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Dual quaternion algebra and its derivations

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Abstract: It is well known that the automorphism group $Aut(H)$ of the algebra of real quaternions H consists entirely of inner automorphisms $i_q : p \to q \cdot p \cdot q^{-1}$ for invertible $q \in H$ and is isomorphic to the group of rotations $SO(3)$. Hence, H has only inner derivations $D = ad(x)$, $x \in H$. See [4] for derivations of various types of quaternions over the reals. Unlike real quaternions, the algebra H_d of dual quaternions has no nontrivial inner derivation. Inspired from almost inner derivations for Lie algebras, which were first introduced in [3] in their study of spectral geometry, we introduce coset invariant derivations for dual quaternion algebra being a derivation that simply keeps every dual quaternion in its coset space. We begin with finding conditions for a linear map on H_d become a derivation and show that the dual quaternion algebra H_d consists of only central derivations. We also show how a coset invariant central derivation of H_d is closely related with its spectrum.

Key words: Dual quaternion, derivation

1. Introduction

Derivations of an algebra give interesting insights for studying its algebraic structure. We have considered derivations of Lie algebras and provided for this purpose a simple computational algorithm in [1]. Recently, we have considered quaternions as a class of Lie algebra and given explicitly in [4] derivations of generalized quaternion algebra over the field of real numbers. In this paper, we consider derivations of the algebra H_d of dual quaternions since for such quaternions we have no inner derivations in contrast to classical real quaternions. We first obtain conditions for a linear mapping on H_d be a derivation and show that dual quaternion algebra consists of only central derivations. This is particularly interesting since a derivation of an algebra is the infinitesimal operation corresponding to an automorphism and thus one might determine through the exponential map central automorphisms, as well. Since in the dual quaternion algebra there is no nontrivial inner derivation and every derivation is central we have found it convenient to put an extra condition on such derivations to explore some algebraic properties. Hence, we introduce derivations that keep every dual quaternion in its coset space (i.e. derivations D such that $Dx \in x \cdot H_d$ or $Dx \in H_d \cdot x$ for every $x \in H_d$) and call them coset invariant. The idea actually comes from almost inner derivations for Lie algebras which were first introduced by Gordon and Wilson in [3] in their study of spectral geometry. We refer the reader to the work [2], where the authors establish some algebraic properties of almost inner derivations for Lie algebras. We show how a coset invariant (central) derivation of H_d is related with its eigenvalues.

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This article is organized as follows: Section 2 gives a brief information on classical quaternions. Section 3 considers derivations of the dual quaternion algebra H_d, which is the main subject of the paper. We then determine explicitly all the derivations of H_d and introduce coset invariant derivations which are central since there is no inner derivation in the dual case. We also show how such a derivation acts on pure and non-pure dual quaternions.

2. Preliminaries

An algebra A over a field K is a vector space over K provided with a bilinear map $\cdot : A \times A \to A$. The real quaternions which we denote by H form an algebra over the reals generated by the units e_0, e_1, e_2, e_3 corresponding to the familiar 1, i, j, k. The quaternion (or Hamiltonian) product " $\cdot$ " is determined by the following rules

$$\begin{aligned} e_1 \cdot e_2 &= -e_2 \cdot e_1 = e_3, \\ e_2 \cdot e_3 &= -e_3 \cdot e_2 = e_1, \\ e_3 \cdot e_1 &= -e_1 \cdot e_3 = e_2, \end{aligned}$$

and

$$e_1^2 = e_2^2 = e_3^2 = e_1 \cdot e_2 \cdot e_3 = -e_0$$

where e_0 acts as identity: $e_0 \cdot e_i = e_i \cdot e_0 = e_i$. We express a quaternion $q \in H$ as

$$q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}.$$

Throughout the manuscript we also use interchangeably one of the interpretations $q = (q_0, q_1, q_2, q_3)$ or simply $q = (q_0, \mathbf{q})$ where $\mathbf{q} = (q_1, q_2, q_3) \in \mathbb{R}^3$. The e_0-component q_0 of q may be thought of a time parameter while the vector part $\mathbf{q}$ (also called pure or imaginary quaternion) represents the space.

