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Commutator subgroups of generalized Hecke and extended generalized Hecke groups, II

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Abstract: Let p_1, \dots, p_n be integers where $n \geq 2$ and each $p_i \geq 2$. Let also $H(p_1, \dots, p_n)$ be the generalized Hecke group associated to all $p_i \geq 2$. In this paper, we study the commutator subgroups $H'(p_1, \dots, p_n)$ and $\overline{H}'(p_1, \dots, p_n)$ of the generalized Hecke group $H(p_1, \dots, p_n)$ and the extended generalized Hecke group $\overline{H}(p_1, \dots, p_n)$. We give the generators and the signatures of $H'(p_1, \dots, p_n)$ and $\overline{H}'(p_1, \dots, p_n)$.

Key words: Generalized Hecke groups, extended generalized Hecke groups, commutator subgroups

1. Introduction

Let p_1, \dots, p_n be integers where $n \geq 2$ and each $p_i \geq 2$. Let us consider the linear fractional transformations

$$X_i(z) = -\frac{1}{z + \lambda_i},$$

where $\lambda_i = 2 \cos(\frac{\pi}{p_i})$ for $p_i \geq 2$ is an integer. Generalized Hecke groups $H(p_1, \dots, p_n)$ are generated by X_i 's and have the presentation

$$H(p_1, \dots, p_n) = \langle X_i : X_i^{p_i} = I \rangle \cong C_{p_1} * \dots * C_{p_n}.$$

and the signature $(0; p_1, \dots, p_n, \infty)$, [8] and [9]. Extended generalized Hecke groups $\overline{H}(p_1, \dots, p_n)$ can be defined by adding the reflection $R(z) = 1/\bar{z}$ to the generators of $H(p_1, \dots, p_n)$. Hence the extended generalized Hecke groups $\overline{H}(p_1, \dots, p_n)$ have a presentation

$$\overline{H}(p_1, \dots, p_n) = \langle X_i, R : X_i^{p_i} = R^2 = I, RX_i = X_i^{-1}R \rangle,$$

or

$$\overline{H}(p_1, \dots, p_n) = \langle X_i, R : X_i^{p_i} = R^2 = (X_i R)^2 = I \rangle \cong D_{p_1} *_{\mathbb{Z}_2} \dots *_{\mathbb{Z}_2} D_{p_n}, [8].$$

Notice that the generalized Hecke group $H(2, 3)$ is the modular group $\Gamma = PSL(2, \mathbb{Z})$. The modular group is the discrete subgroup of $PSL(2, \mathbb{R})$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad S(z) = -\frac{1}{z+1}.$$

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Then the modular group Γ has a presentation

$$\Gamma = \langle T, S \mid T^2 = S^3 = I \rangle \cong C_2 * C_3.$$

Also, if $q \geq 3$ is an integer, then the generalized Hecke group $H(2, q)$ is the Hecke group $H(\lambda_q)$ ([2], [7], [10], [11], [12]). If $p_1 = p$ and $p_2 = q$ are integers where $2 \leq p \leq q$ and $p + q > 4$, then generalized Hecke group $H(p, q)$ is the generalized Hecke group $H_{p,q}$ ([5], [15], [18]).

On the other hand, the extended generalized Hecke group $\overline{H}(2, 3)$ is the extended modular group $\overline{\Gamma}$ (or Π) ([23], [24]). We know that the extended modular group $\Pi = PGL(2, \mathbb{Z})$ is defined by adding the reflection $R(z) = 1/\bar{z}$ to the generators of the modular group Γ . The extended modular group Π has a presentation

$$\Pi = \langle T, S, R \mid T^2 = S^3 = R^2 = (RT)^2 = (RS)^2 = I \rangle \cong D_2 *_{\mathbb{Z}_2} D_3.$$

Also, if $q \geq 3$ is an integer, then the extended generalized Hecke group $\overline{H}(2, q)$ is the extended Hecke group $\overline{H}(\lambda_q)$, ([4], [26], [27]). Finally, if $p_1 = p$ and $p_2 = q$ are integers where $2 \leq p \leq q$ and $p + q > 4$, then the extended generalized Hecke group $\overline{H}(p, q)$ is the extended generalized Hecke group $\overline{H}_{p,q}$, ([6]).

