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A new recursive formula arising from a determinantal expression for weighted Delannoy numbers

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Abstract: In this paper, we obtain a determinantal expression for weighted Delannoy numbers, from which we give a new recurrence relation for it. For the special case of our formula, we compute central weighted Delannoy number in terms of weighted Delannoy number.

Key words: Delannoy number, central Delannoy number, determinantal expression, recurrence relation

1. Introduction

For positive integers a, b and c, let us consider paths that start at the origin, remain in the first quadrant and use only the steps $(1, 0)$ with weight a, $(0, 1)$ with weight b and $(1, 1)$ with weight c. The weight of a path is then the product of the weights of the individual steps in the path. For all nonnegative integers p, q, let $w(p, q|a, b, c)$ denote the total of all of the weights of paths that connect the origin to the point (p, q). The $w(p, q|a, b, c)$ are known as the weighted Delannoy numbers and are given by the recurrence relation

$$\begin{aligned} & w(p+1, q+1|a, b, c) \\ & = a \cdot w(p, q+1|a, b, c) + b \cdot w(p+1, q|a, b, c) + c \cdot w(p, q|a, b, c), \end{aligned} \tag{1.1}$$

subject to the initial conditions $w(p, 0|a, b, c) = a^p$, and $w(0, q|a, b, c) = b^q$ for $p, q \geq 0$.

They possess the closed form expression [6, p. 87]

$$w(p, q|a, b, c) = \sum_{k=0}^{p} \binom{p}{k} \binom{q}{k} a^{p-k} b^{q-k} (ab + c)^k .$$

The recurrence relation (1.1) implies the following generating function for $w(p, q|a, b, c)$:

$$\sum_{p,q \geq 0} w(p, q|a, b, c) x^p y^q = \frac{1}{1 - ax - by - cxy}.$$

Substituting $a = b = c = 1$ yields the classical Delannoy numbers $d(p, q)$, given by the recurrence relation

$$d(p, q) = d(p-1, q) + d(p, q-1) + d(p-1, q-1),$$

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and can be generated by

$$\sum_{p,q\geq 0} d(p,q)x^p y^q = \frac{1}{1-x-y-xy}.$$

The historic significance of these numbers is explained in the paper "Why Delannoy numbers?" [1] by Banderier and Schwer. Various properties of Delannoy numbers and their generalizations have been discussed by many authors (see [4, 5, 7, 8, 10, 11, 17, 29–33]).

For example, Razpet [27] studied a divisibility property of weighted Delannoy numbers $w(p,q|a,b,c)$ and showed that these numbers satisfy a congruence relation for all positive integers a,b and c as follows:

$$w(\alpha p+\beta, \gamma p+\delta|a,b,c) \equiv w(\alpha,\gamma|a,b,c) \cdot w(\beta,\delta|a,b,c) \pmod{p},$$

where p is a prime number and α, β, γ and δ are nonnegative integers with $0 \leq \beta < p,\ 0 \leq \delta < p$.

Also, Noble [12] considered some divisibility properties for central weighted Delannoy numbers $w(p,p|a,b,c)$ by applying a generalization of a method of Stoll and Haible that appears [28] for the asymptotic coefficients.

On the other hand, adopting the determinantal representation, some authors have studied important topics such as Bernoulli numbers and polynomials [18, 21], the Euler numbers and polynomials [34], central Delannoy numbers [16], Horadam polynomials [22], Fibonacci polynomials [25] and obtained many remarkable relations for them. One may consult [9, 13, 14, 19, 20, 23, 24, 26] and closely related references therein.

Very recently, by applying a formula for derivatives of a ratio of two differential functions and a recursive relation of the Hessenberg determinant, a new determinantal expression and a new recursive relation for the Delannoy numbers $d(p,q)$ have been offered in [15].

In this paper, we give a determinantal expression for weighted Delannoy number $w(p,q|a,b,c)$. As an application this representation, we derive a new recursive formula for weighted Delannoy numbers. Consequently, we also deduce similar formula for central weighted Delannoy numbers. We remark that our formulas cover some conclusions in this context.

2. Lemmas

In this section, we give two lemmas. The first one is a simple and useful instrument to express some quantities in mathematics by means of some special determinants, while the second one enables us to derive recursive formulas.

