \mathbb{N}\}$ with $B_i < B_{i+1}$ for all $i \in \mathbb{N}$.

Case 1: $|R_\alpha| = \aleph_0$. Suppose that $R_\alpha = \{x_i : i \in \mathbb{N}\}$ with $x_i < x_{i+1}$ for all $i \in \mathbb{N}$. Let r be the least $q \in \mathbb{N}$ with $\min(\text{im } \beta) \leq \min(x_q\alpha^{-1})$ and let $E = \{x_i : i \geq r\}$. Then $x\alpha^{-1} \subseteq \text{im } \beta$ for all $x \in E$. Therefore, Lemma 2.5 implies that $x \in R_{\beta\alpha}$ and so $E \subseteq R_{\beta\alpha}$. Hence, $|R_{\beta\alpha}| \geq |E| = \aleph_0$.

Suppose $|R_{\alpha\beta}| < \aleph_0$. Then there is $k \in \mathbb{N}$ such that $x\beta^{-1}\alpha^{-1}$ is a convex set for all $x \geq k$. Moreover, $|R_\alpha\beta| = \aleph_0$. Otherwise $|R_\alpha\beta| < \aleph_0$ and so Lemma 2.6 implies $|R_{\alpha\beta}| = \aleph_0$, a contradiction. Therefore, $|R_\alpha\beta \cap \{k, k+1, \ldots\}| = \aleph_0$. Let s be the least $q \in \mathbb{N}$ such that $\min(\text{im } \alpha) < \min(x_q\beta\beta^{-1})$ and let $D = \{x_i : i \geq s\}\beta \cap \{k, k+1, \ldots\}$. Let $x \in D$. Then $x\beta^{-1}\alpha^{-1}$ is a convex set and $x\beta^{-1} \cap R_\alpha \neq \emptyset$. Suppose that $x_j \in x\beta^{-1} \cap R_\alpha$ for some $j \in \mathbb{N}$. If $x\beta^{-1} \cap \text{im } \alpha = \{x_j\}$, then $x\beta^{-1}\alpha^{-1} = x_j\alpha^{-1}$ is not a convex set, a contradiction. Thus, $\left|x\beta^{-1} \cap \text{im } \alpha\right| \geq 3$. Since $x_j\alpha^{-1}$ is not a convex set, we obtain $\left|x_j\alpha^{-1}\right| \geq 2$. Hence, $\left|x\beta^{-1}\alpha^{-1}\right| > 3$. Therefore, $\left|\bigcup_{n>3} M_{\alpha\beta}^n\right| \geq |D| = \aleph_0$.

Case 2: $\left|\bigcup_{n>3} M_\alpha^n\right| = \aleph_0$ and $|R_\alpha| < \aleph_0$. Let $\bigcup_{n>3} M_\alpha^n = \{A_i : i \in \mathbb{N}\}$ with $A_i < A_{i+1}$ for all $i \in \mathbb{N}$. Let r be the least $q \in \mathbb{N}$ such that $\min(\text{im } \beta) \leq \min(A_q)$. Then for $i \geq r$, there is $m_i \in \mathbb{N}$ with $\left(\bigcup_{j=m_i}^{m_i+|A_i|-1} B_j\right)\beta \subseteq A_i$. Hence, there is $D_i \in M_{\beta\alpha}$ with $\left(\bigcup_{j=m_i}^{m_i+|A_i|-1} B_j\right) \subseteq D_i$. Then $|D_i| \geq \left|\bigcup_{j=m_i}^{m_i+|A_i|-1} B_j\right| \geq |A_i| > 3$. This shows that $\left|\bigcup_{n>3} M_{\beta\alpha}^n\right| \geq \left|\{D_i \in M_{\beta\alpha} : \left(\bigcup_{j=m_i}^{m_i+|A_i|-1} B_j\right) \subseteq D_i\}\right| = |\{i \in \mathbb{N} : i \geq r\}| = \aleph_0$.

If $\left|\left(\bigcup_{i\in\mathbb{N}} A_i\right)\alpha\beta\right| = \aleph_0$, then we obtain $\left|\bigcup_{n>3} M_{\alpha\beta}^n\right| = \aleph_0$. Suppose now that $\left|\left(\bigcup_{i\in\mathbb{N}} A_i\right)\alpha\beta\right| < \aleph_0$. Assume $\left|\left(\bigcup_{i\in\mathbb{N}} A_i\right)\alpha\right| < \aleph_0$. Let $X = \{\min(A_i) : i \in \mathbb{N}\}$. Then $|X| = \aleph_0$ and $|X\alpha| < \aleph_0$. So, Lemma 2.6

implies that $|R_\alpha| = \aleph_0$, a contradiction. Hence, $\left|\left(\bigcup_{i\in\mathbb{N}} A_i\right)\alpha\right| = \aleph_0$. Then $|R_{\alpha\beta}| = \aleph_0$ by Lemma 2.6.

Case 3: $|Q_\alpha| = \aleph_0$. Then $|Q_\alpha \cap \operatorname{im}\, \beta\alpha| = \aleph_0$ since rank $\beta\alpha = \aleph_0$. This implies that $|Q_{\beta\alpha}| = \aleph_0$.

Suppose that $|Q_{\alpha\beta}|, |R_{\alpha\beta}| < \aleph_0$. Then $|Q_\alpha\beta| = \aleph_0$. Otherwise $|Q_\alpha\beta| < \aleph_0$ and so Lemma 2.6 implies $|R_{\alpha\beta}| = \aleph_0$, a contradiction. Let $Q_\alpha = \{x_i : i \in \mathbb{N}\}$ with $x_i < x_{i+1}$ for all $i \in \mathbb{N}$. Since $|Q_{\alpha\beta}|, |R_{\alpha\beta}| < \aleph_0$, there is $k \in \mathbb{N}$ such that $x\beta^{-1}\alpha^{-1}$ is a convex set, and $\left|x\beta^{-1}\alpha^{-1}\right| < 3$ or $\left|(x+1)\beta^{-1}\alpha^{-1}\right| < 3$ for all $x \geq k$. Then $|Q_\alpha\beta \cap \{k, k+1, \ldots\}| = \aleph_0$ since $|Q_\alpha\beta| = \aleph_0$. Let $D = Q_\alpha\beta \cap \{k, k+1, \ldots\}$ and let $x \in D$. Then there is $s \in Q_\alpha$ such that $s\beta = x$. Since $s \in Q_\alpha$, we obtain that $\left|s\alpha^{-1}\right|, \left|(s+1)\alpha^{-1}\right| \geq 3$. Assume that $(s+1)\beta \neq x$. Then $(s+1)\beta = x+1$. Otherwise, $(s+1)\beta = x-1$ and thus, there is $t > s+1$ with $t\beta = x$. Hence, $x\beta^{-1} \cap \operatorname{im}\, \alpha$ is not a convex set. Lemma 2.5 implies that $x\beta^{-1}\alpha^{-1}$ is not a convex set, a contradiction to $x \geq k$. Thus, $\left|x\beta^{-1}\alpha^{-1}\right| \geq \left|s\alpha^{-1}\right| \geq 3$ and $\left|(x+1)\beta^{-1}\alpha^{-1}\right| \geq \left|(s+1)\alpha^{-1}\right| \geq 3$, a contradiction to $x \in D$. Hence, $x = s\beta = (s+1)\beta$, that is, $\left|x\beta^{-1}\alpha^{-1}\right| \geq \left|\{s, s+1\}\alpha^{-1}\right| \geq 6$ and so $x\beta^{-1}\alpha^{-1} \in \bigcup_{n>3} M^n_{\alpha\beta}$. Therefore, $\left|\bigcup_{n>3} M^n_{\alpha\beta}\right| \geq |D| = \aleph_0$.

