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ANARGYROS KATSABEKIS

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Hilbert series of tangent cones for Gorenstein monomial curves in $\mathbb{A}^4(K)$

Anargyros KATSABEKIS*
Department of Mathematics, Faculty of Science, Bilkent University, Ankara, Turkey

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Abstract: In this paper, we study the Hilbert series of the tangent cone of Gorenstein monomial curves in the 4-dimensional affine space. We give an explicit formula for the reduced Hilbert series of the tangent cone of a noncomplete intersection Gorenstein monomial curve whose tangent cone is Cohen–Macaulay.

Key words: Hilbert series, tangent cone, Gorenstein monomial curve

1. Introduction

Let $n_1 < n_2 < \cdots < n_d$ be positive integers with $\gcd(n_1, \ldots, n_d) = 1$. Consider the polynomial ring $R = K[x_1, \ldots, x_d]$ in d variables over a field K. We shall denote by $\mathbf{x}^{\mathbf{u}}$ the monomial $x_1^{u_1} \cdots x_d^{u_d}$ of R, with $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{N}^d$, where $\mathbb{N}$ stands for the set of nonnegative integers. Consider the affine monomial curve in the d-dimensional affine space $\mathbb{A}^d(K)$ defined parametrically by $x_1 = t^{n_1}, \ldots, x_d = t^{n_d}$. The toric ideal of C, denoted by I_C, is the kernel of the K-algebra homomorphism $\varphi : R \to K[t]$ given by

$$\varphi(x_i) = t^{n_i} \text{ for all } 1 \leq i \leq d.$$

The ideal I_C is generated by all the binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ such that $\varphi(\mathbf{x}^{\mathbf{u}}) = \varphi(\mathbf{x}^{\mathbf{v}})$ see for example, [7, Lemma 4.1]. Given a polynomial $f \in I_C$, we let f^* be the homogeneous summand of f of least degree. We shall denote by I_C^* the ideal in R generated by the polynomials f^* for $f \in I_C$.

Let $m =< t^{n_1}, \ldots, t^{n_d} >$ be the maximal ideal of the one-dimensional local ring $A = K[[t^{n_1}, \ldots, t^{n_d}]]$. The Hilbert function H_A of A is defined by $\mathrm{H}_A(i) = \dim_{A/m}(m^i/m^{i+1})$ for every $i \in \mathbb{N}$, which coincides, by its definition, with the Hilbert function of the associated graded ring $\mathrm{gr}_m(A) = \bigoplus_{i \geq 0} m^i/m^{i+1}$. It is worth noting that $\mathrm{gr}_m(A)$ is isomorphic to the quotient R/I_C^*. We recall that I_C^* is the defining ideal of the tangent cone of C at the origin.

Given an ideal $J \subset R$, we shall denote by $\mathrm{HS}(R/J, z)$ the Hilbert series of the ring R/J, namely $\mathrm{HS}(R/J, z) = \sum_{i \in \mathbb{N}} \mathrm{H}_{R/J}(i) z^i$ where $\mathrm{H}_{R/J}$ is the Hilbert function of R/J. By the Hilbert–Serre theorem, $\mathrm{HS}(R/J, z)$ is a rational function of the form $\mathrm{HS}(R/J, z) = \frac{p(z)}{(1-z)^d}$ for some $p(z) \in \mathbb{Z}[z]$. In particular, by reducing this rational function we get $\mathrm{HS}(R/J, z) = \frac{h(z)}{(1-z)^e}$ for some $h(z) \in \mathbb{Z}[z]$, where e is the Krull dimension of R/J.

*Correspondence: katsampekis@bilkent.edu.tr
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In this paper, we study the reduced Hilbert series of the ring R/I_C^*. Since the Krull dimension of the above ring is equal to 1, we can write $\mathrm{HS}(R/I_C^*, z) = \frac{h(z)}{1-z}$. We restrict ourselves to Gorenstein noncomplete intersection monomial curves in $\mathbb{A}^4(K)$. Recall that a monomial curve C is called Gorenstein if the associated local ring A is Gorenstein. Furthermore we assume that R/I_C^* is a Cohen–Macaulay ring. The significance of this class is underscored by the following result: If R/I_C^* is Cohen–Macaulay, then the Hilbert function of A is nondecreasing. Our aim is to give an explicit formula for the numerator of the reduced Hilbert series of the ring R/I_C^* depending only on a minimal generating set of I_C.

In [4] Bresinsky provided a minimal generating set of I_C consisting of five generators. Actually, there are 6 permutations of the above generator set. In [1] the authors provided necessary and sufficient conditions for the Cohen–Macaulayness of R/I_C^* in all six permutations. We compute a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$. Using [3, Proposition 2.2] we determine a formula for the numerator of the reduced Hilbert series of $R/\mathrm{LM}(I_C^*)$, where $\mathrm{LM}(I_C^*)$ is the ideal generated by the leading monomials of the polynomials of I_C^* with respect to the aforementioned order. By [5, Theorem 5.2.6] the reduced Hilbert series of R/I_C^* coincides with the reduced Hilbert series of $R/\mathrm{LM}(I_C^*)$.

2. Formulas for the reduced Hilbert series

In this section we first recall Bresinsky's theorem, which gives the explicit description of I_C when C is a Gorenstein noncomplete intersection monomial curve in $\mathbb{A}^4(K)$. By Kunz [6] it is well known that the Gorenstein property of a monomial curve is equivalent to the symmetric property of $\mathbb{N}\{n_1, \ldots, n_4\}$.

Theorem 2.1 *([4]) Let C be a monomial curve having the parametrization*

$$x_1 = t^{n_1}, x_2 = t^{n_2}, x_3 = t^{n_3}, x_4 = t^{n_4}.$$

The semigroup $\mathbb{N}\{n_1, \ldots, n_4\}$ is symmetric and C is a noncomplete intersection curve if and only if I_C is minimally generated by the set

$$\{f_1 = x_1^{a_1} - x_3^{a_{13}}x_4^{a_{14}}, f_2 = x_2^{a_2} - x_1^{a_{21}}x_4^{a_{24}}, f_3 = x_3^{a_3} - x_1^{a_{31}}x_2^{a_{32}}, f_4 = x_4^{a_4} - x_2^{a_{42}}x_3^{a_{43}}, f_5 = x_1^{a_{21}}x_3^{a_{43}} - x_2^{a_{32}}x_4^{a_{14}}\}$$

where the polynomials f_i are unique up to isomorphism, $a_{ij} > 0$ and also

$$a_1 = a_{21} + a_{31}, a_2 = a_{32} + a_{42}, a_3 = a_{13} + a_{43}, a_4 = a_{14} + a_{24}.$$

Remark 2.2 *([2])* Theorem 2.1 implies that for any noncomplete intersection Gorenstein monomial curve in $\mathbb{A}^4(K)$, the variables can be renamed to obtain generators exactly of the given form, and this means that there are six isomorphic possible permutations which can be considered within three cases:

(1) $f_1 = (1, (3, 4))$

   (a) $f_2 = (2, (1, 4))$, $f_3 = (3, (1, 2))$, $f_4 = (4, (2, 3))$, $f_5 = ((1, 3), (2, 4))$

   (b) $f_2 = (2, (1, 3))$, $f_3 = (3, (2, 4))$, $f_4 = (4, (1, 2))$, $f_5 = ((1, 4), (2, 3))$

(2) $f_1 = (1, (2, 3))$

   (a) $f_2 = (2, (3, 4))$, $f_3 = (3, (1, 4))$, $f_4 = (4, (1, 2))$, $f_5 = ((2, 4), (1, 3))$

(b) $f_2 = (2,(1,4))$, $f_3 = (3,(2,4))$, $f_4 = (4,(1,3))$, $f_5 = ((1,2),(4,3))$

(3) $f_1 = (1,(2,4))$

(a) $f_2 = (2,(1,3))$, $f_3 = (3,(1,4))$, $f_4 = (4,(2,3))$, $f_5 = ((1,2),(3,4))$

(b) $f_2 = (2,(3,4))$, $f_3 = (3,(1,2))$, $f_4 = (4,(1,3))$, $f_5 = ((2,3),(1,4))$

Here, the notation $f_i = (i,(j,k))$ and $f_5 = ((i,j),(k,l))$ denote the generators $f_i = x_i^{a_i} - x_j^{a_{ij}} x_k^{a_{ik}}$ and $f_5 = x_i^{a_{ki}} x_j^{a_{lj}} - x_k^{a_{jk}} x_l^{a_{il}}$. Thus, given a Gorenstein monomial curve C, if we have the extra condition $n_1 < n_2 < n_3 < n_4$, then the generator set of I_C is exactly given by one of these six permutations.

In [1] they provided necessary and sufficient conditions for the Cohen–Macaulayness of R/I_C^*. More precisely they proved the following.

Theorem 2.3 *([1]) (1) Suppose that I_C is given as in case 1(a). Then R/I_C^* is Cohen–Macaulay if and only if $a_2 \leq a_{21} + a_{24}$.*
(2) Suppose that I_C is given as in case 1(b). (i) Assume that $a_{42} \leq a_{32}$. Then R/I_C^* is Cohen–Macaulay if and only if

1. $a_2 \leq a_{21} + a_{23}$,
2. $a_{42} + a_{13} \leq a_{21} + a_{34}$, *and*
3. *either* $a_{34} < a_{14}$ *and* $a_3 + a_{13} \leq a_{21} + a_{32} - a_{42} + 2a_{34}$ *or* $a_{14} \leq a_{34}$ *and* $a_3 + a_{13} \leq a_1 + a_{32} + a_{34} - a_{14}$.

(ii) Assume that $a_{32} < a_{42}$ and $a_{14} \leq a_{34}$. Then R/I_C^* is Cohen–Macaulay if and only if

1. $a_2 \leq a_{21} + a_{23}$,
2. $a_{42} + a_{13} \leq a_{21} + a_{34}$, *and*
3. $a_3 + a_{13} \leq a_1 + a_{32} + a_{34} - a_{14}$.

(3) Suppose that I_C is given as in case 2(a). (i) Assume that $a_{34} \leq a_{24}$. Then R/I_C^* is Cohen–Macaulay if and only if

1. $a_3 \leq a_{31} + a_{34}$,
2. $a_{12} + a_{34} \leq a_{41} + a_{23}$, *and*
3. *either* $a_{23} < a_{13}$ *and* $a_2 + a_{12} \leq a_{41} + 2a_{23} + a_{24} - a_{34}$ *or* $a_{13} \leq a_{23}$ *and* $a_2 + a_{12} \leq a_1 + a_{23} - a_{13} + a_{24}$.

(ii) Assume that $a_{24} < a_{34}$ and $a_{13} \leq a_{23}$. Then R/I_C^* is Cohen–Macaulay if and only if

1. $a_3 \leq a_{31} + a_{34}$,
2. $a_{12} + a_{34} \leq a_{41} + a_{23}$, *and*
3. $a_2 + a_{12} \leq a_1 + a_{23} - a_{13} + a_{24}$.

(4) Suppose that I_C is given as in case 2(b). (i) Assume that $a_{24} \leq a_{34}$. Then R/I_C^* is Cohen–Macaulay if and only if

1. $a_2 \leq a_{21} + a_{24}$ *and*
2. *either* $a_{32} < a_{12}$ *and* $a_3 + a_{13} \leq a_{41} + 2a_{32} + a_{34} - a_{24}$ *or* $a_{12} \leq a_{32}$ *and* $a_3 + a_{13} \leq a_1 + a_{32} - a_{12} + a_{34}$.

(ii) Assume that $a_{34} < a_{24}$ and $a_{12} \leq a_{32}$. Then R/I_C^* is Cohen–Macaulay if and only if

1. $a_2 \leq a_{21} + a_{24}$ *and*
2. $a_3 + a_{13} \leq a_1 + a_{32} - a_{12} + a_{34}$.

(5) Suppose that I_C is given as in case 3(a). Then R/I_C^* is Cohen–Macaulay if and only if $a_2 \leq a_{21} + a_{23}$ and $a_3 \leq a_{31} + a_{34}$.

(6) Suppose that I_C is given as in case 3(b). (i) Assume that $a_{43} \leq a_{23}$. Then R/I_C^* is Cohen–Macaulay if and only if

1. $a_{12} + a_{43} \leq a_{31} + a_{24}$ *and*
2. *either* $a_{24} < a_{14}$ *and* $a_2 + a_{12} \leq a_{31} + 2a_{24} + a_{23} - a_{43}$ *or* $a_{14} \leq a_{24}$ *and* $a_2 + a_{12} \leq a_1 + a_{23} + a_{24} - a_{14}$.

(ii) Assume that $a_{23} < a_{43}$ and $a_{14} \leq a_{24}$. Then R/I_C^* is Cohen–Macaulay if and only if

1. $a_{12} + a_{43} \leq a_{31} + a_{24}$ *and*
2. $a_2 + a_{12} \leq a_1 + a_{23} + a_{24} - a_{14}$.

For the rest of this section, we assume that R/I_C^* is a Cohen–Macaulay ring.

In the sequel, we will make repeatedly use of the next proposition.

