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Crossed product of infinite groups and complete rewriting systems

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Abstract: The aim of this paper is to obtain a presentation for crossed product of some infinite groups and then find its complete rewriting system. Hence, we present normal form structure of elements of crossed product of infinite groups which yield solvability of the word problem.

Key words: Crossed product, rewriting system, presentation

1. Introduction and preliminaries

Crossed product construction appears in different areas of algebra such as Lie algebras, C^*-algebras, and group theory. This product has also many applications in other fields of mathematics like group representation theory and topology. Here, we consider crossed product construction from view of Combinatorial Group Theory. This product is more important than known group constructions since it contains direct, semidirect [3, 6], twisted [12], and knit [4] products. Crossed product construction is an important structure from the point of the famous extension problem, which is one of the most interesting problems of algebra and was first stated by Hölder in 1895 [10]. This problem consists of describing and classifying all groups E containing H as a normal subgroup such that $E/H \cong G$. The extension problem has been the starting point of new subjects in mathematics such as cohomology of groups, homological algebra, crossed products of groups acting on algebras, crossed products of Hopf algebras acting on algebras, crossed products for von Neumann algebras etc. In [1, 2] the authors give some results on the crossed product about this extension problem. They also say that the set of these $(E, .)$ group structures is a one-to-one correspondence with the set of all normalized crossed systems (H, G, φ, f). Let H and G be two groups. A crossed system of these groups is a quadruple (H, G, φ, f), where $\varphi : G \to Aut(H)$, $g \mapsto \varphi_g(h)$ and $f : G \times G \to H$ are two maps such that the following compatibility conditions hold:

$$g_1 \triangleleft_\varphi (g_2 \triangleleft_\varphi h) = f(g_1, g_2)((g_1g_2) \triangleleft_\varphi h)f(g_1, g_2)^{-1},$$
$$f(g_1, g_2)f(g_1g_2, g_3) = (g_1 \triangleleft_\varphi f(g_2, g_3))f(g_1, g_2g_3),$$

for all $g_1, g_2, g_3 \in G$, and $h \in H$. The crossed system (H, G, φ, f) is called normalized if $f(1, 1) = 1$. Also φ is called a weak action and f is called an φ-cocycle. (H, G, φ, f) is normalized crossed system then $f(1, g) = f(g, 1) = 1$ and $1 \triangleleft_\varphi h = h$, for any $g \in G$ and $h \in H$. Here, the notation "$\triangleleft$" is defined $g \triangleleft_\varphi h = \varphi_g(h)$ as semidirect product action.

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Let H and G be two groups, $\varphi : G \to Aut(H)$ and $f : G \times G \to H$ be two maps. Let $H\#_{\varphi}^{f}G := H \times G$ as a set with a binary operation defined by the formula:

$$(h_1, g_1)(h_2, g_2) = (h_1(g_1 \triangleleft_{\varphi} h_2)f(g_1, g_2), g_1g_2),$$

for all $h_1, h_2 \in H$ and $g_1, g_2 \in G$. Assume that (H, G, φ, f) is a normalized crossed system. Then $(H\#_{\varphi}^{f}G, \cdot)$ is a group with identity $1_{H\#_{\varphi}^{f}G} = (1, 1)$. Here we recall that we have $(h, g)^{-1} = (f(g^{-1}, g)^{-1}(g^{-1} \triangleleft_{\varphi} h^{-1}), g^{-1})$ for $(h, g) \in H\#_{\varphi}^{f}G$. The group $H\#_{\varphi}^{f}G$ is called the crossed product of H and G associated to the crossed system (H, G, φ, f) [1].

- Let φ and f be trivial maps. i.e. $g \triangleleft_{\varphi} h = h$ and $f(g_1, g_2) = 1$ for all $g_1, g_2 \in G$ and $h \in H$. Then (H, G, φ, f) is called trivial crossed system. The crossed product $H\#_{\varphi}^{f}G := H \times G$ is the direct product of H and G.
- Let $f : G \times G \to H$ be a trivial map. Then (H, G, φ, f) is a crossed system if and only if $\varphi : G \to Aut(H)$ is a homomorphism. In this case, the crossed product $H\#_{\varphi}^{f}G$ is the semidirect product of H by G.
- Let $\varphi : G \to Aut(H)$ be a trivial map. Then (H, G, φ, f) is a crossed system if and only if $Im(f) \subseteq Z(H)$, where $Z(H)$ is the center of H, and $f(g_1, g_2)f(g_1g_2, g_3) = f(g_2, g_3)f(g_1, g_2g_3)$ for all $g_1, g_2, g_3 \in G$, that is $f : G \times G \to Z(H)$ is a 2-cocycle. The crossed product $H\#_{\varphi}^{f}G$ associated to this crossed system is denoted by $H \times^{f} G$ and called the twisted product of H and G.