Lemma 2.1 *[2, p. 40, Entry 5] For two differentiable functions $p(x)$ and $q(x) \neq 0$, we have for $k \geq 0$*

$$\left[\frac{p(x)}{q(x)}\right]^{(k)} = \frac{(-1)^k}{(q(x))^{k+1}} \times \begin{vmatrix} p & q & 0 & \cdots & 0 & 0 \\ p' & q' & q & \cdots & 0 & 0 \\ p'' & q'' & \binom{2}{1}q' & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p^{(k-2)} & q^{(k-2)} & \binom{k-2}{1}q^{(k-3)} & \cdots & q & 0 \\ p^{(k-1)} & q^{(k-1)} & \binom{k-1}{1}q^{(k-2)} & \cdots & \binom{k-1}{k-2}q' & q \\ p^{(k)} & q^{(k)} & \binom{k}{1}q^{(k-1)} & \cdots & \binom{k}{k-2}q'' & \binom{k}{k-1}q' \end{vmatrix}. \tag{2.1}$$

In other words, formula (2.1) can be represented as

$$\frac{d^k}{dx^k}\left[\frac{p(x)}{q(x)}\right] = \frac{(-1)^k}{q^{k+1}}\left|W_{(k+1)\times(k+1)}(x)\right|,$$

where $\left|W_{(k+1)\times(k+1)}(x)\right|$ *denotes the determinant of the matrix*

$$W_{(k+1)\times(k+1)}(x) = \left[U_{(k+1)\times 1}(x) \quad V_{(k+1)\times k}(x)\right].$$

Here $U_{(k+1)\times 1}(x)$ *has the elements* $u_{l,1}(x) = p^{(l-1)}(x)$ *for* $1 \le l \le k+1$ *and* $V_{(k+1)\times k}(x)$ *has the entries of the form*

$$v_{i,j}(x) = \begin{cases} \binom{i-1}{j-1} q^{(i-j)}(x), & \text{if } i-j \ge 0; \\ 0, & \text{if } i-j < 0, \end{cases}$$

for $1 \le i \le k+1$ *and* $1 \le j \le k$.

Lemma 2.2 *[3, p. 222, Theorem] Let* $M_0 = 1$ *and*

$$M_n = \begin{vmatrix} m_{1,1} & m_{1,2} & 0 & \cdots & 0 & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & \cdots & 0 & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ m_{n-2,1} & m_{n-2,2} & m_{n-2,3} & \cdots & m_{n-2,n-1} & 0 \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & m_{n-1,n-1} & m_{n-1,n} \\ m_{n,1} & m_{n,2} & m_{n,3} & \cdots & m_{n,n-1} & m_{n,n} \end{vmatrix}$$

for $n \in \mathbb{N}$, *then, the sequence* M_n *satisfies the following relation with* $M_1 = m_{1,1}$,

$$M_n = \sum_{r=1}^{n} (-1)^{n-r} m_{n,r} \left(\prod_{j=r}^{n-1} m_{j,j+1}\right) M_{r-1}, \qquad \text{for } n \ge 2. \tag{2.2}$$

3. Results

This section is devoted to present the results.

Theorem 3.1 *The weighted Delannoy numbers* $w(p,q|a,b,c)$ *for* $p,q \ge 0$ *can be represented in terms of the following determinant as*

$$w(p,q|a,b,c) = \frac{(-1)^q}{q!}\left| \ L_{(q+1)\times 1}(p) \quad M_{(q+1)\times q}(p) \ \right|_{(q+1)\times(q+1)}, \tag{3.1}$$

where

$$L_{(q+1)\times 1}(p) = \left(\langle p\rangle_0 a^p, \langle p\rangle_1 a^{p-1}c, ..., \langle p\rangle_q a^{p-q}c^q\right)^T,$$

$$M_{(q+1)\times q}(p) = \left((-1)^{i-j}\binom{i-1}{j-1}\langle p+1\rangle_{i-j} b^{i-j}\right)_{\substack{1\le i\le q+1 \\ 1\le j\le q}},$$

with the notation T (transpose of matrix), and

$$\langle x\rangle_n=\prod_{k=0}^{n-1}(x-k)=\begin{cases}x(x-1)...(x-n+1), & \text{if } n\geq 1;\\ 1, & \text{if } n=0,\end{cases}$$

is called the falling factorial of $x\in\mathbb{R}$.