Proposition 2.4 *([3, Proposition 2.2]) Let $I \subset R$ be a monomial ideal and let $I = \langle J, \mathbf{x}^{\mathbf{u}} \rangle$ for a monomial ideal J and a monomial $\mathbf{x}^{\mathbf{u}}$. For an ideal $M \subset R$ denote by $p(M)$ the numerator of the Hilbert series of R/M. Then $p(I) = p(J) - z^{\deg(\mathbf{x}^{\mathbf{u}})} p(J : \langle \mathbf{x}^{\mathbf{u}} \rangle)$.*

Theorem 2.5 *Suppose that I_C is given as in case 1(a). Then the reduced Hilbert series of R/I_C^* is $\mathrm{HS}(R/I_C^*, z) = \frac{h(z)}{1-z}$ for*

$$h(z) = \prod_{i=2}^{4}(1 + z + z^2 + \cdots + z^{a_i - 1}) - z^{a_{14}+a_{32}} \sum_{i=0}^{a_{42}-1} z^i \sum_{i=0}^{a_3-1} z^i \sum_{i=0}^{a_{24}-1} z^i - z^{a_{13}+a_{14}} \sum_{i=0}^{a_{32}-1} z^i \sum_{i=0}^{a_{43}-1} z^i \sum_{i=0}^{a_{24}-1} z^i.$$

Proof. By [2, Lemma 2.7],

$$G = \{x_1^{a_1} - x_3^{a_{13}} x_4^{a_{14}}, x_2^{a_2} - x_1^{a_{21}} x_4^{a_{24}}, x_3^{a_3} - x_1^{a_{31}} x_2^{a_{32}}, x_4^{a_4} - x_2^{a_{42}} x_3^{a_{43}}, x_1^{a_{21}} x_3^{a_{43}} - x_2^{a_{32}} x_4^{a_{14}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$. From [5, Lemma 5.5.11] the ideal I_C^* is generated by the least homogeneous summands of the elements in G. In addition, $\mathrm{LM}(I_C^*) = \langle x_3^{a_{13}} x_4^{a_{14}}, x_2^{a_2}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{32}} x_4^{a_{14}} \rangle$. Let

$$J_0 = \langle \mathrm{LM}(I_C^*) \rangle, J_1 = \langle x_2^{a_2}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{32}} x_4^{a_{14}} \rangle, J_2 = \langle x_2^{a_2}, x_3^{a_3}, x_4^{a_4} \rangle.$$

Then $J_i =< J_{i+1}, q_i >$, where $q_0 = x_3^{a_{13}} x_4^{a_{14}}$ and $q_1 = x_2^{a_{32}} x_4^{a_{14}}$. By Proposition 2.4,

$$p(J_i) = p(J_{i+1}) - z^{\deg(q_i)} p(J_{i+1} :< q_i >), \quad \text{for} \quad 0 \le i \le 1.$$

Note that $\deg(q_0) = a_{13} + a_{14}$ and $\deg(q_1) = a_{14} + a_{32}$. In this case, $J_1 :< q_0 >=< x_2^{a_{32}}, x_3^{a_{43}}, x_4^{a_{24}} >$ and $J_2 :< q_1 >=< x_2^{a_{42}}, x_3^{a_3}, x_4^{a_{24}} >$. We have that $p(J_2) = \prod_{i=2}^4 (1-z^{a_i})$ and $p(J_2 :< q_1 >) = (1-z^{a_{42}})(1-z^{a_3})(1-z^{a_{24}})$, so $p(J_1) = \prod_{i=2}^4 (1-z^{a_i}) - z^{a_{32}+a_{14}}(1-z^{a_{42}})(1-z^{a_3})(1-z^{a_{24}})$. Furthermore $p(J_1 :< q_0 >) = (1-z^{a_{32}})(1-z^{a_{43}})(1-z^{a_{24}})$, so $p(J_0) = \prod_{i=2}^4 (1-z^{a_i}) - z^{a_{14}+a_{32}}(1-z^{a_{42}})(1-z^{a_3})(1-z^{a_{24}}) - z^{a_{13}+a_{14}}(1-z^{a_{32}})(1-z^{a_{43}})(1-z^{a_{24}})$. Now using the fact that $1 - z^b = (1-z)\sum_{i=0}^{b-1} z^i$ we get

$$p(J_0) = (1-z)^3 \sum_{i=0}^{a_2-1} z^i \sum_{i=0}^{a_3-1} z^i \sum_{i=0}^{a_4-1} z^i - (1-z)^3 z^{a_{14}+a_{32}} \sum_{i=0}^{a_{42}-1} z^i \sum_{i=0}^{a_3-1} z^i \sum_{i=0}^{a_{24}-1} z^i -$$

$$(1-z)^3 z^{a_{13}+a_{14}} \sum_{i=0}^{a_{32}-1} z^i \sum_{i=0}^{a_{43}-1} z^i \sum_{i=0}^{a_{24}-1} z^i.$$

Thus $p(J_0) = (1-z)^3 h(z)$, and therefore $\mathrm{HS}(R/\mathrm{LM}(I_C^*), z) = \frac{h(z)}{1-z}$. □

We continue with case 1(b). If $a_3 \le a_{32} + a_{34}$, then we can use [2, Remark 2.9] to find the reduced Hilbert series of R/I_C^*.

Theorem 2.6 *Suppose that I_C is given as in case 1(b) and also that $a_3 \le a_{32} + a_{34}$. Then the reduced Hilbert series of R/I_C^* is $\mathrm{HS}(R/I_C^*, z) = \frac{h(z)}{1-z}$ for*

$$h(z) = \prod_{i=2}^{4} (1 + z + z^2 + \cdots + z^{a_i - 1}) - z^{a_{13}+a_{42}} \sum_{i=0}^{a_{32}-1} z^i \sum_{i=0}^{a_{23}-1} z^i \sum_{i=0}^{a_4-1} z^i - z^{a_{13}+a_{14}} \sum_{i=0}^{a_{42}-1} z^i \sum_{i=0}^{a_{23}-1} z^i \sum_{i=0}^{a_{34}-1} z^i.$$

Proof. By [2, Remark 2.9],

$$G = \{x_1^{a_1} - x_3^{a_{13}} x_4^{a_{14}}, x_2^{a_2} - x_1^{a_{21}} x_3^{a_{23}}, x_3^{a_3} - x_2^{a_{32}} x_4^{a_{34}}, x_4^{a_4} - x_1^{a_{41}} x_2^{a_{42}}, x_1^{a_{21}} x_4^{a_{34}} - x_2^{a_{42}} x_3^{a_{13}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$. From [5, Lemma 5.5.11] the ideal I_C^* is generated by the least homogeneous summands of the elements in G. Also $\mathrm{LM}(I_C^*) =< x_3^{a_{13}} x_4^{a_{14}}, x_2^{a_2}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{42}} x_3^{a_{13}} >$. Let

$$J_0 = \mathrm{LM}(I_C^*), J_1 =< x_2^{a_2}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{42}} x_3^{a_{13}} >, J_2 =< x_2^{a_2}, x_3^{a_3}, x_4^{a_4} > .$$

Then $J_i =< J_{i+1}, q_i >$, where $q_0 = x_3^{a_{13}} x_4^{a_{14}}$ and $q_1 = x_2^{a_{42}} x_3^{a_{13}}$. By Proposition 2.4,

$$p(J_i) = p(J_{i+1}) - z^{\deg(q_i)} p(J_{i+1} :< q_i >), \quad \text{for} \quad 0 \le i \le 1.$$

Note that $\deg(q_0) = a_{13} + a_{14}$ and $\deg(q_1) = a_{13} + a_{42}$. In this case, $J_1 :< q_0 >=< x_2^{a_{42}}, x_3^{a_{23}}, x_4^{a_{34}} >$ and $J_2 :< q_1 >=< x_2^{a_{32}}, x_3^{a_{23}}, x_4^{a_4} >$. We have that $p(J_2) = \prod_{i=2}^4 (1-z^{a_i})$ and $p(J_2 :< q_1 >) = (1 -$

$z^{a_{32}})(1-z^{a_{23}})(1-z^{a_4})$, so $p(J_1)=\prod_{i=2}^{4}(1-z^{a_i})-z^{a_{13}+a_{42}}(1-z^{a_{32}})(1-z^{a_{23}})(1-z^{a_4})$. Furthermore $p(J_1:<q_0>)=(1-z^{a_{42}})(1-z^{a_{23}})(1-z^{a_{34}})$, so

$$p(J_0)=\prod_{i=2}^{4}(1-z^{a_i})-z^{a_{13}+a_{42}}(1-z^{a_{32}})(1-z^{a_{23}})(1-z^{a_4})-z^{a_{13}+a_{14}}(1-z^{a_{42}})(1-z^{a_{23}})(1-z^{a_{34}}).$$

Using the fact that $1-z^b=(1-z)\sum_{i=0}^{b-1}z^i$ we get $p(J_0)=(1-z)^3h(z)$. Thus $\mathrm{HS}(R/\mathrm{LM}(I_C^*),z)=\frac{h(z)}{1-z}$. $\square$

Proposition 2.7 *Suppose that I_C is given as in case 1(b). Let $a_3>a_{32}+a_{34}$, $a_{32}<a_{42}$ and $a_{14}\leq a_{34}$. Then the set*

$$G=\{f_1=x_1^{a_1}-x_3^{a_{13}}x_4^{a_{14}},f_2=x_2^{a_2}-x_1^{a_{21}}x_3^{a_{23}},f_3=x_3^{a_3}-x_2^{a_{32}}x_4^{a_{34}},$$

$$f_4=x_4^{a_4}-x_1^{a_{41}}x_2^{a_{42}},f_5=x_1^{a_{21}}x_4^{a_{34}}-x_2^{a_{42}}x_3^{a_{13}},f_6=x_3^{a_3+a_{13}}-x_1^{a_1}x_2^{a_{32}}x_4^{a_{34}-a_{14}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4>x_3>x_2>x_1$.

Proof. We will apply the standard basis algorithm [5] to the set $G=\{f_1,\dots,f_6\}$. Here $\mathrm{ecart}(g)$ denotes $\deg(g)-\deg(\mathrm{LM}(g))$. We will show that $\mathrm{NF}(\mathrm{spoly}(f_i,f_j)|G)=0$, for all i, j with $1\leq i<j\leq 6$. Here $\mathrm{LM}(f_1)=x_3^{a_{13}}x_4^{a_{14}}$, $\mathrm{LM}(f_2)=x_2^{a_2}$, $\mathrm{LM}(f_3)=x_2^{a_{32}}x_4^{a_{34}}$, $\mathrm{LM}(f_4)=x_4^{a_4}$, $\mathrm{LM}(f_5)=x_2^{a_{42}}x_3^{a_{13}}$ and $\mathrm{LM}(f_6)=x_3^{a_3+a_{13}}$. Therefore $\mathrm{NF}(\mathrm{spoly}(f_i,f_j)|G)=0$ as $\mathrm{LM}(f_i)$ and $\mathrm{LM}(f_j)$ are relatively prime, for $(i,j)\in\{(1,2),(2,4),(2,6),(3,6),(4,5),(4,6)\}$.

- $\mathrm{spoly}(f_1,f_3)=x_3^{a_3+a_{13}}-x_1^{a_1}x_2^{a_{32}}x_4^{a_{34}-a_{14}}=f_6$, so $\mathrm{NF}(\mathrm{spoly}(f_1,f_3)|G)=0$.
- $\mathrm{spoly}(f_1,f_4)=x_1^{a_{41}}x_2^{a_{42}}x_3^{a_{13}}-x_1^{a_1}x_4^{a_{34}}$. It holds that $a_{34}+a_{21}\geq a_{42}+a_{13}$, so $a_{34}+a_{21}+a_{41}\geq a_{41}+a_{42}+a_{13}$ and therefore $a_1+a_{34}\geq a_{41}+a_{42}+a_{13}$. Thus $\mathrm{LM}(\mathrm{spoly}(f_1,f_4))=x_1^{a_{41}}x_2^{a_{42}}x_3^{a_{13}}$. Only $\mathrm{LM}(f_5)$ divides $\mathrm{LM}(\mathrm{spoly}(f_1,f_4))$. Using the fact that $a_{21}=a_1-a_{41}$ we get $\mathrm{ecart}(\mathrm{spoly}(f_1,f_4))=\mathrm{ecart}(f_5)$. Since $\mathrm{spoly}(f_5,\mathrm{spoly}(f_1,f_4))=0$, we get $\mathrm{NF}(\mathrm{spoly}(f_1,f_4)|G)=0$.
- $\mathrm{spoly}(f_1,f_5)=x_1^{a_{21}}x_4^{a_4}-x_1^{a_1}x_2^{a_{42}}$. Since $a_4<a_{41}+a_{42}$, we get that $a_4+a_{21}<a_{21}+a_{41}+a_{42}=a_1+a_{42}$. Thus $\mathrm{LM}(\mathrm{spoly}(f_1,f_5))=x_1^{a_{21}}x_4^{a_4}$. Only $\mathrm{LM}(f_4)$ divides $\mathrm{LM}(\mathrm{spoly}(f_1,f_5))$ and $\mathrm{ecart}(\mathrm{spoly}(f_1,f_5))=\mathrm{ecart}(f_4)$. Then $\mathrm{spoly}(f_4,\mathrm{spoly}(f_1,f_5))=0$ and also $\mathrm{NF}(\mathrm{spoly}(f_1,f_5)|G)=0$.
- $\mathrm{spoly}(f_1,f_6)=x_1^{a_1}x_2^{a_{32}}x_4^{a_{34}}-x_1^{a_1}x_3^{a_3}$. Then $\mathrm{LM}(\mathrm{spoly}(f_1,f_6))=x_1^{a_1}x_2^{a_{32}}x_4^{a_{34}}$ and only $\mathrm{LM}(f_3)$ divides $\mathrm{LM}(\mathrm{spoly}(f_1,f_6))$. Note that $\mathrm{ecart}(\mathrm{spoly}(f_1,f_6))=\mathrm{ecart}(f_3)$. Then $\mathrm{spoly}(f_3,\mathrm{spoly}(f_1,f_6))=0$ and $\mathrm{NF}(\mathrm{spoly}(f_1,f_6)|G)=0$.
- $\mathrm{spoly}(f_2,f_3)=x_2^{a_{42}}x_3^{a_3}-x_1^{a_{21}}x_3^{a_{23}}x_4^{a_{34}}$. Since $a_{42}+a_{13}\leq a_{21}+a_{34}$, we have $a_{42}+a_3\leq a_{21}+a_{34}+a_{23}$. Thus $\mathrm{LM}(\mathrm{spoly}(f_2,f_3))=x_2^{a_{42}}x_3^{a_3}$ and only $\mathrm{LM}(f_5)$ divides $\mathrm{LM}(\mathrm{spoly}(f_2,f_3))$. Using the fact that $a_3=a_{13}+a_{23}$ we get $\mathrm{ecart}(\mathrm{spoly}(f_2,f_3))=\mathrm{ecart}(f_5)$. Then $\mathrm{spoly}(f_5,\mathrm{spoly}(f_2,f_3))=0$ and also $\mathrm{NF}(\mathrm{spoly}(f_2,f_3)|G)=0$.
- $\mathrm{spoly}(f_2,f_5)=x_1^{a_{21}}x_2^{a_{32}}x_4^{a_{34}}-x_1^{a_{21}}x_3^{a_3}$. Since $a_{32}+a_{34}<a_3$, we get $a_{32}+a_{34}+a_{21}<a_3+a_{21}$. Thus $\mathrm{LM}(\mathrm{spoly}(f_2,f_5))=x_1^{a_{21}}x_2^{a_{32}}x_4^{a_{34}}$ and only $\mathrm{LM}(f_3)$ divides $\mathrm{LM}(\mathrm{spoly}(f_2,f_5))$. Furthermore $\mathrm{ecart}(\mathrm{spoly}(f_2,f_5))=\mathrm{ecart}(f_3)$. Then $\mathrm{spoly}(f_3,\mathrm{spoly}(f_2,f_5))=0$ and also $\mathrm{NF}(\mathrm{spoly}(f_2,f_5)|G)=0$.