Now we give the following result as the main application of the crossed product construction on groups. The proof of this result can be found in [1, 2].

Theorem 1.1 *Let E be a group, H be normal subgroup of E, and G be the quotient of E by H. Then there exist two maps $\varphi : G \to Aut(H)$ and $f : G \times G \to H$ such that (H, G, φ, f) is a normalized crossed system and $E \cong H\#_{\varphi}^{f}G$.*

The reader is referred to [7–9, 11] for recent studies on crossed product of groups and its derivations.

Let X be a set and let X^* be the free monoid consisting of all words obtained by the elements of X. A (string) rewriting system on X^* is a subset $R \subseteq X^* \times X^*$ and an element $(u, v) \in R$, also can be written as $u \to v$, is called a rule of R. The idea for a rewriting system is an algorithm for substituting the right-hand side of a rule whenever the left-hand side appears in a word. In general, for a given rewriting system R, we write $x \to y$ for $x, y \in X^*$ if $x = uv_1w$, $y = uv_2w$ and $(v_1, v_2) \in R$. Also, we write $x \to^* y$ if $x = y$ or $x \to x_1 \to x_2 \to \cdots \to y$ for some finite chain of reductions and $\leftrightarrow^*$ is the reflexive, symmetric, and transitive closure of $\to$. Furthermore, an element $x \in X^*$ is called irreducible with respect to R if there is no possible rewriting *(or reduction)* $x \to y$; otherwise, x is called reducible. The rewriting system R is called

- Noetherian if there is no infinite chain of rewritings $x \to x_1 \to x_2 \to \cdots$ for any word $x \in X^*$,
- Confluent if whenever $x \to^* y_1$ and $x \to^* y_2$, there is a $z \in X^*$ such that $y_1 \to^* z$ and $y_2 \to^* z$,
- Complete if R is both Noetherian and confluent.

A critical pair of a rewriting system R is a pair of overlapping rules if one of the (r_1r_2, s), $(r_2r_3, t) \in R$ with $r_2 \neq 1$ or $(r_1r_2r_3, s)$, $(r_2, t) \in R$ forms is satisfied. Also, a critical pair is resolved in R if there is a word z such that $sr_3 \to^* z$ and $r_1t \to^* z$ in the first case or $s \to^* z$ and $r_1tr_3 \to^* z$ in the second. A Noetherian rewriting system is complete if and only if every critical pair is resolved [5, 13].

One can ask a question of what the normal form of elements of a given algebraic structure is. Here, we work on this question focusing on crossed product of some infinite groups to obtain a presentation and its complete rewriting system. To do that, in Section 2, we obtain a presentation for crossed product of some infinite groups and in Section 3, by using the presentation given in Section 2, we get a complete rewriting system for that group. Thus, we present normal form structures of elements of crossed product of that group. Thus, these normal form structures yield solvability of the word problem.

2. A presentation for crossed product of infinite groups

In this section, we give one of the main results of this paper which gives a presentation of crossed product of two infinite groups. To do that, let N be a group of infinite direct sum of copies of $\mathbb{Z}_n$ presented by $N = \langle a_i \, (i \in \mathbb{Z}) \, ; \, a_i^n = 1, \, a_ia_j = a_ja_i \rangle$ and $\mathbb{Z}$ be infinite cyclic group generated by t. Let $\varphi : \mathbb{Z} \to Aut(N)$, $t \mapsto \varphi_t(a_i) = a_{i+1}$ and $f : \mathbb{Z} \times \mathbb{Z} \to N$ be two maps. Then, we call $N\#_{\varphi}^{f}\mathbb{Z}$ as crossed product of N and $\mathbb{Z}$ associated to the crossed system $(N, \mathbb{Z}, \varphi, f)$.

Theorem 2.1 *A group G is isomorphic to crossed product $N\#_{\varphi}^{f}\mathbb{Z}$ if and only if G is a group generated by generators α, t and satisfies the relations*

$$\alpha^n = 1, \quad [t^i\alpha^k t^{-i}, t^j\alpha^l t^{-j}] = 1,$$

for some k, $l \in \mathbb{Z}$ and $(k, n) = (l, n) = 1$.