Denote by H the algebra of all real quaternions which is a four dimensional real vector space with the basis $\mathfrak{B}(H) = \{e_0, e_1, e_2, e_3\}$. The orthonormal basis $\mathfrak{B}(H)$ actually corresponds to the canonical basis of $\mathbb{R}^4$ and hence it might be set

$$e_0 = (1,0,0,0), \quad e_1 = (0,1,0,0), \quad e_2 = (0,0,1,0) \text{ and } e_3 = (0,0,0,1).$$

It should be noted that the quaternion products $e_1 \cdot e_2 = e_3$, $e_2 \cdot e_3 = e_1$ and $e_3 \cdot e_1 = e_2$ are the same as vector product

$$e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2$$

except for the vector product

$$e_1 \times e_1 = e_2 \times e_2 = e_3 \times e_3 = 0$$

while for quaternions, this is $e_i^2 = -e_0$ for any $i = 1,2,3$.

Let $q, p \in H$, where $q = (q_0, \mathbf{q})$ and $p = (p_0, \mathbf{p})$. The conjugate of $q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$ is the quaternion $q^* = q_0 e_0 - q_1 e_1 - q_2 e_2 - q_3 e_3$. Hence the subalgebra $\mathbb{R} = \mathbb{R} \cdot e_0$, which is the center $Z(H)$ of H, can be also interpreted as $\mathbb{R} = \{q \in H : q^* = q\}$. Also, the norm $|q|$ of the quaternion q is given by

$$|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = \sqrt{qq^*}.$$

Since any nonzero quaternion $q \neq 0$ admits a (unique) inverse

$$q^{-1} = \frac{q^*}{|q|^2}$$

it is clear that H is a normed division algebra. Addition (and hence subtraction) is defined componentwise:

$$q + p = (q_0 + p_0, \mathbf{q} + \mathbf{p}).$$

The multiplication rule for quaternions is the same as for polynomials, extended by the multiplicative properties of the quaternion elements e_1, e_2, e_3 as follows: Let $q, p \in H$, where $q = (q_0, \mathbf{q})$ and $p = (p_0, \mathbf{p})$.

$$q \cdot p = (q_0 p_0 - \langle \mathbf{q}, \mathbf{p} \rangle\,,\ q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{q} \times \mathbf{p})$$

where $\langle , \rangle$ and $\times$ stand for the usual scalar and vector product we are familiar from $\mathbb{R}^3$. Multiplication of a quaternion $q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$ by c is given by

$$c \cdot q = c q_0 e_0 + c q_1 e_1 + c q_2 e_2 + c q_3 e_3.$$

It follows that the multiplication of quaternions is anticommutative but associative and distributive over addition.

3. Derivations

Definition 3.1 *(Derivation) A derivation of an algebra A is a linear map $D : A \to A$ such that*

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y) \tag{3.1}$$

for all $x, y \in A$.

A simple example is given by the ring of all polynomials in one variable with real coefficients. One might regard this ring as an algebra over the reals and the mapping that sends a polynomial $P(x)$ into its derivative $D_x(P)$ is a derivation in the above sense. Another example is the algebra A of smooth functions $\mathbb{R} \to \mathbb{R}$ and the usual derivative $D(f) = f'$ where $f \in A$ since $D(f \cdot g) = (Df) \cdot g + f \cdot D(g)$ holds for every $f, g \in A$.

The set of derivations of an algebra A forms a vector space, which we denote by $Der(A)$ and call derivation algebra of A. Recall that $\mathfrak{gl}(A)$ is a Lie algebra with Lie bracket given by $[f, g] = f \circ g - g \circ f$ for all $f, g \in \mathfrak{gl}(A)$. It follows from the definition of derivation and the fact $Der(A) \subset \mathfrak{gl}(A)$ that $Der(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$ since

$$[D_1, D_2](x \cdot y) = [D_1, D_2](x) \cdot y + x \cdot [D_1, D_2](y)$$

for every $D_1, D_2 \in Der(A)$ and $x, y \in A$. This means $[D_1, D_2]$ is also a derivation. Any associative algebra A can be made into a Lie algebra by taking commutator as the Lie bracket $[x, y] = x \cdot y - y \cdot x$ for all $x, y \in A$. It follows that if $D \in Der(A)$, then D is also a derivation of the corresponding Lie algebra $L(A) = \mathfrak{a}$ which means

$$D([x, y]) = [D(x), y] + [x, D(y)] \tag{3.2}$$

for all $x, y \in \mathfrak{a}$. However, there may exist in general an associative algebra A and a derivation of the corresponding Lie algebra which is not a derivation of A. Since we are mostly interested in algebra derivations and not Lie algebra derivations we simply do not mind the converse implication.