- $\text{spoly}(f_3, f_4) = x_1^{a_{41}} x_2^{a_2} - x_3^{a_3} x_4^{a_{14}}$. Suppose that $a_{41}+a_2 \leq a_3+a_{14}$. Then $(a_{41}+a_2)n_2 > a_{41}n_1+a_2n_2$ and also $(a_{41}+a_2)n_2 > a_3n_3+a_{14}n_4$, since $a_{41}n_1+a_2n_2 = a_3n_3+a_{14}n_4$. But $a_3n_3+a_{14}n_4 > (a_3+a_{14})n_3$, so $(a_{41}+a_2)n_2 > (a_3+a_{14})n_3$. Moreover $(a_{41}+a_2)n_2 < (a_{41}+a_2)n_3$ and $(a_{41}+a_2)n_3 \leq (a_3+a_{14})n_3$, hence $(a_{41}+a_2)n_2 < (a_3+a_{14})n_3$ a contradiction. Thus $a_3+a_{14} < a_{41}+a_2$ and therefore $\text{LM}(\text{spoly}(f_3, f_4)) = x_3^{a_3} x_4^{a_{14}}$. Only $\text{LM}(f_1)$ divides $\text{LM}(\text{spoly}(f_3, f_4))$ and $\text{ecart}(\text{spoly}(f_3, f_4)) \leq \text{ecart}(f_1)$, since $a_3 = a_{13}+a_{23}$, $a_2 - a_{23} \leq a_{21}$ and $a_1 = a_{21}+a_{41}$. Let $h = \text{spoly}(f_1, \text{spoly}(f_3, f_4)) = x_1^{a_{41}} x_2^{a_2} - x_1^{a_1} x_3^{a_{23}}$. Since $a_2 \leq a_{21}+a_{23}$, we deduce that $a_2+a_{41} \leq a_{21}+a_{41}+a_{23} = a_1+a_{23}$. Thus $\text{LM}(h) = x_1^{a_{41}} x_2^{a_2}$ and only $\text{LM}(f_2)$ divides $\text{LM}(h)$. Using the fact that $a_1 = a_{21}+a_{41}$ we get $\text{ecart}(h) = \text{ecart}(f_2)$. Then $\text{spoly}(f_2, h) = 0$ and $\text{NF}(\text{spoly}(f_3, f_4)|G) = 0$.

- $\text{spoly}(f_5, f_6) = x_1^{a_1} x_2^{a_2} x_4^{a_{34}-a_{14}} - x_1^{a_{21}} x_3^{a_3} x_4^{a_{34}}$. Recall that $a_3+a_{14} < a_{41}+a_2$. Then $a_1+a_2+a_{34}-a_{14} = a_{21}+a_{41}+a_2+a_{34}-a_{14} > a_{21}+a_3+a_{14}+a_{34}-a_{14} = a_{21}+a_3+a_{34}$ and therefore $\text{LM}(\text{spoly}(f_5, f_6)) = x_1^{a_{21}} x_3^{a_3} x_4^{a_{34}}$. Only $\text{LM}(f_1)$ divides $\text{LM}(\text{spoly}(f_5, f_6))$ and $\text{ecart}(\text{spoly}(f_5, f_6)) \leq \text{ecart}(f_1)$, since $a_3 = a_{13}+a_{23}$ and $a_2-a_{21}-a_{23} \leq 0$. Let $g = \text{spoly}(f_1, \text{spoly}(f_5, f_6)) = x_1^{a_1+a_{21}} x_3^{a_{23}} x_4^{a_{34}-a_{14}} - x_1^{a_1} x_2^{a_2} x_4^{a_{34}-a_{14}}$. Then $\text{LM}(g) = x_1^{a_1} x_2^{a_2} x_4^{a_{34}-a_{14}}$. Only $\text{LM}(f_2)$ divides $\text{LM}(g)$ and $\text{ecart}(g) = \text{ecart}(f_2)$. Then $\text{spoly}(f_2, g) = 0$ and $\text{NF}(\text{spoly}(f_5, f_6)|G) = 0$.

- $\text{spoly}(f_3, f_5) = x_2^{a_{42}-a_{32}} x_3^{a_3+a_{13}} - x_1^{a_{21}} x_4^{2a_{34}}$. We distinguish the following cases:

  1. $\text{LM}(\text{spoly}(f_3, f_5)) = x_1^{a_{21}} x_4^{2a_{34}}$. Only $\text{LM}(f_4)$ divides $\text{LM}(\text{spoly}(f_3, f_5))$ and $\text{ecart}(\text{spoly}(f_3, f_5)) \leq \text{ecart}(f_4)$, since $a_3+a_{13} \leq a_1+a_{32}+a_{34}-a_{14}$, $a_1-a_{21} = a_{41}$ and $a_{14}+a_{34} = a_4$. Let $g = \text{spoly}(f_4, \text{spoly}(f_3, f_5)) = x_1^{a_1} x_2^{a_{42}} x_4^{a_{34}-a_{14}} - x_2^{a_{42}-a_{32}} x_3^{a_3+a_{13}}$. Then $a_3+a_{13}+a_{42}-a_{32} \leq a_1+a_{42}+a_{34}-a_{14}$ and therefore $\text{LM}(g) = x_2^{a_{42}-a_{32}} x_3^{a_3+a_{13}}$. Only $\text{LM}(f_6)$ divides $\text{LM}(g)$ and also $\text{ecart}(g) = \text{ecart}(f_6)$. Finally $\text{spoly}(f_6, g) = 0$ and $\text{NF}(\text{spoly}(f_3, f_5)|G) = 0$.

  2. $\text{LM}(\text{spoly}(f_3, f_5)) = x_2^{a_{42}-a_{32}} x_3^{a_3+a_{13}}$. Only $\text{LM}(f_6)$ divides the monomial $\text{LM}(\text{spoly}(f_3, f_5))$ and $\text{ecart}(\text{spoly}(f_3, f_5)) \leq \text{ecart}(f_6)$. Let $h = \text{spoly}(f_6, \text{spoly}(f_3, f_5)) = x_1^{a_1} x_2^{a_{42}} x_4^{a_{34}-a_{14}} - x_1^{a_{21}} x_4^{2a_{34}}$. Then $a_1+a_{42}+a_{34}-a_{14} \geq a_{21}+2a_{34}$, since $a_1 = a_{21}+a_{41}$, $a_{41}+a_{42} \geq a_4$ and $a_4-a_{14} = a_{34}$. We have that $a_1+a_{42}+a_{34}-a_{14} = a_{21}+a_{41}+a_{42}+a_{34}-a_{14} \geq a_{21}+a_4+a_{34}-a_{14} = a_{21}+a_{14}+a_{34}+a_{34}-a_{14} = a_{21}+2a_{34}$. So $\text{LM}(h) = x_1^{a_{21}} x_4^{2a_{34}}$. Only $\text{LM}(f_4)$ divides $\text{LM}(h)$ and also $\text{ecart}(h) = \text{ecart}(f_4)$. Then $\text{spoly}(f_4, h) = 0$ and $\text{NF}(\text{spoly}(f_3, f_5)|G) = 0$. □

Theorem 2.8 *Suppose that I_C is given as in case 1(b). Let $a_3 > a_{32}+a_{34}$, $a_{32} < a_{42}$ and $a_{14} \leq a_{34}$. Then the reduced Hilbert series of R/I_C^* is $\text{HS}(R/I_C^*, z) = \frac{h(z)}{1-z}$ for*

$$h(z) = \sum_{i=0}^{a_2-1} z^i \sum_{i=0}^{a_3+a_{13}-1} z^i \sum_{i=0}^{a_4-1} z^i - z^{a_{13}+a_{14}} \sum_{i=0}^{a_2-1} z^i \sum_{i=0}^{a_3-1} z^i \sum_{i=0}^{a_{34}-1} z^i -$$

$$z^{a_{13}+a_{42}} \sum_{i=0}^{a_{32}-1} z^i \sum_{i=0}^{a_3-1} z^i \sum_{i=0}^{a_{14}-1} z^i - z^{a_{32}+a_{34}} \sum_{i=0}^{a_{42}-1} z^i \sum_{i=0}^{a_{13}-1} z^i \sum_{i=0}^{a_{14}-1} z^i.$$

Proof. By Proposition 2.7, $G = \{x_1^{a_1} - x_3^{a_{13}} x_4^{a_{14}}, x_2^{a_2} - x_1^{a_{21}} x_3^{a_{23}}, x_3^{a_3} - x_2^{a_{32}} x_4^{a_{34}}, x_4^{a_4} - x_1^{a_{41}} x_2^{a_{42}}, x_1^{a_{21}} x_4^{a_{34}} - x_2^{a_{42}} x_3^{a_{13}}, x_3^{a_3+a_{13}} - x_1^{a_1} x_2^{a_{32}} x_4^{a_{34}-a_{14}}\}$ is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$. Then I_C^* is generated by the least homogeneous summands of the elements in G. Moreover, $\mathrm{LM}(I_C^*) =< x_3^{a_{13}}x_4^{a_{14}}, x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4}, x_2^{a_{32}}x_4^{a_{34}}, x_2^{a_{42}}x_3^{a_{13}} >$. Let

$$J_0 = \mathrm{LM}(I_C^*), J_1 =< x_3^{a_{13}}x_4^{a_{14}}, x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4}, x_2^{a_{42}}x_3^{a_{13}} >,$$

$$J_2 =< x_3^{a_{13}}x_4^{a_{14}}, x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4} >, J_3 =< x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4} > .$$

Then $J_i =< J_{i+1}, q_i >$, where $q_0 = x_2^{a_{32}}x_4^{a_{34}}$, $q_1 = x_2^{a_{42}}x_3^{a_{13}}$ and $q_2 = x_3^{a_{13}}x_4^{a_{14}}$. Therefore

$$p(J_i) = p(J_{i+1}) - z^{\deg(q_i)}p(J_{i+1} :< q_i >), \quad \text{for} \quad 0 \le i \le 2.$$

In this case, $J_1 :< q_0 >=< x_2^{a_{42}}, x_3^{a_{13}}, x_4^{a_{14}} >$, $J_2 :< q_1 >=< x_2^{a_{32}}, x_3^{a_3}, x_4^{a_{14}} >$ and $J_3 :< q_2 >=< x_2^{a_2}, x_3^{a_3}, x_4^{a_{34}} >$. We have that $p(J_1 :< q_0 >) = (1 - z^{a_{42}})(1 - z^{a_{13}})(1 - z^{a_{14}})$, $p(J_2 :< q_1 >) = (1 - z^{a_{32}})(1 - z^{a_3})(1 - z^{a_{14}})$, $p(J_3 :< q_2 >) = (1 - z^{a_2})(1 - z^{a_3})(1 - z^{a_{34}})$ and $p(J_3) = (1 - z^{a_2})(1 - z^{a_3+a_{13}})(1 - z^{a_4})$. Therefore

$$p(J_0) = (1 - z^{a_2})(1 - z^{a_3+a_{13}})(1 - z^{a_4}) - z^{a_{13}+a_{14}}(1 - z^{a_2})(1 - z^{a_3})(1 - z^{a_{34}})-$$
$$z^{a_{13}+a_{42}}(1 - z^{a_{32}})(1 - z^{a_3})(1 - z^{a_{14}}) - z^{a_{32}+a_{34}}(1 - z^{a_{42}})(1 - z^{a_{13}})(1 - z^{a_{14}}).$$

One can easily show that $p(J_0) = (1 - z)^3 h(z)$. Thus $\mathrm{HS}(R/\mathrm{LM}(I_C^*), z) = \frac{h(z)}{1-z}$. □

Proposition 2.9 *Suppose that I_C is given as in case 1(b) and also that $a_3 > a_{32}+a_{34}$. Assume that $a_{42} \le a_{32}$.*
(1) If $a_{34} < a_{14}$, then

$$G = \{f_1 = x_1^{a_1} - x_3^{a_{13}}x_4^{a_{14}}, f_2 = x_2^{a_2} - x_1^{a_{21}}x_3^{a_{23}}, f_3 = x_3^{a_3} - x_2^{a_{32}}x_4^{a_{34}},$$

$$f_4 = x_4^{a_4} - x_1^{a_{41}}x_2^{a_{42}}, f_5 = x_1^{a_{21}}x_4^{a_{34}} - x_2^{a_{42}}x_3^{a_{13}}, f_6 = x_3^{a_3+a_{13}} - x_1^{a_{21}}x_2^{a_{32}-a_{42}}x_4^{2a_{34}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$.
(2) If $a_{14} \le a_{34}$, then

$$G = \{f_1 = x_1^{a_1} - x_3^{a_{13}}x_4^{a_{14}}, f_2 = x_2^{a_2} - x_1^{a_{21}}x_3^{a_{23}}, f_3 = x_3^{a_3} - x_2^{a_{32}}x_4^{a_{34}},$$

$$f_4 = x_4^{a_4} - x_1^{a_{41}}x_2^{a_{42}}, f_5 = x_1^{a_{21}}x_4^{a_{34}} - x_2^{a_{42}}x_3^{a_{13}}, f_6 = x_3^{a_3+a_{13}} - x_1^{a_1}x_2^{a_{32}}x_4^{a_{34}-a_{14}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$.

Proof. (1) We will apply the standard basis algorithm to the set $G = \{f_1, \dots, f_6\}$. We will show that $\mathrm{NF}(\mathrm{spoly}(f_i, f_j)|G) = 0$, for all i, j with $1 \le i < j \le 6$. Here $\mathrm{LM}(f_1) = x_3^{a_{13}}x_4^{a_{14}}$, $\mathrm{LM}(f_2) = x_2^{a_2}$, $\mathrm{LM}(f_3) = x_2^{a_{32}}x_4^{a_{34}}$, $\mathrm{LM}(f_4) = x_4^{a_4}$, $\mathrm{LM}(f_5) = x_2^{a_{42}}x_3^{a_{13}}$ and $\mathrm{LM}(f_6) = x_3^{a_3+a_{13}}$. Therefore $\mathrm{NF}(\mathrm{spoly}(f_i, f_j)|G) = 0$ as $\mathrm{LM}(f_i)$ and $\mathrm{LM}(f_j)$ are relatively prime, for $(i, j) \in \{(1, 2), (2, 4), (2, 6), (3, 6), (4, 5), (4, 6)\}$.