For all three cases, we obtain that $\alpha\beta, \beta\alpha \notin P$. Therefore, we can conclude that $F_\mathbb{N} \setminus P$ is an ideal of $F_\mathbb{N}$.

□

Lemma 2.8 *Let $\alpha \in K_l$ for some $l \in \mathbb{N}$ and let G be a generating set of $F_\mathbb{N}$. Then there are $\gamma_1 \in K_{l_1} \cup K_{\aleph_0}$ and $\gamma_2 \in K_{l_2} \cup K_{\aleph_0}$ for some $l_1, l_2 \in \mathbb{N}$ with $l_1, l_2 > l$ such that $\alpha = \gamma_1\gamma_2$ and $\gamma_1, \gamma_2 \in \langle G \setminus \{\alpha\}\rangle$.*

Proof Since $\alpha \in K_l$, we have $|M^*_\alpha| = \aleph_0$. Suppose that $M^*_\alpha = \{B_i : i \in \mathbb{N}\}$ with $B_i < B_{i+1}$ for all $i \in \mathbb{N}$. Let $\gamma_1 \in \Theta$ be such that im $\gamma_1 = \mathbb{N}$ and $M^*_{\gamma_1} = \{B_i : i \in 2\mathbb{N}\}$. Note that such a γ_1 exists.

Moreover, we define $\gamma_2 : \mathbb{N} \to \mathbb{N}$ by $x\gamma_2 := (\min(x\gamma_1^{-1}))\alpha$ for all $x \in \mathbb{N}$. Let $a, b \in \mathbb{N}$ be such that $a \prec b$. Then a is odd and b is even. Furthermore, $b = a+1$ or $a = b+1$. Suppose now $b = a+1$. Since $\gamma_1 \in \Theta$, we obtain that $\max(a\gamma_1^{-1})$ is odd and $\min(b\gamma_1^{-1})$ is even such that $\max(a\gamma_1^{-1}) + 1 = \min(b\gamma_1^{-1})$. Then $\alpha \in F_\mathbb{N}$ implies that $\max(a\gamma_1^{-1})\alpha \preceq \min(b\gamma_1^{-1})\alpha$. Since $M^*_{\gamma_1} \subseteq M^*_\alpha$, it follows that $\min(a\gamma_1^{-1})\alpha = \max(a\gamma_1^{-1})\alpha$. Hence, $\min(a\gamma_1^{-1})\alpha \preceq \min(b\gamma_1^{-1})\alpha$, that is, $a\gamma_2 \preceq b\gamma_2$. We can show similarly for the case $a = b+1$. Therefore, $\gamma_2 \in F_\mathbb{N}$.

By the definitions of γ_1 and γ_2, it is clear that $\gamma_1\gamma_2 = \alpha$ and that there exist $l_1, l_2 > l$ such that $\gamma_1 \in K_{l_1} \cup K_{\aleph_0}$ and $\gamma_2 \in K_{l_2} \cup K_{\aleph_0}$. Hence, for $i \in \{1, 2\}$, there is $k_i \in \mathbb{N}$ satisfying the following properties:

(a1) $|A| \geq l_i > l$ for all $A \in MS_{\gamma_i} \cap C_{k_i}$;

(a2) $|A| = 3$ for all $A \in M^*_{\gamma_i} \cap C_{k_i}$;

(a3) $\left|x\gamma_i^{-1}\right| < 3$ or $\left|(x+1)\gamma_i^{-1}\right| < 3$ for all $x \geq k_i\gamma_i$;

(a4) $x\gamma_i^{-1}$ is a convex set for all $x \geq k_i\gamma_i$

because $\left|\bigcup_{n=1}^{l_i-1} MS^n_{\gamma_i}\right| < \aleph_0$ with $l_i > l$, $\left|\bigcup_{n>3} M^n_{\gamma_i}\right| < \aleph_0, |Q_{\gamma_i}| < \aleph_0$, and $|R_{\gamma_i}| < \aleph_0$, respectively. It is a consequence of (a4) that $a\gamma_i \leq b\gamma_i$ for all $k_i \leq a < b$, which we will use without further mentioning. Since $\alpha \in K_l$, there is $k \in \mathbb{N}$ satisfying the following properties:

(b1) $\left|MS^l_\alpha \cap C_k\right| = \aleph_0$;

(b2) $|A| = 3$ for all $A \in M^*_\alpha \cap C_k$

because $\left|MS^l_\alpha\right| = \aleph_0$ and $\left|\bigcup_{n>3} M^n_\alpha\right| < \aleph_0$, respectively. Since $\langle G\rangle = F_\mathbb{N}$ and $\gamma_1, \gamma_2 \in P$, there are $\mu_1, \mu_2, \ldots, \mu_{m_1}, \eta_1, \eta_2, \ldots, \eta_{m_2} \in G \cap P$ such that $\gamma_1 = \mu_1\mu_2\cdots\mu_{m_1}$ and $\gamma_2 = \eta_1\eta_2\cdots\eta_{m_2}$ for some $m_1, m_2 \in \mathbb{N}$. By (a1) and (b1), it is clear that $\mu_1 \neq \alpha$ and $\eta_1 \neq \alpha$.