The motivation of this paper is to study the commutator subgroups of the generalized Hecke groups $H(p_1, \dots, p_n)$ and the extended generalized Hecke groups $\overline{H}(p_1, \dots, p_n)$. If $n = 2$, then the commutator subgroups of $H(p_1, p_2)$ and $\overline{H}(p_1, p_2)$ was studied by many authors in [1], [3], [13], [14], [15], [17], [19], [21], [22], [25], [29].

Here, our aim is to generalize the results given in [14] in the case $p_1 = p$ and $p_2 = q$ where $2 \leq p \leq q$ and $p + q > 4$, to the case p_1, \dots, p_n are integers where $n \geq 2$ and each $p_i \geq 2$. To do this, we use the Reidemeister–Schreier method, the permutation method (see, [28]) and the extended Riemann–Hurwitz condition (see, [16]). Here we give the generators and the signatures of the commutator subgroups of $H(p_1, \dots, p_n)$ and $\overline{H}(p_1, \dots, p_n)$. Of course, if we take $n = 2$, $p_1 = p$ and $p_2 = q$, then our results coincide with the results given in [14] for $H_{p,q}$ and $\overline{H}_{p,q}$.

2. Commutator subgroups of $H(p_1, \dots, p_n)$ and $\overline{H}(p_1, \dots, p_n)$

First we study the commutator subgroup $H'(p_1, \dots, p_n)$ of the generalized Hecke group $H(p_1, \dots, p_n)$.

Theorem 2.1 *Let p_1, \dots, p_n be integers where $n \geq 2$ and each $p_i \geq 2$. Then*

$$i) |H(p_1, \dots, p_n) : H'(p_1, \dots, p_n)| = p_1 \cdot p_2 \cdots p_n.$$

$$ii) \text{ The commutator subgroup } H'(p_1, \dots, p_n) \text{ of } H(p_1, \dots, p_n) \text{ is a free group of rank } \sum_{i=1}^{n-1} \sum_{j=i+1}^n (p_i - 1) \cdot (p_j - 1) + 2 \cdot \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^n (p_i - 1) \cdot (p_j - 1) \cdot (p_k - 1) + \cdots + (n-1) \cdot \sum_{i=1}^1 \sum_{j=2}^2 \cdots \sum_{s=n}^n (p_i - 1) \cdot (p_j - 1) \cdots (p_s - 1).$$

Proof *i)* To obtain the quotient group $H(p_1, \dots, p_n)/H'(p_1, \dots, p_n)$, we add the relations $X_i X_j = X_j X_i$ where $i, j = 1, \dots, n$ for $i \neq j$ to the relations of $H(p_1, \dots, p_n)$. Hence we get

$$H(p_1, \dots, p_n)/H'(p_1, \dots, p_n) = \langle X_i : X_i^{p_i} = I, X_i X_j = X_j X_i \rangle \cong C_{p_1} \times \cdots \times C_{p_n}.$$

Therefore we find the index as $|H(p_1, \dots, p_n) : H'(p_1, \dots, p_n)| = p_1 \cdot p_2 \cdots p_n$.

ii) Now we can use the Reidemeister–Schreier method for the generators of $H'(p_1, \dots, p_n)$. First we choose a Schreier transversal Σ for $H'(p_1, \dots, p_n)$. Here Σ consists of the identity element I ; $\sum_{i=1}^n (p_i - 1)$ elements of the form $X_i^{a_i}$ where $1 \leq i \leq n$ and $1 \leq a_i \leq p_i - 1$; $\sum_{i=1}^{n-1} \sum_{j=i+1}^n (p_i - 1)(p_j - 1)$ elements of the form $X_i^{a_i} X_j^{a_j}$ where $1 \leq i < j \leq n$ and for $t = i, j$, $1 \leq a_t \leq p_t - 1$; $\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^n (p_i - 1)(p_j - 1)(p_k - 1)$ elements of the form $X_i^{a_i} X_j^{a_j} X_k^{a_k}$ where $1 \leq i < j < k \leq n$ and for $t = i, j, k$, $1 \leq a_t \leq p_t - 1$; \cdots ; $\sum_{i=1}^1 \sum_{j=2}^2 \cdots \sum_{s=n}^n (p_i - 1)(p_j - 1) \cdots (p_s - 1)$ elements of the form $X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}$ where $1 \leq t \leq n$, $1 \leq a_t \leq p_t - 1$.