- $\mathrm{spoly}(f_1, f_3) = x_3^{a_3+a_{13}}x_4^{a_{14}-a_{34}} - x_1^{a_1}x_2^{a_{32}}$. Then $\mathrm{LM}(\mathrm{spoly}(f_1, f_3)) = x_3^{a_3+a_{13}}x_4^{a_{14}-a_{34}}$. Only $\mathrm{LM}(f_6)$ divides $\mathrm{LM}(\mathrm{spoly}(f_1, f_3))$ and also $\mathrm{ecart}(f_6) \le \mathrm{ecart}(\mathrm{spoly}(f_1, f_3))$, since $a_1 = a_{21} + a_{41}$, $a_4 - a_{42} \le a_{41}$ and $a_4 = a_{14} + a_{34}$. Let $g = \mathrm{spoly}(f_6, \mathrm{spoly}(f_1, f_3)) = x_1^{a_{21}}x_2^{a_{32}-a_{42}}x_4^{a_4} - x_1^{a_1}x_2^{a_{32}}$. Only $\mathrm{LM}(f_4)$ divides $\mathrm{LM}(g) = x_1^{a_{21}}x_2^{a_{32}-a_{42}}x_4^{a_4}$ and also $\mathrm{ecart}(g) = \mathrm{ecart}(f_4)$. Furthermore $\mathrm{spoly}(f_4, g) = 0$ and $\mathrm{NF}(\mathrm{spoly}(f_1, f_3)|G) = 0$.

- $\mathrm{spoly}(f_1, f_4) = x_1^{a_{41}} x_2^{a_{42}} x_3^{a_{13}} - x_1^{a_1} x_4^{a_{34}}$. In this case $\mathrm{LM}(\mathrm{spoly}(f_1, f_4)) = x_1^{a_{41}} x_2^{a_{42}} x_3^{a_{13}}$. Only $\mathrm{LM}(f_5)$ divides $\mathrm{LM}(\mathrm{spoly}(f_1, f_4))$. Using the fact that $a_1 = a_{21} + a_{41}$ we get $\mathrm{ecart}(\mathrm{spoly}(f_1, f_4)) = \mathrm{ecart}(f_5)$. The computation $\mathrm{spoly}(f_5, \mathrm{spoly}(f_1, f_4)) = 0$ implies that $\mathrm{NF}(\mathrm{spoly}(f_1, f_4)|G) = 0$.

- $\mathrm{spoly}(f_1, f_5) = x_1^{a_{21}} x_4^{a_4} - x_1^{a_1} x_2^{a_{42}}$. Only $\mathrm{LM}(f_4)$ divides $\mathrm{LM}(\mathrm{spoly}(f_1, f_5)) = x_1^{a_{21}} x_4^{a_4}$ and $\mathrm{ecart}(\mathrm{spoly}(f_1, f_5)) = \mathrm{ecart}(f_4)$. Then $\mathrm{spoly}(f_4, \mathrm{spoly}(f_1, f_5)) = 0$ and also $\mathrm{NF}(\mathrm{spoly}(f_1, f_5)|G) = 0$.

- $\mathrm{spoly}(f_1, f_6) = x_1^{a_{21}} x_2^{a_{32}-a_{42}} x_4^{a_{14}+2a_{34}} - x_1^{a_1} x_3^{a_3}$. We have that $a_{21} + a_{32} - a_{42} + a_{14} + 2a_{34} < a_1 + a_3$, since $a_{14} + 2a_{34} = a_{34} + a_4$, $a_{32} + a_{34} < a_3$ and $a_4 - a_{42} \le a_{41}$. So $\mathrm{LM}(\mathrm{spoly}(f_1, f_6)) = x_1^{a_{21}} x_2^{a_{32}-a_{42}} x_4^{a_{14}+2a_{34}}$. Only $\mathrm{LM}(f_4)$ divides $\mathrm{LM}(\mathrm{spoly}(f_1, f_6))$ and $\mathrm{ecart}(\mathrm{spoly}(f_1, f_6)) > \mathrm{ecart}(f_4)$, since $a_3 - a_{32} - a_{34} > 0$. Let $h = \mathrm{spoly}(f_4, \mathrm{spoly}(f_1, f_6)) = x_1^{a_1} x_3^{a_3} - x_1^{a_1} x_2^{a_{32}} x_4^{a_{34}}$, then $\mathrm{LM}(h) = x_1^{a_1} x_2^{a_{32}} x_4^{a_{34}}$. Only $\mathrm{LM}(f_3)$ divides $\mathrm{LM}(h)$ and $\mathrm{ecart}(h) = \mathrm{ecart}(f_3)$. Then $\mathrm{spoly}(f_3, h) = 0$ and $\mathrm{NF}(\mathrm{spoly}(f_1, f_6)|G) = 0$.

- $\mathrm{spoly}(f_2, f_3) = x_2^{a_{42}} x_3^{a_3} - x_1^{a_{21}} x_3^{a_{23}} x_4^{a_{34}}$. Then $\mathrm{LM}(\mathrm{spoly}(f_2, f_3)) = x_2^{a_{42}} x_3^{a_3}$ and only $\mathrm{LM}(f_5)$ divides $\mathrm{LM}(\mathrm{spoly}(f_2, f_3))$. Furthermore $\mathrm{ecart}(\mathrm{spoly}(f_2, f_3)) = \mathrm{ecart}(f_5)$. Then $\mathrm{spoly}(f_5, \mathrm{spoly}(f_2, f_3)) = 0$ and also $\mathrm{NF}(\mathrm{spoly}(f_2, f_3)|G) = 0$.

- $\mathrm{spoly}(f_2, f_5) = x_1^{a_{21}} x_2^{a_{32}} x_4^{a_{34}} - x_1^{a_{21}} x_3^{a_3}$. Thus $\mathrm{LM}(\mathrm{spoly}(f_2, f_5)) = x_1^{a_{21}} x_2^{a_{32}} x_4^{a_{34}}$ and only $\mathrm{LM}(f_3)$ divides $\mathrm{LM}(\mathrm{spoly}(f_2, f_5))$. Furthermore $\mathrm{ecart}(\mathrm{spoly}(f_2, f_5)) = \mathrm{ecart}(f_3)$. Then $\mathrm{spoly}(f_3, \mathrm{spoly}(f_2, f_5)) = 0$ and $\mathrm{NF}(\mathrm{spoly}(f_2, f_5)|G) = 0$.

- $\mathrm{spoly}(f_3, f_4) = x_1^{a_{41}} x_2^{a_2} - x_3^{a_3} x_4^{a_{14}}$. Since $a_3 + a_{14} < a_{41} + a_2$, we have that $\mathrm{LM}(\mathrm{spoly}(f_3, f_4)) = x_3^{a_3} x_4^{a_{14}}$. Only $\mathrm{LM}(f_1)$ divides $\mathrm{LM}(\mathrm{spoly}(f_3, f_4))$ and $\mathrm{ecart}(\mathrm{spoly}(f_3, f_4)) \le \mathrm{ecart}(f_1)$. Let $g = \mathrm{spoly}(f_1, \mathrm{spoly}(f_3, f_4)) = x_1^{a_{41}} x_2^{a_2} - x_1^{a_1} x_3^{a_{23}}$. Then $\mathrm{LM}(g) = x_1^{a_{41}} x_2^{a_2}$ and only $\mathrm{LM}(f_2)$ divides $\mathrm{LM}(g)$. Also $\mathrm{ecart}(g) = \mathrm{ecart}(f_2)$. Then $\mathrm{spoly}(f_2, g) = 0$ and $\mathrm{NF}(\mathrm{spoly}(f_3, f_4)|G) = 0$.

- $\mathrm{spoly}(f_3, f_5) = x_1^{a_{21}} x_2^{a_{32}-a_{42}} x_4^{2a_{34}} - x_3^{a_3+a_{13}} = -f_6$. Then $\mathrm{NF}(\mathrm{spoly}(f_3, f_5)|G) = 0$.

- $\mathrm{spoly}(f_5, f_6) = x_1^{a_{21}} x_2^{a_{32}} x_4^{2a_{34}} - x_1^{a_{21}} x_3^{a_3} x_4^{a_{34}}$. Then $\mathrm{LM}(\mathrm{spoly}(f_5, f_6)) = x_1^{a_{21}} x_2^{a_{32}} x_4^{2a_{34}}$. Only $\mathrm{LM}(f_3)$ divides $\mathrm{LM}(\mathrm{spoly}(f_5, f_6))$ and $\mathrm{ecart}(\mathrm{spoly}(f_5, f_6)) = \mathrm{ecart}(f_3)$. We have that $\mathrm{spoly}(f_3, \mathrm{spoly}(f_5, f_6)) = 0$ and $\mathrm{NF}(\mathrm{spoly}(f_5, f_6)|G) = 0$.

(2) It is enough to prove that $\mathrm{NF}(\mathrm{spoly}(f_i, f_j)|G) = 0$, for all i, j with $1 \le i < j \le 6$. Here $\mathrm{LM}(f_1) = x_3^{a_{13}} x_4^{a_{14}}$, $\mathrm{LM}(f_2) = x_2^{a_2}$, $\mathrm{LM}(f_3) = x_2^{a_{32}} x_4^{a_{34}}$, $\mathrm{LM}(f_4) = x_4^{a_4}$, $\mathrm{LM}(f_5) = x_2^{a_{42}} x_3^{a_{13}}$ and $\mathrm{LM}(f_6) = x_3^{a_3+a_{13}}$. So $\mathrm{NF}(\mathrm{spoly}(f_i, f_j)|G) = 0$ as $\mathrm{LM}(f_i)$ and $\mathrm{LM}(f_j)$ are relatively prime, for

$$(i, j) \in \{(1, 2), (2, 4), (2, 6), (3, 6), (4, 5), (4, 6)\}.$$

- $\mathrm{spoly}(f_1, f_3) = x_3^{a_3+a_{13}} - x_1^{a_1} x_2^{a_{32}} x_4^{a_{34}-a_{14}} = f_6$. Then $\mathrm{NF}(\mathrm{spoly}(f_1, f_3)|G) = 0$.

- $\mathrm{spoly}(f_1, f_4) = x_1^{a_{41}} x_2^{a_{42}} x_3^{a_{13}} - x_1^{a_1} x_4^{a_{34}}$. Then $\mathrm{LM}(\mathrm{spoly}(f_1, f_4)) = x_1^{a_{41}} x_2^{a_{42}} x_3^{a_{13}}$ and only $\mathrm{LM}(f_5)$ divides $\mathrm{LM}(\mathrm{spoly}(f_1, f_4))$. Also $\mathrm{ecart}(\mathrm{spoly}(f_1, f_4)) = \mathrm{ecart}(f_5)$. Moreover $\mathrm{spoly}(f_5, \mathrm{spoly}(f_1, f_4)) = 0$ and $\mathrm{NF}(\mathrm{spoly}(f_1, f_4)|G) = 0$.

- $\mathrm{spoly}(f_1, f_5) = x_1^{a_{21}} x_4^{a_4} - x_1^{a_1} x_2^{a_{42}}$. Then $\mathrm{LM}(\mathrm{spoly}(f_1, f_5)) = x_1^{a_{21}} x_4^{a_4}$ and only $\mathrm{LM}(f_4)$ divides $\mathrm{LM}(\mathrm{spoly}(f_1, f_5))$. Also $\mathrm{ecart}(\mathrm{spoly}(f_1, f_5)) = \mathrm{ecart}(f_4)$. Then $\mathrm{spoly}(f_4, \mathrm{spoly}(f_1, f_5)) = 0$ and $\mathrm{NF}(\mathrm{spoly}(f_1, f_5)|G) = 0$.

- $\mathrm{spoly}(f_1,f_6)=x_1^{a_1}x_2^{a_{32}}x_4^{a_{34}}-x_1^{a_1}x_3^{a_3}$. Then $\mathrm{LM}(\mathrm{spoly}(f_1,f_6))=x_1^{a_1}x_2^{a_{32}}x_4^{a_{34}}$ and only $\mathrm{LM}(f_3)$ divides $\mathrm{LM}(\mathrm{spoly}(f_1,f_6))$. Moreover $\mathrm{ecart}(\mathrm{spoly}(f_1,f_6))=\mathrm{ecart}(f_3)$. Then $\mathrm{spoly}(f_3,\mathrm{spoly}(f_1,f_6))=0$ and $\mathrm{NF}(\mathrm{spoly}(f_1,f_6)|G)=0$.

- $\mathrm{spoly}(f_2,f_3)=x_2^{a_{42}}x_3^{a_3}-x_1^{a_{21}}x_3^{a_{23}}x_4^{a_{34}}$. Then $\mathrm{LM}(\mathrm{spoly}(f_2,f_3))=x_2^{a_{42}}x_3^{a_3}$ and only $\mathrm{LM}(f_5)$ divides $\mathrm{LM}(\mathrm{spoly}(f_2,f_3))$. Also $\mathrm{ecart}(\mathrm{spoly}(f_2,f_3))=\mathrm{ecart}(f_5)$. Then $\mathrm{spoly}(f_5,\mathrm{spoly}(f_2,f_3))=0$ and also $\mathrm{NF}(\mathrm{spoly}(f_2,f_3)|G)=0$.

- $\mathrm{spoly}(f_2,f_5)=x_1^{a_{21}}x_2^{a_{32}}x_4^{a_{34}}-x_1^{a_{21}}x_3^{a_3}$. Then $\mathrm{LM}(\mathrm{spoly}(f_2,f_5))=x_1^{a_{21}}x_2^{a_{32}}x_4^{a_{34}}$ and only $\mathrm{LM}(f_3)$ divides $\mathrm{LM}(\mathrm{spoly}(f_2,f_5))$. Furthermore $\mathrm{ecart}(\mathrm{spoly}(f_2,f_5))=\mathrm{ecart}(f_3)$. Then $\mathrm{spoly}(f_3,\mathrm{spoly}(f_2,f_5))=0$ and $\mathrm{NF}(\mathrm{spoly}(f_2,f_5)|G)=0$.