Assume that $\mu_j = \alpha$ for some $j \in \{2, 3, \ldots, m_1\}$. Let $MS^{l,k}_\alpha = \{A \in MS^l_\alpha : \{k\} < A\} = \{A_i : i \in \mathbb{N}\}$ with $A_i < A_{i+1}$ for all $i \in \mathbb{N}$. Let $\delta_1 = \mu_1\mu_2\cdots\mu_{j-1}$. Further, let $\delta_2 = \mu_{j+1}\mu_{j+2}\cdots\mu_{m_1}$ if $j < m_1$ and let $\delta_2 = \mathrm{id}_\mathbb{N}$ if $j = m_1$. Note that $\mathrm{id}_\mathbb{N} \in P$. Let $x \in \mathbb{N}$ be such that $x > k_1 + 3$ and $x\delta_1 \in \{\min(A) : A \in MS^{l,k}_\alpha \setminus \{A_1\}\}$. Then $x\delta_1 = \min(A_r)$ for some $r \geq 2$ and so $A_r = \{x\delta_1, x\delta_1 + 1, \ldots, x\delta_1 + l - 1\}$. So, (b2) implies that $B_1 = \{x\delta_1 - 3, x\delta_1 - 2, x\delta_1 - 1\}, B_2 = \{x\delta_1 + l, x\delta_1 + l + 1, x\delta_1 + l + 2\} \in M_\alpha$. Note that $k < x - 3$.

Since $\{x-3, x-2, x-1, x\}\delta_1$ is a convex set containing $x\delta_1$, we get that $\{x-3, x-2, x-1\}\delta_1 \subseteq B_1$ and so $\{x-3, x-2, x-1\} \subseteq (x-1)\delta_1\alpha\delta_2(\delta_1\alpha\delta_2)^{-1}$. We obtain the equality $\{x-3, x-2, x-1\} = (x-1)\delta_1\alpha\delta_2(\delta_1\alpha\delta_2)^{-1}$ by (a2). Let $D = \{x, x+1, \ldots, x + l_1 - 1\}$. Note that $z\gamma_1\gamma_1^{-1}$ is a convex set for all $z \in D$. By (a3), we can conclude that $\left|x\delta_1\alpha\delta_2(\delta_1\alpha\delta_2)^{-1}\right| = \left|x\gamma_1\gamma_1^{-1}\right| = 1$. Let $A = \{X \in M^*_{\gamma_1} : X \subseteq D \setminus \{x\}\}$. Assume that $A \neq \emptyset$. Then there is $E \in A$ with $E \leq X$ for all $X \in A$. Then $\{x, x+1, \ldots, \min(E) - 1\} \in \bigcup_{n=1}^{l_1-1} MS^n_{\delta_1\alpha\delta_2}$, a contradiction. This implies that $\delta_1|_D$ is injective with $z\delta_1 = x\delta_1 + z - x$ for all $z \in D$. Since $l_1 > l$, we have $x + l \in D$ with $(x+l)\delta_1\alpha\alpha^{-1} = (x\delta_1 + l)\alpha\alpha^{-1} = B_2$. Then $(x+l)\gamma_1\gamma_1^{-1} = (x+l)\delta_1\alpha\delta_2(\delta_1\alpha\delta_2)^{-1} = (x\delta_1 + l)\alpha\delta_2\delta_2^{-1}\alpha^{-1}\delta_1^{-1} \supseteq (x\delta_1 + l)\alpha\alpha^{-1}\delta_1^{-1} = B_2\delta_1^{-1}$. Therefore, $\left|(x+l)\gamma_1\gamma_1^{-1}\right| \geq \left|B_2\delta_1^{-1}\right| \geq |B_2| = 3$, a contradiction. Therefore, we conclude that $\mu_j \neq \alpha$ for all $j \in \{1, 2, \ldots, m_1\}$. Similarly, we can show that $\eta_j \neq \alpha$ for all $j \in \{1, 2, \ldots, m_2\}$. So, $\gamma_1, \gamma_2 \in \langle G \setminus \{\alpha\}\rangle$. □

In particular, Lemma 2.8 shows that G has no common elements to K_l for all $l \in \mathbb{N}$, whenever G is a minimal generating set of $F_\mathbb{N}$. The main result of this section states that there are no minimal generating sets of $F_\mathbb{N}$. If such a one existed, it would have the following necessary condition.

Lemma 2.9 *If G is a minimal generating set of $F_\mathbb{N}$, then $G \cap K_n = \emptyset$ for all $n \in \mathbb{N}$. Moreover, $G \cap P \subseteq K_{\aleph_0}$.*

Proof Assume $G \cap K_l \neq \emptyset$ for some $l \in \mathbb{N}$. Then there exists $\alpha \in G \cap K_l$. By Lemma 2.8, there are $\gamma_1, \gamma_2 \in \langle G \setminus \{\alpha\}\rangle$ with $\alpha = \gamma_1\gamma_2$, that is, $\alpha \in \langle G \setminus \{\alpha\}\rangle$. Since $\langle G\rangle = F_\mathbb{N}$, we obtain $\langle G \setminus \{\alpha\}\rangle = F_\mathbb{N}$. It contradicts to the assumption that G is a minimal generating set of $F_\mathbb{N}$. Therefore, $G \cap K_n = \emptyset$ for all $n \in \mathbb{N}$. Together with $P = \left(\bigcup_{n\in\mathbb{N}} K_n\right) \cup K_{\aleph_0}$, we obtain that $G \cap P = G \cap \left(\left(\bigcup_{n\in\mathbb{N}} K_n\right) \cup K_{\aleph_0}\right) = G \cap K_{\aleph_0} \subseteq K_{\aleph_0}$. □

Theorem 2.10 *There are no minimal generating sets of $F_\mathbb{N}$.*

Proof Assume that there is a minimal generating set G of $F_\mathbb{N}$. By Lemma 2.9, we have $G \cap K_n = \emptyset$ for all $n \in \mathbb{N}$. Now, we define $\alpha : \mathbb{N} \to \mathbb{N}$ by

$$x\alpha := \begin{cases} 2n - 1 & \text{if } x = 4n - 3 \text{ for } n \in \mathbb{N}; \\ 2n & \text{if } x \in \{4n-2, 4n-1, 4n\} \text{ for } n \in \mathbb{N}. \end{cases}$$

Then $M^*_\alpha = \{\{4n-2, 4n-1, 4n\} : n \in \mathbb{N}\}$. It is clear that $\alpha \in P$ since $R_\alpha = Q_\alpha = \bigcup_{n>3} M^n_\alpha = \emptyset$. Since $\alpha \in P$ and $\langle G \rangle = F_\mathbb{N}$, Lemma 2.9 implies that $\alpha = \gamma_1\gamma_2\cdots\gamma_l$ for some $\gamma_1, \gamma_2, \ldots, \gamma_l \in G \cap P \subseteq K_{\aleph_0}$ and for some $l \in \mathbb{N}$. Let $\gamma_0 = \mathrm{id}_\mathbb{N}$ and let $i \in \{1, 2, \ldots, l\}$. Since $\alpha = \gamma_1\gamma_2\cdots\gamma_l$, we obtain the following properties:

(a1) $a\gamma_i \leq b\gamma_i$ for all $1\gamma_0\gamma_1\cdots\gamma_{i-1} \leq a < b$;

(a2) $|B| = 3$ for all $B \in M^*_{\gamma_i} \cap C_{1\gamma_0\gamma_1\cdots\gamma_{i-1}}$

because $R_\alpha = \emptyset$ and $M^*_\alpha = M^3_\alpha$, respectively. Moreover, (a1) provides

(a3) $\gamma_i|_A$ is injective for all $A \in MS_{\gamma_i} \cap C_{1\gamma_0\gamma_1\cdots\gamma_{i-1}}$.