Using the Reidemeister–Schreier method, after some calculations, we have the generators of $H'(p_1, \dots, p_n)$ as follows:

There are $\sum_{i=1}^{n-1} \sum_{j=i+1}^n (p_i - 1)(p_j - 1)$ generators of the form $[X_i^a, X_j^b]$ where $1 \leq i < j \leq n$ and for $t = i, j$, $1 \leq a_t \leq p_t - 1$.

There are $2 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^n (p_i - 1)(p_j - 1)(p_k - 1)$ generators of the form $[X_i^{a_i}, X_j^{a_j} X_k^{a_k}]$ or $[X_i^{a_i} X_j^{a_j}, X_k^{a_k}]$ (for the difference, please see the place of the comma) where $1 \leq i < j < k \leq n$ and for $t = i, j, k$, $1 \leq a_t \leq p_t - 1$.

If we continue similarly, then we find that there are

$$(n - 1) \sum_{i=1}^1 \sum_{j=2}^2 \cdots \sum_{s=n}^n (p_i - 1)(p_j - 1) \cdots (p_s - 1)$$

generators of the form $[X_1^{a_1}, X_2^{a_2} \cdots X_n^{a_n}]$ or $[X_1^{a_1} X_2^{a_2}, \dots, X_n^{a_n}]$ or $[X_1^{a_1} X_2^{a_2} \cdots, X_n^{a_n}]$ where $1 \leq t \leq n$, $1 \leq a_t \leq p_t - 1$. Indeed, from [20], the generators are

$$1 + p_1 p_2 \cdots p_n \left\{ -1 + \sum_{i=1}^n \left(1 - \frac{1}{p_i} \right) \right\}.$$

Also, using the Riemann–Hurwitz formula and the permutation method, we find the signature of $H'(p_1, \dots, p_n)$ as

$$\left(1 + \frac{\left((n - 1) - \sum_{i=1}^n \frac{1}{p_i} \right) \cdot p_1 \cdot p_2 \cdots p_n - \frac{p_1 \cdot p_2 \cdots p_n}{\text{lcm}(p_1, p_2, \dots, p_n)}}{2}; \infty^{\left(\frac{p_1 \cdot p_2 \cdots p_n}{\text{lcm}(p_1, p_2, \dots, p_n)} \right)} \right).$$

□

Example 2.2 Let us consider the generalized Hecke group $H(2, 3, 4)$. Then we have the index $|H(2, 3, 4) : H'(2, 3, 4)| = 24$. Now, we set up a Schreier transversal Σ for $H'(2, 3, 4)$. Σ consists of I ; $\sum_{i=1}^3 (p_i - 1) = 6$ elements of the form $X_1, X_2, X_2^2, X_3, X_3^2, X_3^3$; $\sum_{i=1}^2 \sum_{j=i+1}^3 (p_i - 1)(p_j - 1) = 11$ elements of the form $X_1X_2, X_1X_2^2, X_1X_3, X_1X_3^2, X_1X_3^3, X_2X_3, X_2X_3^2, X_2X_3^3, X_2^2X_3, X_2^2X_3^2, X_2^2X_3^3$ and $\sum_{i=1}^1 \sum_{j=2k=3}^2 \sum_{k=3}^3 (p_i - 1)(p_j - 1)(p_k - 1) = 6$ elements of the form $X_1X_2X_3, X_1X_2X_3^2, X_1X_2X_3^3, X_1X_2^2X_3, X_1X_2^2X_3^2, X_1X_2^2X_3^3$. Using the Reidemeister-Schreier method, we find the generators of $H'(2, 3, 4)$ as follows: $\sum_{i=1}^2 \sum_{j=i+1}^3 (p_i - 1)(p_j - 1) = 11$ generators of the form $[X_1, X_2], [X_1, X_2^2], [X_1, X_3], [X_1, X_3^2], [X_1, X_3^3], [X_2, X_3], [X_2, X_3^2], [X_2, X_3^3], [X_2^2, X_3], [X_2^2, X_3^2], [X_2^2, X_3^3]$; $2 \sum_{i=1}^1 \sum_{j=2k=3}^2 \sum_{k=3}^3 (p_i - 1)(p_j - 1)(p_k - 1) = 12$ generators of the form $[X_1X_2, X_3], [X_1X_2, X_3^2], [X_1X_2, X_3^3], [X_1X_2^2, X_3], [X_1X_2^2, X_3^2], [X_1X_2^2, X_3^3], [X_1X_2^2X_3], [X_1X_2^2X_3^2], [X_1X_2^2X_3^3]$. Finally, using the Riemann-Hurwitz formula, we find the signature of $H'(2, 3, 4)$ as $(11; \infty^{(2)})$.