- $\mathrm{spoly}(f_3,f_4)=x_1^{a_{41}}x_2^{a_2}-x_3^{a_3}x_4^{a_{14}}$. Then $\mathrm{LM}(\mathrm{spoly}(f_3,f_4))=x_3^{a_3}x_4^{a_{14}}$ and only $\mathrm{LM}(f_1)$ divides $\mathrm{LM}(\mathrm{spoly}(f_3,f_4))$. Also $\mathrm{ecart}(\mathrm{spoly}(f_3,f_4))\leq\mathrm{ecart}(f_1)$, since $a_2\leq a_{21}+a_{23}$, $a_1=a_{21}+a_{41}$ and $a_3=a_{13}+a_{23}$. Let $h=\mathrm{spoly}(f_1,\mathrm{spoly}(f_3,f_4))=x_1^{a_{41}}x_2^{a_2}-x_1^{a_1}x_3^{a_{23}}$, then $\mathrm{LM}(h)=x_1^{a_{41}}x_2^{a_2}$ and only $\mathrm{LM}(f_2)$ divides $\mathrm{LM}(h)$. Moreover $\mathrm{ecart}(h)=\mathrm{ecart}(f_2)$. Then $\mathrm{spoly}(f_2,h)=0$ and also $\mathrm{NF}(\mathrm{spoly}(f_3,f_4)|G)=0$.

- $\mathrm{spoly}(f_5,f_6)=x_1^{a_1}x_2^{a_2}x_4^{a_{34}-a_{14}}-x_1^{a_{21}}x_3^{a_3}x_4^{a_{34}}$. Then $\mathrm{LM}(\mathrm{spoly}(f_5,f_6))=x_1^{a_{21}}x_3^{a_3}x_4^{a_{34}}$ and only $\mathrm{LM}(f_1)$ divides $\mathrm{LM}(\mathrm{spoly}(f_5,f_6))$. Furthermore $\mathrm{ecart}(\mathrm{spoly}(f_5,f_6))\leq\mathrm{ecart}(f_1)$, since $a_2\leq a_{21}+a_{23}$ and $a_3=a_{13}+a_{23}$. Let $g=\mathrm{spoly}(f_1,\mathrm{spoly}(f_5,f_6))=x_1^{a_1}x_2^{a_2}x_4^{a_{34}-a_{14}}-x_1^{a_1+a_{21}}x_3^{a_{23}}x_4^{a_{34}-a_{14}}$. Then $\mathrm{LM}(g)=x_1^{a_1}x_2^{a_2}x_4^{a_{34}-a_{14}}$ and only $\mathrm{LM}(f_2)$ divides $\mathrm{LM}(g)$. Also $\mathrm{ecart}(g)=\mathrm{ecart}(f_2)$. Then $\mathrm{spoly}(f_2,g)=0$ and $\mathrm{NF}(\mathrm{spoly}(f_5,f_6)|G)=0$.

- $\mathrm{spoly}(f_3,f_5)=x_1^{a_{21}}x_2^{a_{32}-a_{42}}x_4^{2a_{34}}-x_3^{a_3+a_{13}}$. We distinguish the following cases:
  (1) $\mathrm{LM}(\mathrm{spoly}(f_3,f_5))=x_3^{a_3+a_{13}}$, then only $\mathrm{LM}(f_6)$ divides $\mathrm{LM}(\mathrm{spoly}(f_3,f_5))$ and also $\mathrm{ecart}(\mathrm{spoly}(f_3,f_5))\leq\mathrm{ecart}(f_6)$, since $a_{34}=a_4-a_{14}$, $a_4-a_{42}\leq a_{41}$ and $a_1=a_{21}+a_{41}$. Let $g=\mathrm{spoly}(f_6,\mathrm{spoly}(f_3,f_5))=x_1^{a_{21}}x_2^{a_{32}-a_{42}}x_4^{2a_{34}}-x_1^{a_1}x_2^{a_{32}}x_4^{a_{34}-a_{14}}$. Then $\mathrm{LM}(g)=x_1^{a_{21}}x_2^{a_{32}-a_{42}}x_4^{2a_{34}}$ and only $\mathrm{LM}(f_4)$ divides $\mathrm{LM}(g)$. Also $\mathrm{ecart}(g)=\mathrm{ecart}(f_4)$. We have that $\mathrm{spoly}(f_4,g)=0$ and $\mathrm{NF}(\mathrm{spoly}(f_3,f_5)|G)=0$.
  (2) $\mathrm{LM}(\mathrm{spoly}(f_3,f_5))=x_1^{a_{21}}x_2^{a_{32}-a_{42}}x_4^{2a_{34}}$. Only $\mathrm{LM}(f_4)$ divides $\mathrm{LM}(\mathrm{spoly}(f_3,f_5))$ and also
  $$\mathrm{ecart}(\mathrm{spoly}(f_3,f_5))<\mathrm{ecart}(f_4),$$
  since $a_3+a_{13}\leq a_1+a_{32}+a_{34}-a_{14}$, $a_1=a_{21}+a_{41}$ and $a_4=a_{14}+a_{34}$. Then $\mathrm{spoly}(f_4,\mathrm{spoly}(f_3,f_5))=x_3^{a_3+a_{13}}-x_1^{a_1}x_2^{a_{32}}x_4^{a_{34}-a_{14}}=f_6$. So $\mathrm{NF}(\mathrm{spoly}(f_3,f_5)|G)=0$. $\square$

Theorem 2.10 *Suppose that I_C is given as in case 1(b) and also that $a_3>a_{32}+a_{34}$. Assume that $a_{42}\leq a_{32}$. Then the reduced Hilbert series of R/I_C^* is $\mathrm{HS}(R/I_C^*,z)=\frac{h(z)}{1-z}$ for*

$$h(z)=\sum_{i=0}^{a_2-1}z^i\sum_{i=0}^{a_3+a_{13}-1}z^i\sum_{i=0}^{a_4-1}z^i-z^{a_{13}+a_{42}}\sum_{i=0}^{a_{32}-1}z^i\sum_{i=0}^{a_3-1}z^i\sum_{i=0}^{a_4-1}z^i-$$

$$z^{a_{32}+a_{34}}\sum_{i=0}^{a_{42}-1}z^i\sum_{i=0}^{a_{13}-1}z^i\sum_{i=0}^{a_{14}-1}z^i-z^{a_{13}+a_{14}}\sum_{i=0}^{a_{42}-1}z^i\sum_{i=0}^{a_3-1}z^i\sum_{i=0}^{a_{34}-1}z^i.$$

Proof. By Proposition 2.9, $\mathrm{LM}(I_C^*)$ with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$ is written as $\mathrm{LM}(I_C^*) =< x_3^{a_{13}}x_4^{a_{14}}, x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4}, x_2^{a_{42}}x_3^{a_{13}}, x_2^{a_{32}}x_4^{a_{34}} >$. Let

$$J_0 = \mathrm{LM}(I_C^*), J_1 =< x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4}, x_2^{a_{32}}x_4^{a_{34}}, x_2^{a_{42}}x_3^{a_{13}} >,$$

$$J_2 =< x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4}, x_2^{a_{42}}x_3^{a_{13}} >, J_3 =< x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4} > .$$

Then $J_i =< J_{i+1}, q_i >$, where $q_0 = x_3^{a_{13}}x_4^{a_{14}}$, $q_1 = x_2^{a_{32}}x_4^{a_{34}}$ and $q_2 = x_2^{a_{42}}x_3^{a_{13}}$. So

$$p(J_i) = p(J_{i+1}) - z^{\deg(q_i)}p(J_{i+1} :< q_i >), \quad \text{for} \quad 0 \leq i \leq 2.$$

In this case, $J_1 :< q_0 >=< x_2^{a_{42}}, x_3^{a_3}, x_4^{a_{34}} >$, $J_2 :< q_1 >=< x_2^{a_{42}}, x_3^{a_{13}}, x_4^{a_{14}} >$ and $J_3 :< q_2 >=< x_2^{a_{32}}, x_3^{a_3}, x_4^{a_4} >$. We have that $p(J_1 :< q_0 >) = (1 - z^{a_{42}})(1 - z^{a_3})(1 - z^{a_{34}})$, $p(J_2 :< q_1 >) = (1 - z^{a_{42}})(1 - z^{a_{13}})(1 - z^{a_{14}})$, $p(J_3 :< q_2 >) = (1 - z^{a_{32}})(1 - z^{a_3})(1 - z^{a_4})$ and $p(J_3) = (1 - z^{a_2})(1 - z^{a_3+a_{13}})(1 - z^{a_4})$. So

$$p(J_0) = (1 - z^{a_2})(1 - z^{a_3+a_{13}})(1 - z^{a_4}) - z^{a_{13}+a_{42}}(1 - z^{a_{32}})(1 - z^{a_3})(1 - z^{a_4}) -$$

$$z^{a_{32}+a_{34}}(1 - z^{a_{42}})(1 - z^{a_{13}})(1 - z^{a_{14}}) - z^{a_{13}+a_{14}}(1 - z^{a_{42}})(1 - z^{a_3})(1 - z^{a_{34}}).$$

Thus $p(J_0) = (1 - z)^3 h(z)$, so $\mathrm{HS}(R/\mathrm{LM}(I_C^*), z) = \frac{h(z)}{1-z}$. $\square$

The proof of the next proposition is similar to that of Proposition 2.7 and therefore it is omitted.

Proposition 2.11 *Suppose that I_C is given as in case 2(a). If $a_{24} < a_{34}$ and $a_{13} \leq a_{23}$, then*

$$G = \{x_1^{a_1} - x_2^{a_{12}}x_3^{a_{13}}, x_2^{a_2} - x_3^{a_{23}}x_4^{a_{24}}, x_3^{a_3} - x_1^{a_{31}}x_4^{a_{34}},$$

$$x_4^{a_4} - x_1^{a_{41}}x_2^{a_{42}}, x_1^{a_{41}}x_3^{a_{23}} - x_2^{a_{12}}x_4^{a_{34}}, x_2^{a_2+a_{12}} - x_1^{a_1}x_3^{a_{23}-a_{13}}x_4^{a_{24}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$.

Theorem 2.12 *Suppose that I_C is given as in case 2(a). If $a_{24} < a_{34}$ and $a_{13} \leq a_{23}$, then the reduced Hilbert series of R/I_C^* is $\mathrm{HS}(R/I_C^*, z) = \frac{h(z)}{1-z}$ for*

$$h(z) = \sum_{i=0}^{a_2+a_{12}-1} z^i \sum_{i=0}^{a_3-1} z^i \sum_{i=0}^{a_4-1} z^i - z^{a_{12}+a_{13}} \sum_{i=0}^{a_2-1} z^i \sum_{i=0}^{a_{23}-1} z^i \sum_{i=0}^{a_4-1} z^i -$$

$$z^{a_{12}+a_{34}} \sum_{i=0}^{a_2-1} z^i \sum_{i=0}^{a_{13}-1} z^i \sum_{i=0}^{a_{24}-1} z^i - z^{a_{23}+a_{24}} \sum_{i=0}^{a_{12}-1} z^i \sum_{i=0}^{a_{13}-1} z^i \sum_{i=0}^{a_{34}-1} z^i.$$

Proof. By Proposition 2.11, $G = \{x_1^{a_1} - x_2^{a_{12}}x_3^{a_{13}}, x_2^{a_2} - x_3^{a_{23}}x_4^{a_{24}}, x_3^{a_3} - x_1^{a_{31}}x_4^{a_{34}}, x_4^{a_4} - x_1^{a_{41}}x_2^{a_{42}}, x_1^{a_{41}}x_3^{a_{23}} - x_2^{a_{12}}x_4^{a_{34}}, x_2^{a_2+a_{12}} - x_1^{a_1}x_3^{a_{23}-a_{13}}x_4^{a_{24}}\}$ is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$. Then I_C^* is generated by the least homogeneous summands of the elements in G. In addition, $\mathrm{LM}(I_C^*) =< x_3^{a_{23}}x_4^{a_{24}}, x_2^{a_2+a_{12}}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{12}}x_3^{a_{13}}, x_2^{a_{12}}x_4^{a_{34}} >$. Let

$$J_0 = \mathrm{LM}(I_C^*), J_1 =< x_2^{a_2+a_{12}}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{12}}x_4^{a_{34}}, x_2^{a_{12}}x_3^{a_{13}} >,$$

$$J_2 =< x_2^{a_2+a_{12}}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{12}}x_3^{a_{13}} >, J_3 =< x_2^{a_2+a_{12}}, x_3^{a_3}, x_4^{a_4} > .$$

Then $J_i =< J_{i+1}, q_i >$, where $q_0 = x_3^{a_{23}}x_4^{a_{24}}$, $q_1 = x_2^{a_{12}}x_4^{a_{34}}$ and $q_2 = x_2^{a_{12}}x_3^{a_{13}}$. So

$$p(J_i) = p(J_{i+1}) - z^{\deg(q_i)}p(J_{i+1} :< q_i >), \quad \text{for} \quad 0 \le i \le 2.$$

In this case, $J_1 :< q_0 >=< x_2^{a_{12}}, x_3^{a_{13}}, x_4^{a_{34}} >$, $J_2 :< q_1 >=< x_2^{a_2}, x_3^{a_{13}}, x_4^{a_{24}} >$ and $J_3 :< q_2 >=< x_2^{a_2}, x_3^{a_{23}}, x_4^{a_4} >$. We have that $p(J_1 :< q_0 >) = (1 - z^{a_{12}})(1 - z^{a_{13}})(1 - z^{a_{34}})$, $p(J_2 :< q_1 >) = (1 - z^{a_2})(1 - z^{a_{13}})(1 - z^{a_{24}})$, $p(J_3 :< q_2 >) = (1 - z^{a_2})(1 - z^{a_{23}})(1 - z^{a_4})$ and $p(J_3) = (1 - z^{a_2+a_{12}})(1 - z^{a_3})(1 - z^{a_4})$. Thus

$$p(J_0) = (1 - z^{a_2+a_{12}})(1 - z^{a_3})(1 - z^{a_4}) - z^{a_{12}+a_{13}}(1 - z^{a_2})(1 - z^{a_{23}})(1 - z^{a_4}) -$$

$$z^{a_{12}+a_{34}}(1 - z^{a_2})(1 - z^{a_{13}})(1 - z^{a_{24}}) - z^{a_{23}+a_{24}}(1 - z^{a_{12}})(1 - z^{a_{13}})(1 - z^{a_{34}}).$$

Thus $p(J_0) = (1 - z)^3 h(z)$, and therefore $\mathrm{HS}(R/\mathrm{LM}(I_C^*), z) = \frac{h(z)}{1-z}$. □

The proof of the next proposition is similar to that of Proposition 2.9 and therefore it is omitted.