Let $a_l = 2$ and $a_{l-j} = 2a_{l-j+1} + 3$ for all $j \in \mathbb{N} \setminus \{l, l+1, \ldots\}$. Since $\gamma_i \in K_{\aleph_0}$, there exists $m_i \in \mathbb{N}$ such that $|C| \geq a_i$ for all $C \in MS_{\gamma_i} \cap C_{m_i}$. Let $m^* = \max\{1\gamma_1, 1\gamma_1\gamma_2, \ldots, 1\gamma_1\gamma_2\cdots\gamma_{l-1}, m_1, m_2, \ldots, m_l\}$ and let $y \in \mathbb{N}$ be such that $\{m^*\} < \{y, y\gamma_1, y\gamma_1\gamma_2, \ldots, y\gamma_1\gamma_2\cdots\gamma_{l-1}\}$. Further, let $D_1 \in MS_{\gamma_1} \cap C_y$ and let $x = \min(D_1)$. Since $m^* < y \leq x$, we obtain that $|D_1| \geq a_1$ and $\gamma_1|_{D_1}$ is injective by (a3). Let $j \in \{2, 3, \ldots, l\}$. Then $m^* < y \leq x$ and (a1) imply that $m^* \leq y\gamma_1\gamma_2\cdots\gamma_{j-1} \leq x\gamma_1\gamma_2\cdots\gamma_{j-1}$. Since $a_{j-1} = 2a_j + 3$ and $m^* \leq x\gamma_1\gamma_2\cdots\gamma_{j-1}$, the properties (a2) and (a3) provide that there is a convex set $D_j \subseteq D_{j-1}\gamma_{j-1} \cap E_j$ for some $E_j \in MS_{\gamma_j}$ such that $|D_j| = a_j$ and $\gamma_j|_{D_j}$ is injective. Let $D = D_l\gamma^{-1}_{l-1}\gamma^{-1}_{l-2}\cdots\gamma^{-1}_1$. Since $D\gamma_0\gamma_1\cdots\gamma_{r-1} \subseteq D_r$, $\gamma_r|_{D_r}$ is injective, and $D_r\gamma_r\gamma_r^{-1} = D_r$ for all $1 \leq r \leq l$, we obtain that $|D| = |D_l| = a_l = 2$. Then there is $D' \in MS_{\gamma_1\gamma_2\cdots\gamma_l}$ with $D \subseteq D'$. Thus, $|D'| \geq |D| = 2$, a contradiction to $\alpha = \gamma_1\gamma_2\cdots\gamma_l$ with $MS_\alpha = MS^1_\alpha$. □

Although a minimal generating set of the uncountable semigroup $F_\mathbb{N}$ does not exist, there is an uncountable subsemigroup of $F_\mathbb{N}$ having such one. Let $A \subseteq \mathbb{N}$ and let $\alpha_A \in \Theta$ be such that $\mathrm{im}\ \alpha_A = \mathbb{N}$ and $\left|x\alpha_A^{-1}\right| = 3$ if $x \in A$ and $\left|x\alpha_A^{-1}\right| = 5$ otherwise. Note that such an α_A exists. Further, let $Q := \{\alpha_A : A \subseteq \mathbb{N}\}$. Then $|Q| = 2^{\aleph_0}$, which means that Q is uncountable. For $A, B \subseteq \mathbb{N}$, it is easy to verify that $\left|M^m_{\alpha_A\alpha_B}\right| > 0$ for some $m \geq 9$, that is, $\alpha_A\alpha_B \notin Q$. This shows that Q is a minimal generating set of the semigroup generated by Q. In other words, the uncountable subsemigroup $\langle Q \rangle$ of $F_\mathbb{N}$ has a minimal generating set.

3. Infinite decreasing chains of generating sets of $F_\mathbb{N}$

The previous section shows that there are no minimal generating sets of $F_\mathbb{N}$. Obviously, $F_\mathbb{N}$ itself is the maximum generating set. Both facts provide that $F_\mathbb{N}$ must have infinite decreasing chains of generating sets of $F_\mathbb{N}$. In this section, we will provide such two chains.

Let $\mathrm{Inj}(F_\mathbb{N})$ be the set of all injective transformations in $F_\mathbb{N}$ and let ξ be the transformation on $\mathbb{N}$ defined by $x\xi := x + 2$ for all $x \in \mathbb{N}$. Thus, $\xi^n \in \mathrm{Inj}(F_\mathbb{N})$ with $1\xi^n = 2n + 1$ for all $n \in \mathbb{N}$. Let $\mathcal{B} := \{\alpha \in F_\mathbb{N} : |\mathrm{nb}(\alpha)| = 2, \mathrm{c}(\alpha) = 3, \text{ and } \mathrm{im}\ \alpha = \mathbb{N}\}$. For $n \in \mathbb{N}$, there is exactly one $\beta \in \mathcal{B}$ with $\min(\mathrm{nb}(\beta)) = n$. This transformation will be denoted by β_n. Let $n \in \mathbb{N}$. We put $\mathcal{B}_n := \{\beta_i : i \geq n\}$. Further, we define transformations λ_n and δ_n as follows:

$$x\lambda_n := \begin{cases} n - x + 1 & \text{if } x \in \{1, 2, \ldots, n\}; \\ x - n + 1 & \text{otherwise} \end{cases}$$

and

$$x\delta_n := \begin{cases} m & \text{if } x \in \{1,2,\ldots,n\}; \\ m+x-n & \text{otherwise,} \end{cases}$$

where $m = 1$ if n is odd and $m = 2$ if n is even. It is easy to check that $\delta_n \in F_{\mathbb{N}}$. But $\lambda_n \in F_{\mathbb{N}}$, whenever n is odd. In this case, we observe that $|\mathrm{nb}(\lambda_n)| = 0, |\{1,2,\ldots,n\}\lambda_n| = n$, and $1\lambda_n = n$. If $n \neq 1$, then $(n-1)\lambda_n = 2 = (n+1)\lambda_n$, that is, $\mathrm{c}(\lambda_n) > 0$ and so $\lambda_n \in \Lambda_n$.