Now, we study the commutator subgroups $\overline{H}'(p_1, \dots, p_n)$ of the extended generalized Hecke groups $\overline{H}(p_1, \dots, p_n)$. To do this, firstly, we rename the generators X_i of the extended generalized Hecke group $\overline{H}(p_1, \dots, p_n)$. Let s, t and u be the number of the generators X_i of order 2, of even order ≥ 4 and of odd order ≥ 3 in $\overline{H}(p_1, \dots, p_n)$, respectively. Let us denote the generators of order 2 by A_1, A_2, \dots, A_s ; generators of even order ≥ 4 by B_1, B_2, \dots, B_t ; and generators of odd order ≥ 3 by C_1, C_2, \dots, C_u . Also, let q_k and r_l be the orders of the generators B_k and C_l , respectively, where $0 \leq k \leq t$ and $0 \leq l \leq u$. Then, the extended generalized Hecke group $\overline{H}(p_1, \dots, p_n)$ has a presentation

$$\langle A_j, B_k, C_l, R : A_j^2 = B_k^{q_k} = C_l^{r_l} = R^2 = I, RA_j = A_jR, RB_k = B_k^{-1}R, RC_l = C_l^{-1}R \rangle.$$

Therefore, we can write as

$$\overline{H}(p_1, \dots, p_n) = \overline{H}(\underbrace{2, 2, \dots, 2}_s, q_1, q_2, \dots, q_t, r_1, r_2, \dots, r_u)$$

Thus, we can give the following result:

Theorem 2.3 Let p_1, \dots, p_n be integers for $n \geq 2$ and $p_i \geq 2$. Then

i) $[\overline{H}(p_1, \dots, p_n) : \overline{H}'(p_1, \dots, p_n)] = 2^{s+t+1}$.

ii) The commutator subgroup $\overline{H}'(p_1, \dots, p_n)$ is a group of generators $\sum_{j=2}^s (j-1) \binom{s}{j} + \sum_{k=1}^t (2k-1) \binom{t}{k} + \sum_{j=1}^s \sum_{k=1}^t (2k+j-1) \binom{s}{j} \binom{t}{k} + 2^{s+t}u$.

Proof i) The quotient group $\overline{H}(p_1, \dots, p_n)/\overline{H}'(p_1, \dots, p_n)$ is the group obtained by adding the relations $RB_k = B_kR, RC_l = C_lR$, where $0 \leq k \leq t$ and $0 \leq l \leq u$ to the relations in $\overline{H}(p_1, \dots, p_n)$. Thus the quotient

or $B_1B_2 \cdots B_t B_{t-1}^{-1} (B_1 \cdots B_{t-2} B_t)^{-1}$ or $B_1B_2 \cdots B_t^2 (B_1B_2 \cdots B_{t-1})^{-1}$. If $s \geq 1$ and $t \geq 1$, then there are $2 \cdot \binom{s}{1} \binom{t}{1}$ generators of the form $A_d B_g A_d B_g^{-1}$, or $A_d B_g^2 A_d$ where $1 \leq d \leq s$ and $1 \leq g \leq t$; $3 \cdot \binom{s}{2} \binom{t}{1}$ generators of the form $A_d A_e B_g A_d (A_e B_g)^{-1}$ or $A_d A_e B_g A_e (A_d B_g)^{-1}$ or $A_d A_e B_g^2 (A_d A_e)^{-1}$ where $1 \leq d < e \leq s$ and $1 \leq g \leq t$; $4 \cdot \binom{s}{1} \binom{t}{2}$ generators of the form $A_d B_g B_h A_d (B_g B_h)^{-1}$ or $A_d B_g B_h B_g (A_d B_h)^{-1}$ or $A_d B_g B_h B_g^{-1} (A_d B_h)^{-1}$ or $A_d B_g B_h^2 (A_d B_g)^{-1}$ where $1 \leq d \leq s$ and $1 \leq g < h \leq t$; \dots ; $(2t+s-1) \cdot \binom{s}{s} \binom{t}{t}$ generators of the form