Proposition 2.13 *Suppose that I_C is given as in case 2(a) and also that $a_{34} \le a_{24}$. (1) If $a_{23} < a_{13}$, then*

$$G = \{x_1^{a_1} - x_2^{a_{12}}x_3^{a_{13}}, x_2^{a_2} - x_3^{a_{23}}x_4^{a_{24}}, x_3^{a_3} - x_1^{a_{31}}x_4^{a_{34}},$$

$$x_4^{a_4} - x_1^{a_{41}}x_2^{a_{42}}, x_1^{a_{41}}x_3^{a_{23}} - x_2^{a_{12}}x_4^{a_{34}}, x_2^{a_2+a_{12}} - x_1^{a_{41}}x_3^{2a_{23}}x_4^{a_{24}-a_{34}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$.
(2) If $a_{13} \le a_{23}$, then

$$G = \{x_1^{a_1} - x_2^{a_{12}}x_3^{a_{13}}, x_2^{a_2} - x_3^{a_{23}}x_4^{a_{24}}, x_3^{a_3} - x_1^{a_{31}}x_4^{a_{34}},$$

$$x_4^{a_4} - x_1^{a_{41}}x_2^{a_{42}}, x_1^{a_{41}}x_3^{a_{23}} - x_2^{a_{12}}x_4^{a_{34}}, x_2^{a_2+a_{12}} - x_1^{a_1}x_3^{a_{23}-a_{13}}x_4^{a_{24}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$.

Theorem 2.14 *Suppose that I_C is given as in case 2(a) and also that $a_{34} \le a_{24}$. Then the reduced Hilbert series of R/I_C^* is $\mathrm{HS}(R/I_C^*, z) = \frac{h(z)}{1-z}$ for*

$$h(z) = \sum_{i=0}^{a_2+a_{12}-1} z^i \sum_{i=0}^{a_3-1} z^i \sum_{i=0}^{a_4-1} z^i - z^{a_{12}+a_{13}} \sum_{i=0}^{a_2-1} z^i \sum_{i=0}^{a_{23}-1} z^i \sum_{i=0}^{a_4-1} z^i -$$

$$z^{a_{12}+a_{34}} \sum_{i=0}^{a_2-1} z^i \sum_{i=0}^{a_{13}-1} z^i \sum_{i=0}^{a_{24}-1} z^i - z^{a_{23}+a_{24}} \sum_{i=0}^{a_{12}-1} z^i \sum_{i=0}^{a_{13}-1} z^i \sum_{i=0}^{a_{34}-1} z^i.$$

Proof. By Proposition 2.13, $\mathrm{LM}(I_C^*)$ with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$ is written as $\mathrm{LM}(I_C^*) =< x_3^{a_{23}}x_4^{a_{24}}, x_2^{a_2+a_{12}}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{12}}x_4^{a_{34}}, x_2^{a_{12}}x_3^{a_{13}} >$. Let

$$J_0 = \mathrm{LM}(I_C^*), J_1 =< x_2^{a_2+a_{12}}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{12}}x_4^{a_{34}}, x_2^{a_{12}}x_3^{a_{13}} >,$$

$$J_2 =< x_2^{a_2+a_{12}}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{12}}x_3^{a_{13}} >, J_3 =< x_2^{a_2+a_{12}}, x_3^{a_3}, x_4^{a_4} > .$$

Then $J_i =< J_{i+1}, q_i >$, where $q_0 = x_3^{a_{23}}x_4^{a_{24}}$, $q_1 = x_2^{a_{12}}x_4^{a_{34}}$ and $q_2 = x_2^{a_{12}}x_3^{a_{13}}$. Thus

$$p(J_i) = p(J_{i+1}) - z^{\deg(q_i)}p(J_{i+1} :< q_i >), \quad \text{for} \quad 0 \le i \le 2.$$

In this case, $J_1 :< q_0 >=< x_2^{a_{12}}, x_3^{a_{13}}, x_4^{a_{34}} >$, $J_2 :< q_1 >=< x_2^{a_2}, x_3^{a_{13}}, x_4^{a_{24}} >$ and $J_3 :< q_2 >=< x_2^{a_2}, x_3^{a_{23}}, x_4^{a_4} >$. Thus $p(J_0) = (1-z^{a_2+a_{12}})(1-z^{a_3})(1-z^{a_4}) - z^{a_{12}+a_{13}}(1-z^{a_2})(1-z^{a_{23}})(1-z^{a_4}) - z^{a_{12}+a_{34}}(1-z^{a_2})(1-z^{a_{13}})(1-z^{a_{24}}) - z^{a_{23}+a_{24}}(1-z^{a_{12}})(1-z^{a_{13}})(1-z^{a_{34}})$. So $p(J_0) = (1-z)^3h(z)$, and therefore $HS(R/\mathrm{LM}(I_C^*), z) = \frac{h(z)}{1-z}$. □

We continue with case 2(b). If $a_3 \le a_{32} + a_{34}$, then we can use [2, Remark 2.9] to find the reduced Hilbert series of R/I_C^*.

Theorem 2.15 *Suppose that I_C is given as in case 2(b) and also that $a_3 \le a_{32}+a_{34}$. Then the reduced Hilbert series of R/I_C^* is $\mathrm{HS}(R/I_C^*, z) = \frac{h(z)}{1-z}$ for*

$$h(z) = \prod_{i=2}^{4}(1+z+z^2+\cdots+z^{a_i-1}) - z^{a_{12}+a_{13}}\sum_{i=0}^{a_{32}-1} z^i \sum_{i=0}^{a_{43}-1} z^i \sum_{i=0}^{a_4-1} z^i - z^{a_{13}+a_{24}}\sum_{i=0}^{a_{12}-1} z^i \sum_{i=0}^{a_{43}-1} z^i \sum_{i=0}^{a_{34}-1} z^i.$$

Proof. By [2, Remark 2.9],

$$G = \{x_1^{a_1} - x_2^{a_{12}}x_3^{a_{13}}, x_2^{a_2} - x_1^{a_{21}}x_4^{a_{24}}, x_3^{a_3} - x_2^{a_{32}}x_4^{a_{34}}, x_4^{a_4} - x_1^{a_{41}}x_3^{a_{43}}, x_1^{a_{41}}x_2^{a_{32}} - x_3^{a_{13}}x_4^{a_{24}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$. So $\mathrm{LM}(I_C^*) =< x_3^{a_{13}}x_4^{a_{24}}, x_2^{a_2}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{12}}x_3^{a_{13}} >$. Let

$$J_0 =< \mathrm{LM}(I_C^*) >, J_1 =< x_2^{a_2}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{12}}x_3^{a_{13}} >, J_2 =< x_2^{a_2}, x_3^{a_3}, x_4^{a_4} > .$$

Then $J_i =< J_{i+1}, q_i >$, where $q_0 = x_3^{a_{13}}x_4^{a_{24}}$ and $q_1 = x_2^{a_{12}}x_3^{a_{13}}$. Thus

$$p(J_i) = p(J_{i+1}) - z^{\deg(q_i)}p(J_{i+1} :< q_i >), \quad \text{for} \quad 0 \le i \le 1.$$

In this case, $J_1 :< q_0 >=< x_2^{a_{12}}, x_3^{a_{43}}, x_4^{a_{34}} >$ and $J_2 :< q_1 >=< x_2^{a_{32}}, x_3^{a_{43}}, x_4^{a_4} >$. We have that $p(J_1 :< q_0 >) = (1-z^{a_{12}})(1-z^{a_{43}})(1-z^{a_{34}})$, $p(J_2 :< q_1 >) = (1-z^{a_{32}})(1-z^{a_{43}})(1-z^{a_4})$ and $p(J_2) = (1-z^{a_2})(1-z^{a_3})(1-z^{a_4})$. So $p(J_0) = (1-z^{a_2})(1-z^{a_3})(1-z^{a_4}) - z^{a_{12}+a_{13}}(1-z^{a_{32}})(1-z^{a_{43}})(1-z^{a_4}) - z^{a_{13}+a_{24}}(1-z^{a_{12}})(1-z^{a_{43}})(1-z^{a_{34}})$. So $p(J_0) = (1-z)^3h(z)$. Thus $\mathrm{HS}(R/\mathrm{LM}(I_C^*), z) = \frac{h(z)}{1-z}$. □

The proof of the next proposition is similar to that of Proposition 2.7 and therefore it is omitted.

Proposition 2.16 *Suppose that I_C is given as in case 2(b) and also that $a_3 > a_{32} + a_{34}$. If $a_{34} < a_{24}$ and $a_{12} \le a_{32}$, then*

$$G = \{x_1^{a_1} - x_2^{a_{12}}x_3^{a_{13}}, x_2^{a_2} - x_1^{a_{21}}x_4^{a_{24}}, x_3^{a_3} - x_2^{a_{32}}x_4^{a_{34}},$$

$$x_4^{a_4} - x_1^{a_{41}}x_3^{a_{43}}, x_1^{a_{41}}x_2^{a_{32}} - x_3^{a_{13}}x_4^{a_{24}}, x_3^{a_3+a_{13}} - x_1^{a_1}x_2^{a_{32}-a_{12}}x_4^{a_{34}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$.

Theorem 2.17 *Suppose that I_C is given as in case 2(b) and also that $a_3 > a_{32} + a_{34}$. Assume that $a_{34} < a_{24}$ and $a_{12} \leq a_{32}$. Then the reduced Hilbert series of R/I_C^* is $\mathrm{HS}(R/I_C^*, z) = \frac{h(z)}{1-z}$ for*

$$h(z) = \sum_{i=0}^{a_2-1} z^i \sum_{i=0}^{a_3+a_{13}-1} z^i \sum_{i=0}^{a_4-1} z^i - z^{a_{12}+a_{13}} \sum_{i=0}^{a_{32}-1} z^i \sum_{i=0}^{a_3-1} z^i \sum_{i=0}^{a_4-1} z^i -$$

$$z^{a_{32}+a_{34}} \sum_{i=0}^{a_{12}-1} z^i \sum_{i=0}^{a_{13}-1} z^i \sum_{i=0}^{a_{24}-1} z^i - z^{a_{13}+a_{24}} \sum_{i=0}^{a_{12}-1} z^i \sum_{i=0}^{a_3-1} z^i \sum_{i=0}^{a_{34}-1} z^i.$$

Proof. By Proposition 2.16, $G = \{x_1^{a_1} - x_2^{a_{12}} x_3^{a_{13}}, x_2^{a_2} - x_1^{a_{21}} x_4^{a_{24}}, x_3^{a_3} - x_2^{a_{32}} x_4^{a_{34}}, x_4^{a_4} - x_1^{a_{41}} x_3^{a_{43}}, x_1^{a_{41}} x_2^{a_{32}} - x_3^{a_{13}} x_4^{a_{24}}, x_3^{a_3+a_{13}} - x_1^{a_1} x_2^{a_{32}-a_{12}} x_4^{a_{34}}\}$ is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$. Then $\mathrm{LM}(I_C^*)$ with respect to the aforementioned order is written as $\mathrm{LM}(I_C^*) =< x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4}, x_2^{a_{32}} x_4^{a_{34}}, x_2^{a_{42}} x_3^{a_{13}}, x_3^{a_{13}} x_4^{a_{24}} >$. Let

$$J_0 = \mathrm{LM}(I_C^*), J_1 =< x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4}, x_2^{a_{12}} x_3^{a_{13}}, x_2^{a_{32}} x_4^{a_{34}} >,$$

$$J_2 =< x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4}, x_2^{a_{12}} x_3^{a_{13}} >, J_3 =< x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4} > .$$

Note that $J_i =< J_{i+1}, q_i >$, where $q_0 = x_3^{a_{13}} x_4^{a_{24}}$, $q_1 = x_2^{a_{32}} x_4^{a_{34}}$ and $q_2 = x_2^{a_{12}} x_3^{a_{13}}$. So

$$p(J_i) = p(J_{i+1}) - z^{\deg(q_i)} p(J_{i+1} :< q_i >), \quad \text{for} \quad 0 \leq i \leq 2.$$

In this case, $J_1 :< q_0 >=< x_2^{a_{12}}, x_3^{a_3}, x_4^{a_{34}} >$, $J_2 :< q_1 >=< x_2^{a_{12}}, x_3^{a_{13}}, x_4^{a_{24}} >$ and $J_3 :< q_2 >=< x_2^{a_{32}}, x_3^{a_3}, x_4^{a_4} >$. We have that $p(J_1 :< q_0 >) = (1 - z^{a_{12}})(1 - z^{a_3})(1 - z^{a_{34}})$, $p(J_2 :< q_1 >) = (1 - z^{a_{12}})(1 - z^{a_{13}})(1 - z^{a_{24}})$, $p(J_3 :< q_2 >) = (1 - z^{a_{32}})(1 - z^{a_3})(1 - z^{a_4})$ and $p(J_3) = (1 - z^{a_2})(1 - z^{a_3+a_{13}})(1 - z^{a_4})$. So

$$p(J_0) = (1 - z^{a_2})(1 - z^{a_3+a_{13}})(1 - z^{a_4}) - z^{a_{12}+a_{13}}(1 - z^{a_{32}})(1 - z^{a_3})(1 - z^{a_4}) -$$

$$z^{a_{32}+a_{34}}(1 - z^{a_{12}})(1 - z^{a_{13}})(1 - z^{a_{24}}) - z^{a_{13}+a_{24}}(1 - z^{a_{12}})(1 - z^{a_3})(1 - z^{a_{34}}).$$

Thus $p(J_0) = (1 - z)^3 h(z)$. So $\mathrm{HS}(R/\mathrm{LM}(I_C^*), z) = \frac{h(z)}{1-z}$. □

The proof of the next proposition is similar to that of Proposition 2.9 and therefore it is omitted.

Proposition 2.18 *Suppose that I_C is given as in case 2(b) and also that $a_3 > a_{32} + a_{34}$. Assume that $a_{24} \leq a_{34}$. (1) If $a_{32} < a_{12}$, then*

$$G = \{x_1^{a_1} - x_2^{a_{12}} x_3^{a_{13}}, x_2^{a_2} - x_1^{a_{21}} x_4^{a_{24}}, x_3^{a_3} - x_2^{a_{32}} x_4^{a_{34}},$$

$$x_4^{a_4} - x_1^{a_{41}} x_3^{a_{43}}, x_1^{a_{41}} x_2^{a_{32}} - x_3^{a_{13}} x_4^{a_{24}}, x_3^{a_3+a_{13}} - x_1^{a_{41}} x_2^{2a_{32}} x_4^{a_{34}-a_{24}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$.