A particular class of derivations are so called inner derivations.

Definition 3.2 *(Inner derivation) Let A be an algebra and $x \in A$. By an inner derivation associated to x we mean the map*

$$D = ad(x) : A \longrightarrow A, \qquad y \longmapsto x \cdot y - y \cdot x,$$

for every $y \in A$.

Let $ad(A) = \{ad(x) : x \in A\}$ denote the set of inner derivations of A which is a subalgebra (resp. ideal) of $Der(A)$. In this paper, we mainly consider A as the algebra H_d of dual quaternions. It is known that $ad(H_d)$ can be determined from the equation

$$ad(H_d) \simeq \frac{H_d}{Z(H_d)}$$

which immediately implies $ad(H_d) = \{0\}$ since H_d is commutative and hence $Z(H_d) = H_d$. It is also well known that the automorphism group $Aut(H)$ of classical quaternion algebra H over $\mathbb{R}$ consists entirely of inner automorphisms $i_q : p \rightarrow q \cdot p \cdot q^{-1}$ for invertible $q \in H$ and is isomorphic to the group of rotations $SO(3) = \{A \in GL(3,\mathbb{R}) : A^T A = 1\}$, where $GL(3,\mathbb{R})$ denotes the general linear group of 3×3 invertible matrices with real entries. Moreover, the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ is semisimple and it follows that H has only inner derivations $D = ad(e_i)$ for $i = 1,2,3$. Hence we have $Der(H) = ad(H)$ while $ad(H_d)$ for dual quaternion algebra is trivial as we have just seen above. Actually, $Der(H_d)$ consists of only central derivations as we will see below.

Definition 3.3 *A derivation D of an algebra A is called* central *if it maps A into its center $Z(A)$.*

Denote by $C(A)$ the set of all central derivations of an algebra A. It is clear that $C(A)$ is a subalgebra of $Der(A)$ and every element of $C(A)$ sends the derivation algebra of A to 0. Note also that a central derivation of an algebra A commutes with every inner derivation of A since $C(A) = Cent_{Der(A)}(ad(A))$ where for any ideal I of A

$$Cent_A(I) = \{x \in A : x \cdot y - y \cdot x = 0, \forall y \in I\}$$

denotes the centralizer of I in A. Studying central derivations goes back to sixties. See for example Togo [4]. We show in the subsequent section that the algebra H_d of dual quaternions has only central derivations. See Proposition 3.4. First we give a brief exposition on dual quaternion algebra.

3.1. Dual quaternions

Denote by H_d the algebra of dual quaternions for which we have the following relations between its units:

$$e_i^2 = 0, \forall i \text{ and } e_i e_j = 0 \text{ for } 1 \leq i < j \leq 3.$$

The quaternion product on H_d is defined by

$$x \cdot y = (x_0 y_0)e_0 + (x_0 y_1 + y_0 x_1)e_1 + (x_0 y_2 + y_0 x_2)e_2 + (x_0 y_3 + y_3 x_0)e_3$$

for all $x, y \in H_d$. It follows at once that the center $Z(H_d) = H_d$ since it is commutative. Now, let D be a derivation of H_d. Thus, D admits a matrix representation with respect to the basis $\mathfrak{B}(H_d) = \{e_0, e_1, e_2, e_3\}$

which is the 4×4 matrix $[D] = (d_{ij})^T$ whose entries are defined by the equations

$$\begin{aligned} D(e_0) &= d_{11}e_0 + d_{12}e_1 + d_{13}e_2 + d_{14}e_3 \\ D(e_1) &= d_{21}e_0 + d_{22}e_1 + d_{23}e_2 + d_{24}e_3 \\ D(e_2) &= d_{31}e_0 + d_{32}e_1 + d_{33}e_2 + d_{34}e_3 \\ D(e_3) &= d_{41}e_0 + d_{42}e_1 + d_{43}e_2 + d_{44}e_3. \end{aligned}$$