$$\begin{aligned} &A_1 A_2 \cdots A_s B_1 B_2 \cdots B_t A_1 (A_2 \cdots A_s B_1 B_2 \cdots B_t)^{-1} \text{ or} \\ &A_1 A_2 \cdots A_s B_1 B_2 \cdots B_t A_2 (A_1 A_3 \cdots A_s B_1 B_2 \cdots B_t)^{-1} \text{ or} \\ &\quad \vdots \\ &A_1 A_2 \cdots A_s B_1 B_2 \cdots B_t A_s (A_1 A_2 \cdots A_{s-1} B_1 B_2 \cdots B_t)^{-1} \text{ or} \\ &A_1 A_2 \cdots A_s B_1 B_2 \cdots B_t B_1 (A_1 A_2 \cdots A_s B_2 \cdots B_t)^{-1} \text{ or} \\ &A_1 A_2 \cdots A_s B_1 B_2 \cdots B_t B_1^{-1} (A_1 A_2 \cdots A_s B_2 \cdots B_t)^{-1} \text{ or} \\ &A_1 A_2 \cdots A_s B_1 B_2 \cdots B_t B_2 (A_1 A_2 \cdots A_s B_1 B_3 \cdots B_t)^{-1} \text{ or} \\ &A_1 A_2 \cdots A_s B_1 B_2 \cdots B_t B_2^{-1} (A_1 A_2 \cdots A_s B_1 B_3 \cdots B_t)^{-1} \text{ or} \\ &\quad \vdots \\ &A_1 A_2 \cdots A_s B_1 B_2 \cdots B_t B_{t-1} (A_1 A_2 \cdots A_s B_1 \cdots B_{t-2} B_t)^{-1} \text{ or} \\ &A_1 A_2 \cdots A_s B_1 B_2 \cdots B_t B_{t-1}^{-1} (A_1 A_2 \cdots A_s B_1 \cdots B_{t-2} B_t)^{-1} \text{ or} \\ &A_1 A_2 \cdots A_s B_1 B_2 \cdots B_t^2 (A_1 A_2 \cdots A_s B_1 \cdots B_{t-2} B_{t-1})^{-1}. \end{aligned}$$

Also there are $u \cdot \binom{s+t}{0}$ generators of the form C_l where $1 \leq l \leq u$; $u \cdot \binom{s+t}{1}$ generators of the form $A_d C_l A_d$ or $B_g C_l B_g^{-1}$ where $1 \leq d \leq s$, $1 \leq g \leq t$ and $1 \leq l \leq u$; $u \cdot \binom{s+t}{2}$ generators of the form $A_d A_e C_l (A_d A_e)^{-1}$ or $B_g B_h C_l (B_g B_h)^{-1}$ or $A_d B_g C_l (A_d B_g)^{-1}$ where $1 \leq d < e \leq s$, $1 \leq g < h \leq t$ and $1 \leq l \leq u$; $u \cdot \binom{s+t}{3}$ generators of the form $A_d A_e A_f C_l (A_d A_e A_f)^{-1}$ or $A_d A_e B_g C_l (A_d A_e B_g)^{-1}$ or $A_d B_g B_h C_l (A_d B_g B_h)^{-1}$ or $B_g B_h B_m C_l (B_g B_h B_m)^{-1}$ where $1 \leq d < e < f \leq s$, $1 \leq g < h < m \leq t$ and $1 \leq l \leq u$; \dots ; $u \cdot \binom{s+t}{s+t}$ generators of the form $A_1 A_2 \cdots A_s B_1 B_2 \cdots B_t C_l (A_1 A_2 \cdots A_s B_1 B_2 \cdots B_t)^{-1}$ where $1 \leq l \leq u$.

Also, using the Riemann–Hurwitz formula and permutation method, the signature of $\overline{H}^l(p_1, \dots, p_n)$ is

$$\begin{cases} (1 + \frac{2^{s+t-1}(s+t-3)}{2}; (q_k/2)^{(2^{s+t-1})}, r_l^{(2^{s+t})}, \infty^{(2^{s+t-1})}), & \text{if } s \geq 1 \text{ and } t \geq 1, \\ (0; r_l^{(2)}, \infty), & \text{if } s = 1 \text{ and } t = 0, \\ (0; (q_1/2), r_l^{(2)}, \infty), & \text{if } s = 0 \text{ and } t = 1, \\ (0; r_l, \infty), & \text{if } s = 0 \text{ and } t = 0, \end{cases} \tag{2.2}$$

where $0 \leq k \leq t$ and $0 \leq l \leq u$. □

Example 2.4 Let us consider the generalized Hecke group $\overline{H}(2, 2, 3, 4, 5)$. Since $s = 2$, $t = 1$ and $u = 2$, we take the generators as A_1, A_2, B_1, C_1, C_2 . Here, there are the relations $A_1^2 = A_2^2 = B_1^4 = C_1^3 = C_2^5$.