Lemma 3.1 *Let $n \in \mathbb{N}$. Then $\delta_m \in \langle \mathcal{B}_n \cup \Lambda_n \cup \{\xi\}\rangle$ for all $m \in \mathbb{N}$.*

Proof Let $m \in \mathbb{N}, m_1 = \max\{m,n\}$, and $m_2 = 2m_1 + 1$. Then we can calculate that

$$\delta_m = \begin{cases} \xi\beta_1 & \text{if } m = n = 1; \\ \xi^{m_1}\beta_{m_2-2}\lambda_{m_2-2} & \text{if } m = 1, n > 1; \\ \xi^{m_1}\beta_{m_2}^{k_1}\lambda_{m_2} & \text{if } m = 2k_1+1 \text{ for some } k_1 \in \mathbb{N}; \\ \xi^{m_1}\beta_{m_2-1}^{k_2}\lambda_{m_2-2} & \text{if } m = 2k_2 \text{ for some } k_2 \in \mathbb{N}. \end{cases}$$

Clearly, $\beta_1 \in \mathcal{B}_1$. If $n + m > 2$, then $m_2 - 2 > n$, which implies that $\beta_{m_2-2}, \beta_{m_2-1}, \beta_{m_2} \in \mathcal{B}_n$ and $\lambda_{m_2-2}, \lambda_{m_2} \in \Lambda_n$. Altogether, we obtain $\delta_m \in \langle \mathcal{B}_n \cup \Lambda_n \cup \{\xi\}\rangle$. □

Let $n \in \mathbb{N}$. We define a transformation α_n on $\mathbb{N}$ by $x\alpha_n := x$ if $x \in \mathbb{N}\setminus\{n, n+1, \ldots\}$ and $x\alpha_n := n$ otherwise. It is clear that $\alpha_n \in F_{\mathbb{N}}$. Then we put $\mathcal{A}_n := \{\alpha_i : i \geq n\}$. Further, let

$$\Delta := \{\alpha \in F_{\mathbb{N}} : |M_\alpha^*| = \aleph_0\}$$

and $\Delta_n := \Delta \cap \Omega_n = \{\alpha \in F_{\mathbb{N}} : 1\alpha \geq n, |\{1,2,\ldots,n\}\alpha| = n, \text{ and } |M_\alpha^*| = \aleph_0\}$.

Lemma 3.2 *Let $\alpha \in F_{\mathbb{N}} \setminus \Delta$. Then $\alpha \in \langle \mathcal{A}_n \cup \mathcal{B}_n \cup \Lambda_n \cup \{\xi\}\rangle$ for all $n \in \mathbb{N}$.*

Proof Since $\alpha \in F_{\mathbb{N}} \setminus \Delta$, we have $|M_\alpha^*| < \aleph_0$. Let $n \in \mathbb{N}$ and let $k_1 \in \mathbb{N} \setminus \{1,2,\ldots,n\}$ be odd. Further, let $k' = \frac{1}{2}(k_1 - 1)$.

Case 1: $|M_\alpha^*| = 0$. Then $|\mathrm{nb}(\alpha)| = 0$. Thus, x and $x\alpha$ have the same parity for all $x \in \mathbb{N}$. We define $\gamma : \mathbb{N} \to \mathbb{N}$ by

$$x\gamma := \begin{cases} 1\alpha + k_1 - x & \text{if } x \in \{1,2,\ldots,k_1-1\}; \\ (x-k_1+1)\alpha & \text{otherwise.} \end{cases}$$

Then $|\mathrm{nb}(\gamma)| = 0, \mathrm{c}(\gamma) > 0, 1\gamma = 1\alpha + k_1 - 1 > n$, and $|\{1,2,\ldots,n\}\gamma| = n$, that is, $\gamma \in \Lambda_n$. So, we obtain $\alpha = \xi^{k_1'}\gamma \in \langle \Lambda_n \cup \{\xi\}\rangle$.

Case 2: $|M_\alpha^*| = m$ for some $m \in \mathbb{N}$. Suppose now $M_\alpha^* = \{A_i : 1 \leq i \leq m\}$ for some $m \in \mathbb{N}$ with $A_i < A_j$ for all $1 \leq i < j \leq m$. It follows $|A_i| < \aleph_0$ for all $i \in \mathbb{N} \setminus \{m, m+1, \ldots\}$. Let

$$p_i = \min(A_i) \text{ for all } i \in \{1,2,\ldots,m\}$$

and

$$m_i = \max(A_i) \text{ for all } i \in \mathbb{N} \setminus \{m, m+1, \ldots\}.$$

Further, let $k_{i+1} = k_i + p_{i+1} - m_i$ for all $i \in \mathbb{N} \setminus \{m, m+1, \ldots\}$.
Case 2.1: $m = 1$. If $1 \notin A_1$ and $|A_1| < \aleph_0$, then $|A_1| = 2l_1 + 1$ for some $l_1 \in \mathbb{N}$. We define a transformation $\gamma^{'}$ on $\mathbb{N}$ as follows:

$$\gamma^{'} := \begin{cases} \delta_{|A_1|}\xi^{k'} & \text{if } 1 \in A_1 \text{ and } |A_1| < \aleph_0; \\ \alpha_1 \xi^{k'} & \text{if } 1 \in A_1 \text{ and } |A_1| = \aleph_0; \\ \xi^{k'}\beta^{l_1}_{k_1+p_1-1} & \text{if } 1 \notin A_1 \text{ and } |A_1| < \aleph_0; \\ \xi^{k'}\alpha_{k_1+p_1-1} & \text{if } 1 \notin A_1 \text{ and } |A_1| = \aleph_0. \end{cases}$$