Proof From Lemma 2.1, we have

$$\frac{\partial^p}{\partial x^p}\left[\frac{1}{1-ax-by-cxy}\right]$$

$$=\frac{(-1)^p}{(1-ax-by-cxy)^{(p+1)}}$$

$$\times\begin{vmatrix}1 & 1-ax-by-cxy & 0 & 0 & \dots & 0\\ 0 & -(a+cy) & 1-ax-by-cxy & 0 & \dots & 0\\ 0 & 0 & -\binom{2}{1}(a+cy) & 1-ax-by-cxy & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & ... & 0\\ 0 & 0 & 0 & 0 & \dots & 1-ax-by-cxy\\ 0 & 0 & 0 & 0 & \dots & -\binom{p}{1}(a+cy)\end{vmatrix}$$

$$=\frac{p!\,(a+cy)^p}{(1-ax-(b+cx)\,y)^{(p+1)}}$$

and

$$\frac{\partial^{p+q}}{\partial y^q\partial x^p}\left[\frac{1}{1-ax-by-cxy}\right]=p!\frac{\partial^q}{\partial y^q}\frac{(a+cy)^p}{(1-ax-(b+cx)\,y)^{p+1}}$$

$$=p!\frac{(-1)^q}{(1-ax-(b+cx)\,y)^{(p+1)(q+1)}}$$

$$\times\begin{vmatrix}(a+cy)^p & (1-ax-(b+cx)\,y)^{p+1}\\ \langle p\rangle_1(a+cy)^{p-1}c & -\langle p+1\rangle_1(b+cx)(1-ax-(b+cx)\,y)^p\\ \langle p\rangle_2(a+cy)^{p-2}c^2 & \langle p+1\rangle_2(b+cx)^2(1-ax-(b+cx)\,y)^{p-1}\\ \vdots & \vdots\\ \langle p\rangle_{q-2}(a+cy)^{p-q+2}c^{q-2} & (-1)^{q-2}\langle p+1\rangle_{q-2}(b+cx)^{q-2}(1-ax-(b+cx)\,y)^{p-q+3}\\ \langle p\rangle_{q-1}(a+cy)^{p-q+1}c^{q-1} & (-1)^{q-1}\langle p+1\rangle_{q-1}(b+cx)^{q-1}(1-ax-(b+cx)\,y)^{p-q+2}\\ \langle p\rangle_q(a+cy)^{p-q}c^q & (-1)^q\langle p+1\rangle_q(b+cx)^q(1-ax-(b+cx)\,y)^{p-q+1}\end{vmatrix}$$

$$\begin{matrix}0\\ (1-ax-(b+cx)\,y)^{p+1}\\ -\binom{2}{1}\langle p+1\rangle_1(b+cx)(1-ax-(b+cx)\,y)^p\\ \vdots\\ (-1)^{q-3}\binom{q-2}{1}\langle p+1\rangle_{q-3}(b+cx)^{q-3}(1-ax-(b+cx)\,y)^{p-q+4}\\ (-1)^{q-2}\binom{q-1}{1}\langle p+1\rangle_{q-2}(b+cx)^{q-2}(1-ax-(b+cx)\,y)^{p-q+3}\\ (-1)^{q-1}\binom{q}{1}\langle p+1\rangle_{q-1}(b+cx)^{q-1}(1-ax-(b+cx)\,y)^{p-q+2}\end{matrix}$$

$$\begin{array}{cc}
0 & \cdots \\
0 & \cdots \\
(1-ax-(b+cx)y)^{p+1} & \cdots \\
\vdots & \ddots \\
(-1)^{q-4}\binom{q-2}{2}\langle p+1\rangle_{q-4}(b+cx)^{q-4}(1-ax-(b+cx)y)^{p-q+5} & \ldots \\
(-1)^{q-3}\binom{q-1}{2}\langle p+1\rangle_{q-3}(b+cx)^{q-3}(1-ax-(b+cx)y)^{p-q+4} & \cdots \\
(-1)^{q-2}\binom{q}{2}\langle p+1\rangle_{q-2}(b+cx)^{q-2}(1-ax-(b+cx)y)^{p-q+3} & \cdots
\end{array}$$

$$\left.\begin{array}{cc}
0 & 0 \\
0 & 0 \\
0 & 0 \\
\vdots & \vdots \\
(1-ax-(b+cx)y)^{p+1} & 0 \\
-\binom{q-1}{q-2}\langle p+1\rangle_{1}(b+cx)(1-ax-(b+cx)y)^{p} & (1-ax-(b+cx)y)^{p+1} \\
\binom{q}{q-2}\langle p+1\rangle_{2}(b+cx)^{2}(1-ax-(b+cx)y)^{p-1} & -\binom{q}{q-1}\langle p+1\rangle_{1}(b+cx)(1-ax-(b+cx)y)^{p}
\end{array}\right|$$