Then the index is 16 and the Schreier transversal Σ is $\{A_1, A_2, B_1, A_1A_2, A_1B_1, A_2B_1, A_1A_2B_1, A_1R, A_2R, B_1R, A_1A_2R, A_1B_1R, A_2B_1R, A_1A_2B_1R, I, R\}$. If we use the Reidemeister–Schreier method and make the required calculations, then we get one generator of the form $A_1A_2A_1A_2$; one generator of the form B_1^2 ; four generators of the form $A_1B_1A_1B_1^{-1}, A_1B_1^2A_1, A_2B_1A_2B_1^{-1}, A_2B_1^2A_2$; three generators of the form $A_1A_2B_1A_1(A_2B_1)^{-1}, A_1A_2B_1A_2(A_1B_1)^{-1}, A_1A_2B_1^2(A_1A_2)^{-1}$; two generators of the form C_1, C_2 ; six generators of the form $A_1C_1A_1, A_2C_1A_2, A_1C_2A_1, A_2C_2A_2, B_1C_1B_1^{-1}, B_1C_2B_1^{-1}$; six generators of the form $A_1A_2C_1(A_1A_2)^{-1}, A_1A_2C_2(A_1A_2)^{-1}, A_1B_1C_1(A_1B_1)^{-1}, A_1B_1C_2(A_1B_1)^{-1}, A_2B_1C_1(A_2B_1)^{-1}, A_2B_1C_2(A_2B_1)^{-1}$ and two generators of the form $A_1A_2B_1C_1(A_1A_2B_1)^{-1}, A_1A_2B_1C_2(A_1A_2B_1)^{-1}$. Totally, there are 25 generators of $\overline{H}'(2, 2, 3, 4, 5)$. Also, the signature of $\overline{H}'(2, 2, 3, 4, 5)$ is $(1; 2^{(4)}, 3^{(8)}, 5^{(8)}, \infty^{(4)})$.

Example 2.5 Let us consider the generalized Hecke group $\overline{H}(5, 5, 7, 8)$. Since $s = 0, t = 1$ and $u = 3$, we take the generators as B_1, C_1, C_2, C_3 . Thus we have the relations $B_1^8 = C_1^5 = C_2^5 = C_3^7$. Here the index is 4. Then we can determine the Schreier transversal $\Sigma = \{B_1, B_1R, I, R\}$. If we use the Reidemeister–Schreier method and make the required calculations, then we find the generators of $\overline{H}'(5, 5, 7, 8)$ as one generator of the form B_1^2 ; three generators of the form C_1, C_2, C_3 ; and finally, three generators of the form $B_1C_1B_1^{-1}, B_1C_2B_1^{-1}, B_1C_3B_1^{-1}$. Therefore, there are seven generators of $\overline{H}'(5, 5, 7, 8)$. Also, we obtain the signature of $\overline{H}'(5, 5, 7, 8)$ as $(0; 5^{(4)}, 7^{(2)}, 4, \infty)$.

From Eq. (2.1), if $s + t \leq 1$, then the commutator subgroup $\overline{H}'(p_1, \dots, p_n)$ is isomorphic to the free product of some finite cyclic groups. Thus, we can study the second commutator subgroup $\overline{H}''(p_1, \dots, p_n)$ using Theorem 1.1:

Corollary 2.6 *i) If $s = 0$ and $t = 0$, then $\overline{H}''(p_1, \dots, p_n) (\cong H'(p_1, \dots, p_n))$ is a free group of rank $1 + r_1 \cdot r_2 \cdots r_u \{-1 + \sum_{l=1}^u (1 - \frac{1}{r_l})\}$.*

ii) If $s = 1$ and $t = 0$, then $\overline{H}''(p_1, \dots, p_n)$ is a free group of rank $1 + r_1^2 \cdot r_2^2 \cdots r_u^2 \{-1 + 2 \sum_{l=1}^u (1 - \frac{1}{r_l})\}$.

iii) If $s = 0$ and $t = 1$, then $\overline{H}''(p_1, \dots, p_n)$ is a free group of rank $1 + (q_1/2)r_1^2 \cdot r_2^2 \cdots r_u^2 \{-\frac{2}{q_1} + 2 \sum_{l=1}^u (1 - \frac{1}{r_l})\}$.

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