It is clear that $\gamma^{'} \in \Theta, M_\alpha = M_{\gamma^{'}}$, and $1\gamma^{'} \geq k_1 > n$. Then Corollary 2.3 implies that there exists $\gamma^{''} \in \Lambda_n$ with $\alpha = \gamma^{'}\gamma^{''}$. Since $\gamma^{'} \in \langle \mathcal{A}_n \cup \mathcal{B}_n \cup \Lambda_n \cup \{\xi\}\rangle$, we obtain that $\alpha = \gamma^{'}\gamma^{''} \in \langle \mathcal{A}_n \cup \mathcal{B}_n \cup \Lambda_n \cup \{\xi\}\rangle$.
Case 2.2: $m > 1$. If $1 \notin A_1$, then $|A_1| = 2l_1 + 1$ for some $l_1 \in \mathbb{N}$. In the case $|A_m| < \aleph_0$, we obtain that $|A_m| = 2l_m + 1$ for some $l_m \in \mathbb{N}$. We define transformations $\gamma_1, \gamma_2, \ldots, \gamma_m$ on $\mathbb{N}$ as follows:

$$\gamma_1 := \begin{cases} \delta_{m_1}\xi^{k_1^{'}} & \text{if } 1 \in A_1; \\ \xi^{k_1^{'}}\beta^{l_1}_{k_1+p_1-1} & \text{otherwise,} \end{cases}$$

for $i \in \mathbb{N} \setminus \{1, m, m+1, \ldots\}$, we put

$$\gamma_i := \begin{cases} \beta^{l_i}_{k_i} & \text{if } 1 \in A_1 \text{ and } m_1 \text{ is odd;} \\ \beta^{l_i}_{k_i+1} & \text{if } 1 \in A_1 \text{ and } m_1 \text{ is even;} \\ \beta^{l_i}_{k_i+p_1-1} & \text{if } 1 \notin A_1, \end{cases}$$

and

$$\gamma_m := \begin{cases} \beta^{l_m}_{k_m} & \text{if } 1 \in A_1, m_1 \text{ is odd, and } |A_m| < \aleph_0; \\ \alpha_{k_m} & \text{if } 1 \in A_1, m_1 \text{ is odd, and } |A_m| = \aleph_0; \\ \beta^{l_m}_{k_m+1} & \text{if } 1 \in A_1, m_1 \text{ is even, and } |A_m| < \aleph_0; \\ \alpha_{k_m+1} & \text{if } 1 \in A_1, m_1 \text{ is even, and } |A_m| = \aleph_0; \\ \beta^{l_m}_{k_m+p_1-1} & \text{if } 1 \notin A_1 \text{ and } |A_m| < \aleph_0; \\ \alpha_{k_m+p_1-1} & \text{if } 1 \notin A_1 \text{ and } |A_m| = \aleph_0. \end{cases}$$

Let $\alpha^* = \gamma_1\gamma_2\cdots\gamma_m$. By straightforward calculations, we obtain that $\alpha^* \in \Theta$, $M_\alpha = M_{\alpha^*}$, and $1\alpha^* \geq k_1 > n$. Then Corollary 2.3 implies that there exists $\alpha^{'} \in \Lambda_n$ with $\alpha = \alpha^*\alpha^{'}$. By the definition of γ_1 and Lemma 3.1, we get $\gamma_1 \in \langle \mathcal{B}_n \cup \Lambda_n \cup \{\xi\}\rangle$. For $i \in \{2, 3, \ldots, m\}$, we obtain that $\gamma_i \in \langle \mathcal{A}_n \cup \mathcal{B}_n\rangle$ since $k_i > n$. Therefore, $\alpha = \alpha^*\alpha^{'} \in \langle \mathcal{A}_n \cup \mathcal{B}_n \cup \Lambda_n \cup \{\xi\}\rangle$. □

Both previous lemmas lead to the definition of an infinite decreasing chain $\{H_n : n \in \mathbb{N}\}$ of generating sets of $F_\mathbb{N}$, where $H_n := \mathcal{A}_n \cup \mathcal{B}_n \cup \Lambda_n \cup \Delta_n \cup \{\xi\}$. It is worth mentioning that the intersection of the H_i's gives the singleton set $\{\xi\}$, which is not a generating set of $F_\mathbb{N}$. It is easy to verify that $\xi \notin \langle \mathcal{A}_n \cup \mathcal{B}_n \cup \Lambda_n \cup \Delta_n\rangle$. Therefore, the relative rank of $F_\mathbb{N}$ modulo $\mathcal{A}_n \cup \mathcal{B}_n \cup \Lambda_n \cup \Delta_n$ is one.

Theorem 3.3 $\langle H_n\rangle = F_\mathbb{N}$ *for all* $n \in \mathbb{N}$.

Proof Let $n \in \mathbb{N}$. It is a consequence of Lemma 3.2 that

$$\langle \mathcal{A}_n \cup \mathcal{B}_n \cup \Lambda_n \cup \Delta \cup \{\xi\}\rangle = F_\mathbb{N}.$$

In order to show $\langle H_n\rangle = F_{\mathbb{N}}$, it is enough to prove $\Delta \setminus \Delta_n \subseteq \langle H_n\rangle$. Let $\alpha \in \Delta \setminus \Delta_n$. Then $|M_\alpha^*| = \aleph_0$ and so $|M_\alpha| = \aleph_0$. Suppose that $M_\alpha = \{A_i : i \in \mathbb{N}\}$ with $A_i < A_{i+1}$ for all $i \in \mathbb{N}$. Let $p_i = \min(A_i)$ for all $i \in \mathbb{N}$ and let $k_1 \in \mathbb{N}$ be odd such that $k_1 > n$.

Case 1: $|\{1, 2, \ldots, n\}\alpha| = n$. We define $\gamma : \mathbb{N} \to \mathbb{N}$ by $x\gamma := k_1 + i - 1$ for all $x \in A_i, i \in \mathbb{N}$. It is obvious that $\gamma \in \Theta, M_\gamma^* = M_\alpha^*, 1\gamma = k_1 > n$, and $|\{1, 2, \ldots, n\}\gamma| = n$. This means $\gamma \in \Delta_n$. Moreover, Corollary 2.3 implies that there exists $\gamma' \in \Lambda_n$ with $\gamma\gamma' = \alpha$. Therefore, $\alpha \in \langle H_n\rangle$.

Case 2: $|\{1, 2, \ldots, n\}\alpha| < n$. Let s be the smallest natural number r such that $n < p_r$ and $A_r \in M_\alpha^*$. Then we define $\gamma_0 : \mathbb{N} \to \mathbb{N}$ by

$$x\gamma_0 := \begin{cases} k_1 + x - 1 & \text{if } x \in \{1, 2, \ldots, p_s - 1\}; \\ k_1 + p_s + i - 2 & \text{if } x \in A_{s+i-1} \text{ for } i \in \mathbb{N}. \end{cases}$$