Each column of the above matrix is of course a dual quaternion. In order to obtain D in the matrix form it suffices to know the Leibnitz rule in (3.1) only between the basis elements. First we notice a simple observation: If D is a derivation of H_d, then the first column of the matrix representation of D consists of only zeros. Indeed, e_0 acts as an identity element and we have for any $i = 1, 2, 3$ that

$$D(e_0 \cdot e_i) = D(e_0) \cdot e_i + e_0 \cdot D(e_i) \Longleftrightarrow D(e_i) = D(e_0) \cdot e_i + D(e_i)$$

which occurs if and only if

$$D(e_0) \cdot e_i = 0, \qquad \forall i = 1, 2, 3.$$

Actually, it is enough to evaluate the latter equation only for e_0 since $D(e_0) \cdot e_0 = 0$ already implies

$$d_{11} = d_{12} = d_{13} = d_{14} = 0.$$

A simple computation yields for $1 \leq i, j \leq 3$ that

$$D(e_i \cdot e_j) = \begin{cases} 2d_{(i+1),1}e_i & \text{if } i = j \\ d_{(j+1),1}e_i + d_{(i+1),1}e_j & \text{if } i \neq j \end{cases}$$

and hence we have obtained a typical derivation of dual quaternion algebra as in the proposition below:

Proposition 3.4 *Any derivation D of the algebra H_d of dual quaternions is of the form*

$$D = \begin{pmatrix} 0 & O \\ O^T & A \end{pmatrix} \in M_4(\mathbb{R}),$$

where $O = (0\ \ 0\ \ 0)$ and $A = (a_{ij})_{1 \leq i,j \leq 3}$ is a submatrix with real entries and the dimension of $Der(H_d)$ is 9. Moreover,

$$Der(H_d) = C(H_d),$$

that is, any derivation is central.

Proof $Der(H_d)$ has no nontrivial inner derivation since $ad(e_i) = 0$ for every i. Thus, the algebra H_d possesses entirely noninner derivations. If $D \in Der(H_d)$, then we have $D(H_d) \in \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3 \subset Z(H_d)$ which means every derivation is central and hence we write $Der(H_d) = C(H_d)$. □

The fact that dual quaternion algebra has only central derivations enables us to connect such derivations to central automorphisms.

Definition 3.5 *An automorphism σ of an algebra A is called* central *if and only if it commutes with every inner automorphism, or $x^{-1} \cdot \sigma(x) \in Z(A)$ for each $x \in A$ where by $Z(A)$ we mean the center of A as usual.*

Denote by $Aut_c(A)$ the set of central automorphisms of an algebra A, which is a subgroup of the group $Aut(A)$ of all A-automorphisms. Since a derivation of an algebra is the infinitesimal operation corresponding to an automorphism it follows that $Aut_c(H_d)$ might be obtained through the exponential image of the algebra of central derivations. Hence, it is enough to select a derivation of H_d and simply take its matrix exponential to generate the corresponding central automorphism in H_d.

Also, apart from the description of D it is immediate that we have a particular advantage of considering a derivation of dual quaternions essentially as a 3×3 matrix. Although one might have either $\det(A) = 0$ or $\det A \neq 0$, this has absolutely no effect in $\det D$ which is always zero. However, the task of finding eigenvalues of D might be simply reduced to that of A:

Corollary 3.6 *Both A and D have the same eigenvalues λ (except possibly for $\lambda = 0$) such that $u = (x, y, z)$ is the corresponding eigenvector of A if and only if $v = (t, u) \in \mathbb{R}^4$ with $\lambda t = 0$ is the corresponding eigenvector of D.*