$$\to(-1)^{q}p!\left|\begin{array}{ccc}
a^{p} & 1 & 0 \\
\langle p\rangle_{1}a^{p-1}c & -\langle p+1\rangle_{1}b & 1 \\
\langle p\rangle_{2}a^{p-2}c^{2} & \langle p+1\rangle_{2}b^{2} & -\binom{2}{1}\langle p+1\rangle_{1}b \\
\vdots & \vdots & \vdots \\
\langle p\rangle_{q-2}a^{p-q+2}c^{q-2} & (-1)^{q-2}\langle p+1\rangle_{q-2}b^{q-2} & (-1)^{q-3}\binom{q-2}{1}\langle p+1\rangle_{q-3}b^{q-3} \\
\langle p\rangle_{q-1}a^{p-q+1}c^{q-1} & (-1)^{q-1}\langle p+1\rangle_{q-1}b^{q-1} & (-1)^{q-2}\binom{q-1}{1}\langle p+1\rangle_{q-2}b^{q-2} \\
\langle p\rangle_{q}a^{p-q}c^{q} & (-1)^{q}\langle p+1\rangle_{q}b^{q} & (-1)^{q-1}\binom{q}{1}\langle p+1\rangle_{q-1}b^{q-1}
\end{array}\right.$$

$$\left.\begin{array}{cccc}
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 \\
1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
(-1)^{q-4}\binom{q-2}{2}\langle p+1\rangle_{q-4}b^{q-4} & \ldots & 1 & 0 \\
(-1)^{q-3}\binom{q-1}{2}\langle p+1\rangle_{q-3}b^{q-3} & \cdots & -\binom{q-1}{q-2}\langle p+1\rangle_{1}b & 1 \\
(-1)^{q-2}\binom{q}{2}\langle p+1\rangle_{q-2}b^{q-2} & \cdots & \binom{q}{q-2}\langle p+1\rangle_{2}b^{2} & -\binom{q}{q-1}\langle p+1\rangle_{1}b
\end{array}\right|,$$

as $x,y\to 0$. So, we have

$$\begin{aligned}
w(p,q|a,b,c) &= \frac{1}{p!q!}\frac{\partial^{p+q}}{\partial y^{q}\partial x^{p}}\left[\frac{1}{1-ax-by-cxy}\right] \\
&= \frac{(-1)^{q}}{q!}\left|\begin{array}{cc}\left(\langle p\rangle_{ij}a^{p-ij}c^{ij}\right)_{\substack{0\le i\le q\\ j=1}} & \left((-1^{i-j}\binom{i-1}{j-1}\langle p+1\rangle_{i-j}b^{i-j}\right)_{\substack{1\le i\le q+1\\ 1\le j\le q}}\end{array}\right|_{(q+1)\times(q+1)},
\end{aligned}$$

which completes the proof. □

Remark 3.2 *For the special case* $a=b=c=1$, *Equation 3.1 reduces to Theorem 2.1 of [15].*

Theorem 3.3 *For* $p,q\ge 0$, *the weighted Delannoy numbers* $w(p,q|a,b,c)$ *satisfy the following recursive formula*

$$w(p,q|a,b,c)$$

$$= \binom{p}{q} a^{p-q} c^{q} + (-1)^{q+1} \sum_{r=0}^{q-1} (-1)^{r} \binom{p+1}{q-r} b^{q-r} w(p, r|a, b, c). \tag{3.2}$$

For $p = q = n$, *we have the following relation for the central weighted Delannoy numbers:*

$$w(n, n|a, b, c) = c^{n} + (-1)^{n+1} \sum_{r=0}^{n-1} (-1)^{r} \binom{n+1}{r+1} b^{n-r} w(n, r|a, b, c). \tag{3.3}$$

Proof Applying (2.2) to the determinantal expressions (3.1), we get

$$\begin{aligned} & w(p, n-1|a, b, c) \\ & = \frac{\langle p \rangle_{n-1}}{(n-1)!} a^{p-n+1} c^{n-1} + (-1)^{n} \sum_{r=2}^{n} (-1)^{r} \frac{\langle p+1 \rangle_{n-r+1}}{(n-r+1)!} b^{n-r+1} w(p, r-2|a, b, c), \end{aligned}$$

which can be simplified as (3.2). Taking $p = q = n$ yields the recurrence formula (3.3) for central weighted Delannoy numbers in terms of weighted Delannoy numbers. So, the proof is completed. □

Remark 3.4 *If we set* $a = b = c = 1$, *Equations 3.2 and 3.3 coincide with identities (3.1) and (3.2) of [15], respectively.*

Remark 3.5 *We notice that a combinatorial proof of our recursive formula (3.2) for weighted Delannoy numbers* $w(p, q|a, b, c)$ *can be achieved by using the principle of inclusion and exclusion.*

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