Note that $\gamma_0 \in \Delta_n$ since $1\gamma_0 = k_1 > n, |\{1, 2, \ldots, n\}\gamma_0| = n$, and $\left|M_{\gamma_0}^*\right| = |M_\alpha^*| - s = \aleph_0$. If $s = \min\{i \in \mathbb{N} : A_i \in M_\alpha^*\}$, then $M_{\gamma_0} = M_\alpha$ and so we put $\beta := \gamma_0$. Suppose $s > \min\{i \in \mathbb{N} : A_i \in M_\alpha^*\}$. Let $\{C \in M_\alpha^* : C < A_s\} = \{B_i : 1 \leq i \leq m\}$ for some $m \in \mathbb{N}$ with $B_i < B_j$ for all $1 \leq i < j \leq m$. For $i \in \mathbb{N} \setminus \{1, m+1, m+2, \ldots\}$, there is $l_i \in \mathbb{N}$ with $|B_i| = 2l_i + 1$. Moreover, there is $l_1 \in \mathbb{N}$ with $|B_1| = 2l_1 + 1$ or $|B_1| = 2l_1$, depending on the parity of $|B_1|$. Let $q_i = \min(B_i)$ and $m_i = \max(B_i)$ for all $i \in \{1, 2, \ldots, m\}$. Further, let $k_{j+1} = k_j + q_{j+1} - m_j$ for all $j \in \mathbb{N} \setminus \{m, m+1, \ldots\}$. For $i \in \{1, 2, \ldots, m\}$, we define $\gamma_i : \mathbb{N} \to \mathbb{N}$ as follows:

$$\gamma_i := \begin{cases} \beta_{k_i}^{l_i} & \text{if } 1 \in B_1 \text{ and } |B_1| \text{ is odd;} \\ \beta_{k_i-1}^{l_i} & \text{if } 1 \in B_1 \text{ and } |B_1| \text{ is even;} \\ \beta_{k_i+q_1-1}^{l_i} & \text{if } 1 \notin B_1. \end{cases}$$

In this case, we put $\beta := \gamma_0\gamma_1\gamma_2\cdots\gamma_m$. By straightforward calculations, we obtain that $\beta \in \Theta, M_\beta = M_\alpha$, and $1\beta \geq k_1 - 1 \geq n$. Then Corollary 2.3 implies that there exists $\beta' \in \Lambda_n$ such that $\beta\beta' = \alpha$. Therefore, $\alpha = \beta\beta' \in \langle H_n\rangle$. □

It is easy to see that $\Omega_{n+1} \subsetneq \Omega_n, \mathcal{A}_{n+1} \subsetneq \mathcal{A}_n$, and $\mathcal{B}_{n+1} \subsetneq \mathcal{B}_n$ for all $n \in \mathbb{N}$. Therefore, we can conclude that $\{H_n : n \in \mathbb{N}\}$ is an infinite decreasing chain of generating sets of $F_{\mathbb{N}}$.

Recall that $F_{\mathbb{N}} = \Theta\Lambda_n$ for any $n \in \mathbb{N}$, where Θ is a subsemigroup of $F_{\mathbb{N}}$. This means that we can generate any element in $F_{\mathbb{N}}$ by elements from Θ and Λ_n. Now, let

$$\Gamma := \{\alpha \in \Theta : \text{rank } \alpha = \aleph_0 \text{ and there exists } b \in \text{im } \alpha \text{ with } \left|b\alpha^{-1}\right| \geq 3\}.$$

We will generate the elements in $F_{\mathbb{N}}$ by elements from the proper subsemigroup Γ of $F_{\mathbb{N}}, \Lambda_n$, and the additional transformation ξ, for any $n \in \mathbb{N}$. Moreover, Λ_n is covered by the semigroup Λ.

Proposition 3.4 *Λ and Γ are subsemigroups of $F_{\mathbb{N}}$.*

Proof Let $\alpha, \beta \in \Lambda$. Then $|\text{nb}(\alpha)| = |\text{nb}(\beta)| = 0$ and $\text{c}(\alpha), \text{c}(\beta) > 0$. This means $M_\alpha^* = M_\beta^* = \emptyset$. Assume $\left|M_{\alpha\beta}^*\right| > 0$. Then there exists $D \in M_{\alpha\beta}^*$, that is, $|D| > 1$ and $|D\alpha\beta| = 1$. Since D is a convex set and $|D| > 1$, there is $a \in \mathbb{N}$ such that $\{a, a+1\} \subseteq D$. Since $|\text{nb}(\alpha)| = 0$, we obtain that $a\alpha = b$ and $(a+1)\alpha = c$ for some $b, c \in \mathbb{N}$ such that $|b-c| = 1$. Since $|\{b,c\}\beta| = |\{a,a+1\}\alpha\beta| \leq |D\alpha\beta| = 1$ and $|b-c| = 1$, we obtain $|\mathrm{nb}(\beta)| \neq 0$, a contradiction. Therefore, $M^*_{\alpha\beta} = \emptyset$, that is, $|\mathrm{nb}(\alpha\beta)| = 0$. Together with $0 < \mathrm{c}(\alpha) \leq \mathrm{c}(\alpha\beta)$, we obtain that $\alpha\beta \in \Lambda$.

Now, let $\alpha, \beta \in \Gamma$. Then $\alpha, \beta \in \Theta$ and rank α = rank $\beta = \aleph_0$. It is clear that rank $\alpha\beta = \aleph_0$ and $\alpha\beta \in \Theta$. Furthermore, there is $a \in \mathbb{N}$ with $\left|a\alpha^{-1}\right| \geq 3$. Then $\left|a\beta(\alpha\beta)^{-1}\right| = \left|a\beta\beta^{-1}\alpha^{-1}\right| \geq \left|a\alpha^{-1}\right| \geq 3$. Altogether, we conclude that $\alpha\beta \in \Gamma$. □

We are going to establish a second infinite decreasing chain of generating sets of $F_{\mathbb{N}}$, which are subsets of the union of the three semigroups $\{\xi\}, \Lambda$, and Γ. Let $n \in \mathbb{N}$ and let G_n be the set of all $\alpha \in F_{\mathbb{N}}$ satisfying at least one of the following three properties:

(g1) $\alpha = \xi$;

(g2) $\alpha \in \Lambda_n$;

(g3) $\alpha \in \Theta_n$ such that $|M^*_\alpha| \in \{1, \aleph_0\}$ and $M^*_\alpha = M^3_\alpha$.

Clearly, $G_n \subseteq \Gamma \cup \Lambda_n \cup \{\xi\}$.

Theorem 3.5 *$\langle G_n \rangle = F_{\mathbb{N}}$ for all $n \in \mathbb{N}$.*

Proof Let $n \in \mathbb{N}$. By the definition of G_n, we have $\Lambda_n \cup \{\xi\} \subseteq G_n$. We will show that $\mathcal{A}_n, \mathcal{B}_n, \Delta_n \subseteq \langle G_n \rangle$.