Proof Because the characteristic polynomials of A and D satisfy

$$\det(D - \lambda I_4) = -\lambda \det(A - \lambda I_3) \tag{3.3}$$

we conclude that A and D have, indeed, the same eigenvalues. In particular, $\lambda = 0$ is always an eigenvalue of D. Regarding the corresponding eigenvectors, let $0 \neq \lambda$ be a common eigenvalue for A and D with the corresponding eigenvectors u and v, respectively. Since

$$D - \lambda I_4 = \begin{pmatrix} -\lambda & O \\ O^T & A - \lambda I_3 \end{pmatrix},$$

it follows immediately that $(D - \lambda I_4)v = 0$ if and only if

$$\begin{pmatrix} -\lambda & O \\ O^T & A - \lambda I_3 \end{pmatrix} \begin{pmatrix} t \\ u \end{pmatrix} = 0,$$

where $(A - \lambda I_3)u = 0$ and $\lambda t = 0$ for any $t \in \mathbb{R}$. □

Since there is no nontrivial inner derivation for dual quaternion algebra but only central derivations we find it convenient to report that (central) derivations of dual quaternions satisfy interesting properties if we assume an extra condition on $Der(H_d)$ as follows:

$$D(x) \in x \cdot H_d, \qquad \forall x \in H_d.$$

Actually, we have inspired by the definition of almost inner derivations which were introduced for Lie algebras by Gordon and Wilson in [3]. We recall that a derivation D of a Lie algebra $\mathfrak{g}$ is called almost inner if $D(x) \in [\mathfrak{g}, x]$ for every $x \in \mathfrak{g}$. Note that $ad\,\mathfrak{g} \cdot x = [\mathfrak{g}, x]$ is the orbit of $ad\,\mathfrak{g}$ under the action of $Der(\mathfrak{g})$ on $\mathfrak{g}$ which in turn is obtained by differentiating the action of $Aut(G)$ on $\mathfrak{g}$. Of course, we could treat H_d with the usual commutator rule as a Lie algebra and consider almost inner derivations for the corresponding Lie algebra $L(H_d) = \mathfrak{h}_d$. However, commutativity implies that a derivation D of $\mathfrak{h}_d$ is almost inner if $D(x) = 0$ for every $x \in \mathfrak{h}_d$, that is, if D annihilates $\mathfrak{h}_d$. Looking at the Proposition 3.4, we see that $D = 0$ is the only possibility. Hence there is no almost inner derivation for the (Lie) algebra of dual quaternions. That is why we manipulate the definition of almost inner derivation and require D to keep each dual quaternion x in its coset space $x \cdot H_d$ rather than belonging to the adjoint orbit $ad\,\mathfrak{h}_d \cdot x$. See the definition below.

Definition 3.7 *We call a derivation D of H_d coset invariant if $D(x) \in x \cdot H_d$ (resp. $D(x) \in H_d \cdot x$) for all $x \in H_d$.*

We denote by $CI(H_d)$ the set of all coset invariant (central) derivations of H_d. Note that if $D_1, D_2 \in CI(H_d)$ are such that $D_1D_2 \in Der(H_d)$, then we can try to see whether or not D_1D_2 is coset invariant. In general, D_1D_2 can fail to be a derivation. However, thanks to Proposition 3.4 no need to worry about it for dual quaternions. In fact, given arbitrary $x \in H_d$ there exist $y, z \in H_d$ such that $D_1(x) = x \cdot y$ and $D_2(x) = x \cdot z$. Thus

$$\begin{aligned}(D_1D_2)(x) &= D_1(x \cdot z) = D_1(x) \cdot z + x \cdot D_1(z) \\ &= (x \cdot y) \cdot z + x \cdot (z \cdot w)\end{aligned}$$

for some $w \in H_d$. It follows from bilinearity of the product $\cdot$ that

$$\begin{aligned}(D_1D_2)(x) &= x \cdot (y \cdot z) + x \cdot (z \cdot w) \\ &= x \cdot t\end{aligned}$$

for some $t = y \cdot z + z \cdot w \in H_d$. That is, $D_1, D_2 \in CI(H_d)$ always yields $D_1D_2 \in CI(H_d)$.

Now, we pay attention to some equations which are apparently simple but at the same time could be useful for finding eigenvalues and the corresponding eigenvectors of both A and D. Hence we find it convenient to state the following

Proposition 3.8 *Let $D \in CI(H_d)$ and let $x = (x_0, x_1, x_2, x_3) \in H_d$ be arbitrary. Then there exists $y = (y_0, y_1, y_2, y_3) \in H_d$ for which*

$$A\mathbf{x} = y_0\mathbf{x} + x_0\mathbf{y}$$

and

$$Dx = y_0x + x_0y,$$

where $\mathbf{x} = (x_1, x_2, x_3)^T$ and $\mathbf{y} = (y_1, y_2, y_3)^T$ are imaginary parts of x and y, respectively.