(2) If $a_{12} \leq a_{32}$, then

$$G = \{x_1^{a_1} - x_2^{a_{12}} x_3^{a_{13}}, x_2^{a_2} - x_1^{a_{21}} x_4^{a_{24}}, x_3^{a_3} - x_2^{a_{32}} x_4^{a_{34}},$$

$$x_4^{a_4} - x_1^{a_{41}} x_3^{a_{43}}, x_1^{a_{41}} x_2^{a_{32}} - x_3^{a_{13}} x_4^{a_{24}}, x_3^{a_3+a_{13}} - x_1^{a_1} x_2^{a_{32}-a_{12}} x_4^{a_{34}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$.

Theorem 2.19 *Suppose that I_C is given as in case 2(b) and also that $a_3 > a_{32}+a_{34}$. Assume that $a_{24} \leq a_{34}$. Then the reduced Hilbert series of R/I_C^* is* $\mathrm{HS}(R/I_C^*, z) = \frac{h(z)}{1-z}$ *for*

$$h(z) = \sum_{i=0}^{a_2-1} z^i \sum_{i=0}^{a_3+a_{13}-1} z^i \sum_{i=0}^{a_4-1} z^i - z^{a_{13}+a_{24}} \sum_{i=0}^{a_2-1} z^i \sum_{i=0}^{a_3-1} z^i \sum_{i=0}^{a_{34}-1} z^i -$$

$$z^{a_{12}+a_{13}} \sum_{i=0}^{a_{32}-1} z^i \sum_{i=0}^{a_3-1} z^i \sum_{i=0}^{a_{24}-1} z^i - z^{a_{32}+a_{34}} \sum_{i=0}^{a_{12}-1} z^i \sum_{i=0}^{a_{13}-1} z^i \sum_{i=0}^{a_{24}-1} z^i.$$

Proof. By Proposition 2.18, $\mathrm{LM}(I_C^*)$ with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$ is written as $\mathrm{LM}(I_C^*) =< x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4}, x_3^{a_{13}}x_4^{a_{24}}, x_2^{a_{12}}x_3^{a_{13}}, x_2^{a_{32}}x_4^{a_{34}} >$. Let

$$J_0 = \mathrm{LM}(I_C^*), J_1 =< x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4}, x_3^{a_{13}}x_4^{a_{24}}, x_2^{a_{12}}x_3^{a_{13}} >,$$

$$J_2 =< x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4}, x_3^{a_{13}}x_4^{a_{24}} >, J_3 =< x_2^{a_2}, x_3^{a_3+a_{13}}, x_4^{a_4} > .$$

Then $J_i =< J_{i+1}, q_i >$, where $q_0 = x_2^{a_{32}}x_4^{a_{34}}$, $q_1 = x_2^{a_{12}}x_3^{a_{13}}$ and $q_2 = x_3^{a_{13}}x_4^{a_{24}}$. So

$$p(J_i) = p(J_{i+1}) - z^{\deg(q_i)}p(J_{i+1} :< q_i >), \text{ for } 0 \leq i \leq 2.$$

In this case, $J_1 :< q_0 >=< x_2^{a_{12}}, x_3^{a_{13}}, x_4^{a_{24}} >$, $J_2 :< q_1 >=< x_2^{a_{32}}, x_3^{a_3}, x_4^{a_{24}} >$ and $J_3 :< q_2 >=< x_2^{a_2}, x_3^{a_3}, x_4^{a_{34}} >$. We have that $p(J_1 :< q_0 >) = (1-z^{a_{12}})(1-z^{a_{13}})(1-z^{a_{24}})$, $p(J_2 :< q_1 >) = (1-z^{a_{32}})(1-z^{a_3})(1-z^{a_{24}})$, $p(J_3 :< q_2 >) = (1-z^{a_2})(1-z^{a_3})(1-z^{a_{34}})$ and $p(J_3) = (1-z^{a_2})(1-z^{a_3+a_{13}})(1-z^{a_4})$. Thus

$$p(J_0) = (1-z^{a_2})(1-z^{a_3+a_{13}})(1-z^{a_4}) - z^{a_{13}+a_{24}}(1-z^{a_2})(1-z^{a_3})(1-z^{a_{34}}) -$$

$$z^{a_{12}+a_{13}}(1-z^{a_{32}})(1-z^{a_3})(1-z^{a_{24}}) - z^{a_{32}+a_{34}}(1-z^{a_{12}})(1-z^{a_{13}})(1-z^{a_{24}}).$$

So $p(J_0) = (1-z)^3 h(z)$, and therefore $\mathrm{HS}(R/\mathrm{LM}(I_C^*), z) = \frac{h(z)}{1-z}$. □

Theorem 2.20 *Suppose that I_C is given as in case 3(a). Then the reduced Hilbert series of R/I_C^* is* $\mathrm{HS}(R/I_C^*, z) = \frac{h(z)}{1-z}$ *for*

$$h(z) = \prod_{i=2}^{4}(1+z+z^2+\cdots+z^{a_i-1}) - z^{a_{12}+a_{14}} \sum_{i=0}^{a_{42}-1} z^i \sum_{i=0}^{a_3-1} z^i \sum_{i=0}^{a_{34}-1} z^i - z^{a_{14}+a_{23}} \sum_{i=0}^{a_{12}-1} z^i \sum_{i=0}^{a_{43}-1} z^i \sum_{i=0}^{a_{34}-1} z^i.$$

Proof. By [2, Remark 2.9],

$$G = \{x_1^{a_1} - x_2^{a_{12}}x_4^{a_{14}}, x_2^{a_2} - x_1^{a_{21}}x_3^{a_{23}}, x_3^{a_3} - x_1^{a_{31}}x_4^{a_{34}}, x_4^{a_4} - x_2^{a_{42}}x_3^{a_{43}}, x_3^{a_{23}}x_4^{a_{14}} - x_1^{a_{31}}x_2^{a_{42}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$. Then $\mathrm{LM}(I_C^*) =< x_2^{a_2}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{12}}x_4^{a_{14}}, x_3^{a_{23}}x_4^{a_{14}} >$. Let

$$J_0 = \mathrm{LM}(I_C^*), J_1 =< x_2^{a_2}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{12}}x_4^{a_{14}} >, J_2 =< x_2^{a_2}, x_3^{a_3}, x_4^{a_4} > .$$

Note that $J_i =< J_{i+1}, q_i >$, where $q_0 = x_3^{a_{23}} x_4^{a_{14}}$ and $q_1 = x_2^{a_{12}} x_4^{a_{14}}$. So

$$p(J_i) = p(J_{i+1}) - z^{\deg(q_i)} p(J_{i+1} :< q_i >), \text{ for } 0 \le i \le 1.$$

In this case, $J_1 :< q_0 >=< x_2^{a_{12}}, x_3^{a_{43}}, x_4^{a_{34}} >$ and $J_2 :< q_1 >=< x_2^{a_{42}}, x_3^{a_3}, x_4^{a_{34}} >$. We have that $p(J_1 :< q_0 >) = (1-z^{a_{12}})(1-z^{a_{43}})(1-z^{a_{34}})$, $p(J_2 :< q_1 >) = (1-z^{a_{42}})(1-z^{a_3})(1-z^{a_{34}})$ and $p(J_2) = (1-z^{a_2})(1-z^{a_3})(1-z^{a_4})$. So $p(J_0) = (1-z^{a_2})(1-z^{a_3})(1-z^{a_4}) - z^{a_{12}+a_{14}}(1-z^{a_{42}})(1-z^{a_3})(1-z^{a_{34}}) - z^{a_{14}+a_{23}}(1-z^{a_{12}})(1-z^{a_{43}})(1-z^{a_{34}})$. So $p(J_0) = (1-z)^3 h(z)$, and therefore $\mathrm{HS}(R/\mathrm{LM}(I_C^*), z) = \frac{h(z)}{1-z}$. □

The proof of the next proposition is similar to that of Proposition 2.7 and therefore it is omitted.

Proposition 2.21 *Suppose that I_C is given as in case 3(b). If $a_{23} < a_{43}$ and $a_{14} \le a_{24}$, then*

$$G = \{x_1^{a_1} - x_2^{a_{12}} x_4^{a_{14}}, x_2^{a_2} - x_3^{a_{23}} x_4^{a_{24}}, x_3^{a_3} - x_1^{a_{31}} x_2^{a_{32}},$$

$$x_4^{a_4} - x_1^{a_{41}} x_3^{a_{43}}, x_1^{a_{31}} x_4^{a_{24}} - x_2^{a_{12}} x_3^{a_{43}}, x_2^{a_2+a_{12}} - x_1^{a_1} x_3^{a_{23}} x_4^{a_{24}-a_{14}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$.

Theorem 2.22 *Suppose that I_C is given as in case 3(b). If $a_{23} < a_{43}$ and $a_{14} \le a_{24}$, then the reduced Hilbert series of R/I_C^* is $\mathrm{HS}(R/I_C^*, z) = \frac{h(z)}{1-z}$ for*

$$h(z) = \sum_{i=0}^{a_2+a_{12}-1} z^i \sum_{i=0}^{a_3-1} z^i \sum_{i=0}^{a_4-1} z^i - z^{a_{12}+a_{43}} \sum_{i=0}^{a_2-1} z^i \sum_{i=0}^{a_{23}-1} z^i \sum_{i=0}^{a_4-1} z^i -$$

$$z^{a_{12}+a_{14}} \sum_{i=0}^{a_2-1} z^i \sum_{i=0}^{a_{43}-1} z^i \sum_{i=0}^{a_{24}-1} z^i - z^{a_{23}+a_{24}} \sum_{i=0}^{a_{12}-1} z^i \sum_{i=0}^{a_{43}-1} z^i \sum_{i=0}^{a_{14}-1} z^i.$$

Proof. By Proposition 2.21, $G = \{x_1^{a_1} - x_2^{a_{12}} x_4^{a_{14}}, x_2^{a_2} - x_3^{a_{23}} x_4^{a_{24}}, x_3^{a_3} - x_1^{a_{31}} x_2^{a_{32}}, x_4^{a_4} - x_1^{a_{41}} x_3^{a_{43}}, x_1^{a_{31}} x_4^{a_{24}} - x_2^{a_{12}} x_3^{a_{43}}, x_2^{a_2+a_{12}} - x_1^{a_1} x_3^{a_{23}} x_4^{a_{24}-a_{14}}\}$ is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4 > x_3 > x_2 > x_1$. Then

$$\mathrm{LM}(I_C^*) =< x_2^{a_2+a_{12}}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{12}} x_4^{a_{14}}, x_2^{a_{12}} x_3^{a_{43}}, x_3^{a_{23}} x_4^{a_{24}} > .$$

Let

$$J_0 = \mathrm{LM}(I_C^*), J_1 =< x_2^{a_2+a_{12}}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{12}} x_3^{a_{43}}, x_2^{a_{12}} x_4^{a_{14}} >,$$

$$J_2 =< x_2^{a_2+a_{12}}, x_3^{a_3}, x_4^{a_4}, x_2^{a_{12}} x_3^{a_{43}} >, J_3 =< x_2^{a_2+a_{12}}, x_3^{a_3}, x_4^{a_4} > .$$

Note that $J_i =< J_{i+1}, q_i >$, where $q_0 = x_3^{a_{23}} x_4^{a_{24}}$, $q_1 = x_2^{a_{12}} x_4^{a_{14}}$ and $q_2 = x_2^{a_{12}} x_3^{a_{43}}$. So

$$p(J_i) = p(J_{i+1}) - t^{\deg(q_i)} p(J_{i+1} :< q_i >), \text{ for } 0 \le i \le 2.$$

In this case, $J_1 :< q_0 >=< x_2^{a_{12}}, x_3^{a_{43}}, x_4^{a_{14}} >$, $J_2 :< q_1 >=< x_2^{a_2}, x_3^{a_{43}}, x_4^{a_{24}} >$ and $J_3 :< q_2 >=< x_2^{a_2}, x_3^{a_{23}}, x_4^{a_4} >$. We have that $p(J_1 :< q_0 >) = (1-z^{a_{12}})(1-z^{a_{43}})(1-z^{a_{14}})$, $p(J_2 :< q_1 >) = (1-z^{a_2})(1-z^{a_{43}})(1-z^{a_{24}})$, $p(J_3 :< q_2 >) = (1-z^{a_2})(1-z^{a_{23}})(1-z^{a_4})$ and $p(J_3) = (1-z^{a_2+a_{12}})(1-z^{a_3})(1-z^{a_4})$. So

$$p(J_0) = (1-z^{a_2+a_{12}})(1-z^{a_3})(1-z^{a_4}) - z^{a_{12}+a_{43}}(1-z^{a_2})(1-z^{a_{23}})(1-z^{a_4}) -$$

$$z^{a_{12}+a_{14}}(1-z^{a_2})(1-z^{a_{43}})(1-z^{a_{24}})-z^{a_{23}+a_{24}}(1-z^{a_{12}})(1-z^{a_{43}})(1-z^{a_{14}}).$$

Thus $p(J_0)=(1-z)^3h(z)$, and therefore $\mathrm{HS}(R/\mathrm{LM}(I_C^*),z)=\frac{h(z)}{1-z}$. □

The proof of the next proposition is similar to that of Proposition 2.9 and therefore it is omitted.

Proposition 2.23 *Suppose that I_C is given as in case 3(b) and also that $a_{43}\leq a_{23}$. (1) If $a_{24}<a_{14}$, then*

$$G=\{x_1^{a_1}-x_2^{a_{12}}x_4^{a_{14}},x_2^{a_2}-x_3^{a_{23}}x_4^{a_{24}},x_3^{a_3}-x_1^{a_{31}}x_2^{a_{32}},$$

$$x_4^{a_4}-x_1^{a_{41}}x_3^{a_{43}},x_1^{a_{31}}x_4^{a_{24}}-x_2^{a_{12}}x_3^{a_{43}},x_2^{a_2+a_{12}}-x_1^{a_{31}}x_3^{a_{23}-a_{43}}x_4^{2a_{24}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4>x_3>x_2>x_1$.
(2) If $a_{14}\leq a_{24}$, then

$$G=\{x_1^{a_1}-x_2^{a_{12}}x_4^{a_{14}},x_2^{a_2}-x_3^{a_{23}}x_4^{a_{24}},x_3^{a_3}-x_1^{a_{31}}x_2^{a_{32}},$$

$$x_4^{a_4}-x_1^{a_{41}}x_3^{a_{43}},x_1^{a_{31}}x_4^{a_{24}}-x_2^{a_{12}}x_3^{a_{43}},x_2^{a_2+a_{12}}-x_1^{a_1}x_3^{a_{23}}x_4^{a_{24}-a_{14}}\}$$

is a standard basis for I_C with respect to the negative degree reverse lexicographic term ordering with $x_4>x_3>x_2>x_1$.