Let $\alpha \in \mathcal{A}_n$. Then $\alpha = \alpha_k$ for some $k \geq n$, and $x\alpha = x$ if $x \in \mathbb{N} \setminus \{k, k+1, \ldots\}$ and $x\alpha = k$ otherwise. Let l be the least even natural number r such that $r > k$. We define transformations γ_1 and γ_2 on $\mathbb{N}$ as follows:

$$x\gamma_1 := \begin{cases} l+x & \text{if } x \in \mathbb{N} \setminus \{k, k+1, \ldots\}; \\ l+k & \text{if } x \in \{k, k+2, k+4, \ldots\}; \\ l+k+1 & \text{if } x \in \{k+1, k+3, k+5, \ldots\} \end{cases}$$

and

$$x\gamma_2 := \begin{cases} l+x & \text{if } x \in \{1, 2, \ldots, l+k-1\}; \\ 2l+k & \text{if } x \in \{l+k, l+k+1, l+k+2\}; \\ l+x-2 & \text{if } x \in \mathbb{N} \setminus \{1, 2, \ldots, l+k+2\}. \end{cases}$$

Then $\gamma_1 \in \Lambda_n$ and γ_2 satisfies (g3). By straightforward calculations, we obtain $\gamma_1\gamma_2\lambda_{2l+1} = \alpha$. Since $1\lambda_{2l+1} = 2l+1 > n$, we have $\lambda_{2l+1} \in \Lambda_n$. This shows $\mathcal{A}_n \subseteq \langle G_n \rangle$.

Let $\alpha \in \mathcal{B}_n$. Then $\alpha = \beta_k$ for some $k \geq n$, that is,

$$x\alpha = \begin{cases} x & \text{if } x \in \mathbb{N} \setminus \{k, k+1, \ldots\}; \\ k & \text{if } x \in \{k, k+1, k+2\}; \\ x-2 & \text{if } x \in \mathbb{N} \setminus \{1, 2, \ldots, k+2\}. \end{cases}$$

Let l be again the least even natural number r such that $r > k$ and define $\gamma : \mathbb{N} \to \mathbb{N}$ by $x\gamma := x\alpha + l$ for all $x \in \mathbb{N}$. Then γ satisfies (g3). It is easy to see that $\gamma\lambda_{l+1} = \alpha$. Since $1\lambda_{l+1} = l+1 > n$, we obtain $\lambda_{l+1} \in \Lambda_n$, that is, $\mathcal{B}_n \subseteq \langle G_n \rangle$.

Let $\alpha \in \Delta_n$. Then $1\alpha \geq n, |\{1,2,\ldots,n\}\alpha| = n$, and $|M^*_\alpha| = \aleph_0$. Suppose $M^*_\alpha = \{A_i : i \in \mathbb{N}\}$ with $A_i < A_{i+1}$ for all $i \in \mathbb{N}$. It follows that $|A_i| < \aleph_0$ for all $i \in \mathbb{N}$. For $i \in \mathbb{N}$, let $p_i = \min(A_i)$ and $l_i = |A_i|$. Let l be now the least even natural number r such that $r > 1\alpha$. Further, let $k_2 = l + p_2$ and $k_i = l + p_i - \Sigma_{j=2}^{i-1}(l_j - 3)$ for all $i \in \mathbb{N} \setminus \{1,2\}$. Note that if l_1 is even, then $p_1 = 1$. Put $c = 1$ if l_1 is even and $c = 0$ otherwise. We define transformations γ_1, γ_2, and γ_3 on $\mathbb{N}$ as follows:

$$x\gamma_1 := \begin{cases} x & \text{if } x \in \{1,2,\ldots,p_2-1\};\\ k_i & \text{if } x \in \{p_i, p_i+2, \ldots, p_i+l_i-3\};\\ k_i+1 & \text{if } x \in \{p_i+1, p_i+3, \ldots, p_i+l_i-2\};\\ k_i+2 & \text{if } x = p_i+l_i-1;\\ l+x-\Sigma_{j=1}^{i}(l_j-3) & \text{if } x \in \{p_i+l_i, p_i+l_i+1, \ldots, p_{i+1}-1\}, \end{cases}$$

$$x\gamma_2 := \begin{cases} l+x+l_1-3+c & \text{if } x \in \{1,2,\ldots,l+p_1-1-c\};\\ 2l+p_1+l_1-3 & \text{if } x \in \{l+p_1-c, l+p_1+2-c, \ldots, l+p_1+l_1-3\};\\ 2l+p_1+l_1-2 & \text{if } x \in \{l+p_1+1-c, l+p_1+3-c, \ldots, l+p_1+l_1-2\};\\ l+x & \text{if } x \in \{l+p_1+l_1-1, l+p_1+l_1, \ldots\}, \end{cases}$$

and

$$x\gamma_3 := \begin{cases} l+x & \text{if } x \in \{1,2,\ldots,2l+p_1+l_1-4\};\\ 3l+p_1+l_1-3 & \text{if } x \in \{2l+p_1+l_1-3, 2l+p_1+l_1-2, 2l+p_1+l_1-1\};\\ l+x-2 & \text{if } x \in \{2l+p_1+l_1, 2l+p_1+l_1+1, \ldots, l+k_2-1\};\\ 2l+k_i-2(i-1) & \text{if } x \in \{l+k_i, l+k_i+1, l+k_i+2\};\\ l+x-2i & \text{if } x \in \{l+k_i+3, l+k_i+4, \ldots, l+k_{i+1}-1\} \end{cases}$$

for all $i \in \mathbb{N} \setminus \{1\}$. It is easy to verify that $\gamma_1, \gamma_2 \in \Lambda_n$ and γ_3 satisfies (g3). By straightforward calculations, we obtain that $\gamma_1\gamma_2\gamma_3 \in \Theta, M_{\gamma_1\gamma_2\gamma_3} = M_\alpha$, and $1\gamma_1\gamma_2\gamma_3 \geq 2l + l_1 - 2 \geq l > n$. Then Corollary 2.3 implies that there exists $\gamma_4 \in \Lambda_n$ such that $\gamma_1\gamma_2\gamma_3\gamma_4 = \alpha$. Therefore, $\Delta_n \subseteq \langle G_n \rangle$.

Altogether, we have shown $H_n = \mathcal{A}_n \cup \mathcal{B}_n \cup \Lambda_n \cup \Delta_n \cup \{\xi\} \subseteq \langle G_n \rangle$. By Proposition 3.3, we obtain $\langle G_n \rangle = F_{\mathbb{N}}$. □

Let $n \in \mathbb{N}$. Since $\Omega_{n+1} \subsetneq \Omega_n$, we can conclude that $G_{n+1} \subsetneq G_n$. This shows that $\{G_n : n \in \mathbb{N}\}$ is an infinite decreasing chain of generating sets of $F_{\mathbb{N}}$. Moreover, $\bigcap_{n\in\mathbb{N}} G_n = \{\xi\}$ because any transformation $\alpha \in F_{\mathbb{N}} \setminus \{\xi\}$ is not in $G_{1\alpha+1}$. In other words, the relative rank of $F_{\mathbb{N}}$ modulo G_n is one.

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