Proof By Proposition 3.4, being a derivation D is such that $D(e_0) = 0$, $D(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3$, $D(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3$ and $D(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3$. We recall that the multiplicative structure in H_d is defined by

$$x \cdot y = (x_0y_0)e_0 + (x_0y_1 + x_1y_0)e_1 + (x_0y_2 + x_2y_0)e_2 + (x_0y_3 + x_3y_0)e_3.$$

Since D is coset invariant it follows that $D(x) = x \cdot y$ for some $y = (y_0, y_1, y_2, y_3) \in H_d$ implies both

$$0 = x_0y_0 \tag{3.4}$$

and

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = x_0y_1 + x_1y_0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = x_0y_2 + x_2y_0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = x_0y_3 + x_3y_0 \end{array}. \tag{3.5}$$

This immediately yields that $A\mathbf{x} = y_0\mathbf{x} + x_0\mathbf{y}$. Since the first row and column of D consists of only zeros, it follows at once that the equation $A\mathbf{x} = y_0\mathbf{x} + x_0\mathbf{y}$ together with $x_0y_0 = 0$ can be extended to the one defined by $Dx = y_0x + x_0y$. □

Lemma 3.9 *Let $D \in CI(H_d)$ and let $x = (x_0, x_1, x_2, x_3) \in H_d$ be arbitrary. Denote by $[D] = (d_{ij})^T$ the matrix of D. If x is not a pure dual quaternion (i.e. $x_0 \neq 0$), then there exists a (unique) pure dual quaternion $y = (0, y_1, y_2, y_3) \in H_d$ with nonzero components*

$$y_i = \frac{1}{x_0}\sum_{j=1}^{3} d_{i+1,j+1}\ x_j, \qquad 1 \leq i \leq 3,$$

such that $D(x) = x \cdot y$.

Proof Note that $x_0 \neq 0$ together with the condition $x_0 y_0 = 0$ implies necessarily that $y_0 = 0$ and hence $A\mathbf{x} = x_0\mathbf{y} + y_0\mathbf{x} = x_0\mathbf{y}$. Since we are looking for y for which $D(x) = x \cdot y$, it makes sense to organize the collection of equations in (3.5) of the preceding Proposition and state as the following nonhomogeneous linear system Σ_L:

$$\Sigma_L : (x_0 I_3)y = A\mathbf{x}.$$

It follows that if $x \in H_d$ is such that $x_0 \neq 0$ then $\det(x_0 I_3) \neq 0$ and hence Σ_L has a unique non-trivial solution $y = \frac{1}{x_0}A\mathbf{x}$ (resp. unique pure dual quaternion $y = (0, \mathbf{y}) \in H_d$ with $y = (y_i)_{1\leq i\leq 3}$, such that $y_i = (1/x_0)\Sigma_{j=1}^{3} d_{i+1,j+1} x_j$). □

Remark 3.10 *The Lemma above says actually that if x is not selected to be a pure dual quaternion then it follows for any derivation $D \in Der(H_d)$ that we have $Dx \in x \cdot H_d$, i.e. every derivation is coset invariant and hence $Der(H_d) = CI(H_d)$ on imaginary dual quaternions. If x is a pure dual quaternion then $x_0 = 0$ should imply $y_0 = 0$ or $y_0 \neq 0$. This means in the case $x_0 = y_0 = 0$ one gets for any $D \in Der(H_d)$ that $Dx = 0 = x \cdot 0 \in x \cdot H_d$ (that is, $Dx = x \cdot y$ for $y = (0,0,0,0) \in H_d$). If $x_0 = 0$ and $y_0 \neq 0$ then $Dx = y_0 x = x y_0$ implies $y_0 = d_{22} = d_{33} = d_{44}$ and hence only a derivation of the form*

$$D = \begin{pmatrix} 0 & O \\ O^T & y_0 I_3 \end{pmatrix}$$

acts as a coset invariant derivation since $Dx = xy_0 = x \cdot y$ for $y = (y_0, 0, 0, 0) \in H_d$.