Theorem 2.24 *Suppose that I_C is given as in case 3(b) and also that $a_{43}\leq a_{23}$. Then the reduced Hilbert series of R/I_C^* is $\mathrm{HS}(R/I_C^*,z)=\frac{h(z)}{1-z}$ for*

$$h(z)=\sum_{i=0}^{a_2+a_{12}-1}z^i\sum_{i=0}^{a_3-1}z^i\sum_{i=0}^{a_4-1}z^i-z^{a_{12}+a_{43}}\sum_{i=0}^{a_2-1}z^i\sum_{i=0}^{a_{23}-1}z^i\sum_{i=0}^{a_4-1}z^i-$$

$$z^{a_{12}+a_{14}}\sum_{i=0}^{a_2-1}z^i\sum_{i=0}^{a_{43}-1}z^i\sum_{i=0}^{a_{24}-1}z^i-z^{a_{23}+a_{24}}\sum_{i=0}^{a_{12}-1}z^i\sum_{i=0}^{a_{43}-1}z^i\sum_{i=0}^{a_{14}-1}z^i.$$

Proof. By Proposition 2.23, $\mathrm{LM}(I_C^*)$ with respect to the negative degree reverse lexicographic term ordering with $x_4>x_3>x_2>x_1$ is written as $\mathrm{LM}(I_C^*)=<x_2^{a_2+a_{12}},x_3^{a_3},x_4^{a_4},x_2^{a_{12}}x_3^{a_{43}},x_2^{a_{12}}x_4^{a_{14}},x_3^{a_{23}}x_4^{a_{24}}>$. Let

$$J_0=\mathrm{LM}(I_C^*),J_1=<x_2^{a_2+a_{12}},x_3^{a_3},x_4^{a_4},x_2^{a_{12}}x_3^{a_{43}},x_2^{a_{12}}x_4^{a_{14}}>,$$

$$J_2=<x_2^{a_2+a_{12}},x_3^{a_3},x_4^{a_4},x_2^{a_{12}}x_3^{a_{43}}>,J_3=<x_2^{a_2+a_{12}},x_3^{a_3},x_4^{a_4}>.$$

Then $J_i=<J_{i+1},q_i>$, where $q_0=x_3^{a_{23}}x_4^{a_{24}}$, $q_1=x_2^{a_{12}}x_4^{a_{14}}$ and $q_2=x_2^{a_{12}}x_3^{a_{43}}$. Thus

$$p(J_i)=p(J_{i+1})-z^{\deg(q_i)}p(J_{i+1}:<q_i>),\ \text{for}\ 0\leq i\leq 2.$$

In this case, $J_1:<q_0>=<x_2^{a_{12}},x_3^{a_{43}},x_4^{a_{14}}>$, $J_2:<q_1>=<x_2^{a_2},x_3^{a_{43}},x_4^{a_{24}}>$ and $J_3:<q_2>=<x_2^{a_2},x_3^{a_{23}},x_4^{a_4}>$. So

$$p(J_0)=(1-z^{a_2+a_{12}})(1-z^{a_3})(1-z^{a_4})-z^{a_{12}+a_{43}}(1-z^{a_2})(1-z^{a_{23}})(1-z^{a_4})-$$

$$z^{a_{12}+a_{14}}(1-z^{a_2})(1-z^{a_{43}})(1-z^{a_{24}})-z^{a_{23}+a_{24}}(1-z^{a_{12}})(1-z^{a_{43}})(1-z^{a_{14}}).$$

Now we have that $p(J_0)=(1-z)^3h(z)$. Therefore $\mathrm{HS}(R/\mathrm{LM}(I_C^*),z)=\frac{h(z)}{1-z}$. □

3. Examples

In this section, we give some examples showing how the formulas in the previous section can be used to compute $\mathrm{HS}(R/I_C^*, z)$.

Example 3.1 Let $m \geq 5$ be an integer such that $\gcd(51, 13m+1, 14m+5, 32m+26) = 1$. Consider the monomial curve $C(m)$ defined parametrically by $x_1 = t^{51}$, $x_2 = t^{13m+1}$, $x_3 = t^{14m+5}$ and $x_4 = t^{32m+26}$. The ideal $I_{C(m)}$ is minimally generated by $x_1^{2m+1} - x_3^5 x_4$, $x_2^5 - x_1^m x_3$, $x_3^6 - x_2^4 x_4$, $x_4^2 - x_1^{m+1} x_2$ and $x_1^m x_4 - x_2 x_3^5$. Thus we are in case 1(b) of Remark 2.2 and from Theorem 2.3 we deduce that $R/I_{C(m)}^*$ is Cohen–Macaulay. By Theorem 2.10, the numerator of the reduced Hilbert series of $R/I_{C(m)}^*$ is

$$h(z) = (1+z)\sum_{i=0}^{4} z^i \sum_{i=0}^{10} z^i - z^6(1+z)\sum_{i=0}^{3} z^i \sum_{i=0}^{5} z^i - z^5 \sum_{i=0}^{4} z^i - z^6 \sum_{i=0}^{5} z^i.$$

We have that

$$(1+z)\sum_{i=0}^{4} z^i \sum_{i=0}^{10} z^i = 1 + 3z + 5z^2 + 7z^3 + 9z^4 + \sum_{i=5}^{10} 10z^i + 9z^{11} + 7z^{12} + 5z^{13} + 3z^{14} + z^{15}$$

and also

$$z^6(1+z)\sum_{i=0}^{3} z^i \sum_{i=0}^{5} z^i = z^6 + 3z^7 + 5z^8 + 7z^9 + 8z^{10} + 8z^{11} + 7z^{12} + 5z^{13} + 3z^{14} + z^{15}.$$

So $h(z) = \sum_{i=0}^{4}(2i+1)z^i + \sum_{i=5}^{9}(19-2i)z^i + z^{10}$, and therefore

$$\mathrm{HS}(R/I_{C(m)}^*, z) = \frac{\sum_{i=0}^{4}(2i+1)z^i + \sum_{i=5}^{9}(19-2i)z^i + z^{10}}{1-z}.$$

Example 3.2 Let $m \geq 4$ be an integer such that $\gcd(3m+19, 13m+36, 3m^2+15m+21, 4m^2+17m+9) = 1$. Consider the monomial curve $C(m)$ defined parametrically by $x_1 = t^{3m+19}$, $x_2 = t^{13m+36}$, $x_3 = t^{3m^2+15m+21}$ and $x_4 = t^{4m^2+17m+9}$. The ideal $I_{C(m)}$ is minimally generated by $x_1^{m+3} - x_2 x_3$, $x_2^{m+2} - x_3^3 x_4$, $x_3^4 - x_1^3 x_4^3$, $x_4^4 - x_1^m x_2^{m+1}$ and $x_1^m x_3^3 - x_2 x_4^3$. Thus we are in case 2(a) of Remark 2.2 and from Theorem 2.3 we deduce that $R/I_{C(m)}^*$ is Cohen–Macaulay. By Theorem 2.12, the numerator of the reduced Hilbert series of $R/I_{C(m)}^*$ is

$$h(z) = \sum_{i=0}^{m+2} z^i \left(\sum_{i=0}^{3} z^i\right)^2 - z^2 \sum_{i=0}^{m+1} z^i \sum_{i=0}^{2} z^i \sum_{i=0}^{3} z^i - z^4 \sum_{i=0}^{m+1} z^i - z^4 \sum_{i=0}^{2} z^i.$$

We have that $\left(\sum_{i=0}^{3} z^i\right)^2 = 1 + 2z + 3z^2 + 4z^3 + 3z^4 + 2z^5 + z^6$ and also

$$\sum_{i=0}^{m+2} z^i \left(\sum_{i=0}^{3} z^i\right)^2 = 1 + 3z + 6z^2 + 10z^3 + 13z^4 + 15z^5 + \sum_{i=6}^{m+2} 16z^i +$$

$$15z^{m+3}+13z^{m+4}+10z^{m+5}+6z^{m+6}+3z^{m+7}+z^{m+8}.$$

In addition,

$$\sum_{i=0}^{m+1}z^i\sum_{i=0}^{2}z^i\sum_{i=0}^{3}z^i=\sum_{i=0}^{m+1}z^i(1+2z+3z^2+3z^3+2z^4+z^5)=1+3z+6z^2+$$

$$9z^3+11z^4+\sum_{i=5}^{m+1}12z^i+11z^{m+2}+9z^{m+3}+6z^{m+4}+3z^{m+5}+z^{m+6},$$

so

$$z^2\sum_{i=0}^{m+1}z^i\sum_{i=0}^{2}z^i\sum_{i=0}^{3}z^i=z^2+3z^3+6z^4+9z^5+11z^6+\sum_{i=7}^{m+3}12z^i+11z^{m+4}+$$

$$9z^{m+5}+6z^{m+6}+3z^{m+7}+z^{m+8}.$$

Thus $h(z)=\sum_{i=0}^{3}(2i+1)z^i+5z^4+4z^5+\sum_{i=6}^{m+2}3z^i+2z^{m+3}+z^{m+4}$, and therefore

$$\mathrm{HS}(R/I^*_{C(m)},z)=\frac{\sum_{i=0}^{3}(2i+1)z^i+5z^4+4z^5+\sum_{i=6}^{m+2}3z^i+2z^{m+3}+z^{m+4}}{1-z}.$$

Example 3.3 Let $m\geq 7$ be an integer such that $\gcd(2m+21,14m+5,2m^2+m+3,2m^2+7m-5)=1$. Consider the monomial curve $C(m)$ defined parametrically by $x_1=t^{2m+21}$, $x_2=t^{14m+5}$, $x_3=t^{2m^2+m+3}$ and $x_4=t^{2m^2+7m-5}$. The ideal $I_{C(m)})$ is minimally generated by $x_1^m-x_2x_4$, $x_2^{m+1}-x_3^5x_4^2$, $x_3^7-x_1x_2^m$, $x_4^3-x_1^{m-1}x_3^2$ and $x_1x_4^2-x_2x_3^2$. Thus we are in case 3(b) of Remark 2.2 and from Theorem 2.3 we deduce that $R/I^*_{C(m)}$ is Cohen–Macaulay. By Theorem 2.24, the numerator of the reduced Hilbert series of $R/I^*_{C(m)}$ is

$$h(z)=\sum_{i=0}^{m+1}z^i\sum_{i=0}^{6}z^i\sum_{i=0}^{2}z^i-z^3\sum_{i=0}^{m}z^i\sum_{i=0}^{4}z^i\sum_{i=0}^{2}z^i-z^2(1+z)^2\sum_{i=0}^{m}z^i-z^7(1+z).$$

We have that

$$\sum_{i=0}^{6}z^i\sum_{i=0}^{2}z^i=1+2z+\sum_{i=2}^{6}3z^i+2z^7+z^8$$

and also

$$\sum_{i=0}^{m+1}z^i\sum_{i=0}^{6}z^i\sum_{i=0}^{2}z^i=1+3z+6z^2+9z^3+12z^4+15z^5+18z^6+20z^7+\sum_{i=8}^{m+1}21z^i+$$

$$20z^{m+2}+18z^{m+3}+15z^{m+4}+12z^{m+5}+9z^{m+6}+6z^{m+7}+3z^{m+8}+z^{m+9}.$$

In addition,

$$z^3\sum_{i=0}^{4}z^i\sum_{i=0}^{2}z^i=z^3+2z^4+\sum_{i=5}^{7}3z^i+2z^8+z^9$$

and also

$$z^3\sum_{i=0}^{m}z^i\sum_{i=0}^{4}z^i\sum_{i=0}^{2}z^i=z^3+3z^4+6z^5+9z^6+12z^7+14z^8+\sum_{i=9}^{m+3}15z^i+$$

$$14z^{m+4} + 12z^{m+5} + 9z^{m+6} + 6z^{m+7} + 3z^{m+8} + z^{m+9}.$$

Moreover

$$z^2(1+z)^2 \sum_{i=0}^{m} z^i = (z^2 + 2z^3 + z^4) \sum_{i=0}^{m} z^i = z^2 + 3z^3 + \sum_{i=4}^{m+2} 4z^i + 3z^{m+3} + z^{m+4}.$$

Thus $h(z) = 1 + 3z + \sum_{i=2}^{6} 5z^i + 3z^7 + \sum_{i=8}^{m+1} 2z^i + z^{m+2}$, and therefore

$$\mathrm{HS}(R/I^*_{C(m)}, z) = \frac{1 + 3z + \sum_{i=2}^{6} 5z^i + 3z^7 + \sum_{i=8}^{m+1} 2z^i + z^{m+2}}{1-z}.$$

References

[1] Arslan F, Katsabekis A, Nalbandiyan M. On the Cohen-Macaulayness of tangent cones of monomial curves in $\mathbb{A}^4(K)$. Turkish Journal of Mathematics 2019; 43: 1425-1446.

[2] Arslan F, Mete P. Hilbert functions of Gorenstein monomial curves. Proceedings of the American Mathematical Society 2007; 135: 1993-2002.

[3] Bayer D, Stillman M. Computation of Hilbert functions. Journal of Symbolic Computation 1992; 14: 31-50.

[4] Bresinsky H. Symmetric semigroups of integers generated by 4 elements. Manuscripta Mathematica 1975; 17 (3): 205-219.

[5] Greuel G-M, Pfister G. A Singular Introduction to Commutative Algebra. Berlin, Germany: Springer-Verlag, 2002.

[6] Kunz E. The value semigroup of one dimensional Gorenstein ring. Proceedings of the American Mathematical Society 1970; 25: 748-751.

[7] Sturmfels B. Gröbner Bases and Convex Polytopes. University Lecture Series Vol. 8. Providence, RI, USA: American Mathematical Society, 1996.