In view of Proposition 3.8, everything indicates that the equation(s) $A\mathbf{x} = y_0\mathbf{x} + x_0\mathbf{y}$ and/or $Dx = y_0 x + x_0 y$ can be analyzed by means of its terms, one of which is for determining explicitly the existing y when D is a coset invariant derivation. This is already done in Lemma 3.9. And the other term serves to obtain eigenvalues of D. Hence, we present below one more result to express eigenvectors of coset invariant derivations for dual quaternions.

Theorem 3.11 *Let $D \in CI(H_d)$ and let $x = (x_0, x_1, x_2, x_3) \in H_d$ be arbitrary. If x is such that $x_0 = 0$ (i.e. x is a pure dual quaternion) then there exists a $y = (y_0, y_1, y_2, y_3) \in H_d$ such that $Re(y) = y_0 \in \mathrm{Spect}(D)$ with the corresponding eigenvector x.*

Proof $x_0 = 0$ implies either $y_0 = 0$ or $y_0 \neq 0$ since $x_0 y_0 = 0$ as we have already noted before. Hence, we have $Dx = x_0 y + y_0 x = y_0 x$ with $y_0 = 0$ or $y_0 \neq 0$, where $x = (0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$. This can be also interpreted as $(D - y_0 I_4)x = 0$ with $y_0 = 0$ or $y_0 \neq 0$. Since always $\det(D - y_0 I_4)\,|_{y_0=0} = \det D = 0$

we obtain $y_0 = 0 \in \mathrm{Spect}(D)$ for any derivation $D \in Der(H_d)$. In case $y_0 \neq 0$ we have again $y_0 \in \mathrm{Spect}(D)$ but only for the derivation $D \in Der(H_d)$ with the submatrix $A = y_0 I_3$. □

Remark 3.12 *If $D \in CI(H_d)$, then for a given $x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3 \in H_d$ we write $D(x) = x \cdot y$ for some $y \in H_d$ and we have seen that $Re(y)$ of y might be related with the spectrum of D. On the other hand, if we consider the 4×4 real matrix representation*

$$x \sim X = \begin{pmatrix} x_0 & 0 & 0 & 0 \\ x_1 & x_0 & 0 & 0 \\ x_2 & 0 & x_0 & 0 \\ x_3 & 0 & 0 & x_0 \end{pmatrix}$$

of x left multiplication we also see that in case x is not a pure dual quaternion, then $Re(x)$ of x is the only eigenvalue of the matrix X.

We end the paper mentioning shortly about parabolic quaternions since this class of quaternions as well as H_d does not provide inner derivations. We will content ourselves providing only derivations of such quaternions without further discussions. Denote by H_p the algebra of parabolic quaternions whose units satisfy

$$\begin{aligned} e_1^2 &= e_3^2 = 0, \quad e_2^2 = e_0 \\ e_1 \cdot e_2 &= e_3, \quad e_2 \cdot e_3 = e_1, \quad e_3 \cdot e_1 = 0. \end{aligned}$$

Although $ad(H_p) = \{0\}$ there may exists a derivation in general. Repeating what we have done so far, it is not difficult to see that we obtain a derivation D of H_p as the following matrix:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & b & 0 & a \end{pmatrix}.$$

Hence, the dimension of $Der(H_p)$ is 2. Note that since $D^T = D$, the transpose of D is also a derivation (the same is true also for H_d) which does not happen in general. Observe also that there are typically two generators of the algebra $Der(H_p)$ given by

$$D_1 = \begin{pmatrix} A & O \\ O & A \end{pmatrix}, \qquad D_2 = \begin{pmatrix} O & A \\ A & O \end{pmatrix},$$

where $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and O (zero matrix) are 2×2 matrices. It follows that D_1D_2 is a derivation (resp. $D_2D_1 \in Der(H_p)$).

On the other hand, if A is an algebra over a commutative field K with characteristic $p \neq 0$ then it follows by the very definition of derivation that one has the Leibnitz formula for pth power of a derivation D, that is,

$$D^p(x \cdot y) = D^p(x) \cdot y + x \cdot D^p(y)$$

for every $x, y \in A$. Here, we have in particular that $D_1^p = D_1$ (resp. $D_2^p = D_2$) since the matrix A is idempotent. One may also observe that the derivation $[D_1, D_2]$ of the corresponding Lie algebra $\mathfrak{h}_p$ is 0.

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