Proof Suppose that the group G is isomorphic to crossed product $N\#_{\varphi}^{f}\mathbb{Z}$. Thus, there exists a normal subgroup N of G such that $G/N \cong \mathbb{Z}$. It follows that $N = \langle a_i \, (i \in \mathbb{Z}) \, ; \, a_i^n = 1, \, a_ia_j = a_ja_i \rangle$ and there exists $t \in G$ such that $G/N = \{t^kN; k \in \mathbb{Z}\}$. Since $N \trianglelefteq G$, we obtain that $ta_it^{-1} \in N$ for $t \in G$. That is, there exist $0 \leq m_i < n$ such that we have

$$ta_it^{-1} = a_{i+1}^{m_i}. \tag{2.1}$$

Thus, we get

$$G \cong \left\langle a_i \, (i \in \mathbb{Z}), t \, ; \, a_i^n = 1, \, a_ia_j = a_ja_i, \, ta_it^{-1} = a_{i+1}^{m_i} \, ((m_i, n) = 1) \right\rangle. \tag{2.2}$$

Now, by using some indices defined on a_i, we write the relation $ta_it^{-1} = a_{i+1}^{m_i}$ $((m_i, n) = 1)$ given in (2.2) more clearly. Thus, we get

$$a_1 = t^{-1}a_2^{m_1}t \ ((m_1, n) = 1), \qquad a_2 = t^{-1}a_3^{m_2}t \ ((m_2, n) = 1),$$
$$a_3 = t^{-1}a_4^{m_3}t \ ((m_3, n) = 1), \quad \cdots, \quad a_r = t^{-1}a_{r+1}^{m_r}t \ ((m_r, n) = 1).$$

As seen above, each generator a_i is obtained by using a_{i+1}. By using these equalities and the relation $a_ia_j = a_ja_i$ given in (2.2), we get

$[t^i a_{i+1}^{m_i\cdots m_2 m_1} t^{-i}, t^j a_{j+1}^{m_j\cdots m_2 m_1} t^{-j}] = 1$, where $(\prod_{s=1}^{i} m_s, n) = 1$. Thus, for each m_s, we have $(m_s, n) = 1$. Let $\alpha = (\cdots, 1, 1, 1, a_i, 1, 1, 1, \cdots)$, by using the relation $a_i^n = 1$ in (2.2), we get $\alpha^n = 1$. By using the above relation $[t^i a_{i+1}^{m_i\cdots m_2 m_1} t^{-i}, t^j a_{j+1}^{m_j\cdots m_2 m_1} t^{-j}] = 1$, we also obtain $[t^i \alpha^k t^{-i}, t^j \alpha^l t^{-j}] = 1$, where $(k, n) = (l, n) = 1$.

Conversely, now let $G \cong \langle \alpha, t\,;\, \alpha^n = 1,\ [t^i\alpha^k t^{-i}, t^j \alpha^l t^{-j}] = 1 \rangle$ for some $k, l \in \mathbb{Z}$ and $(k, n) = (l, n) = 1$. By Theorem 1.1, we need to prove that $N \trianglelefteq G$ and $G/N \cong \mathbb{Z}$. For any $g' \in G$ we have $g' = x_1 x_2 \cdots x_k$, for some $k \in \mathbb{N}$ and $x_i \in \{\alpha, \alpha^{-1}, t, t^{-1}\}$ $(1 \leq i \leq k)$. That is, to prove that $N \trianglelefteq G$ we only need to show that $t^{-1}\alpha^x t \in N$ and $t\alpha^x t^{-1} \in N$ for any $x \in \mathbb{Z}$. From (2.1), by obtaining a general form, we write that $t\alpha t^{-1} = \alpha^m$. By induction, we obtain that $t\alpha^x t^{-1} = \alpha^{mx} \in N$. Since $(m, n) = 1$, there exist $\gamma, \beta \in \mathbb{Z}$ such that $\gamma m + \beta n = 1$. We obtain from (2.1) that $\alpha = t^{-1}\alpha^m t$ and it follows from here that $\alpha^\gamma = t^{-1}\alpha^{\gamma m} t$. Since $t^{-1}\alpha^{\beta n} t = 1$ we obtain that $t^{-1}\alpha^{\gamma m + \beta n} t = \alpha^\gamma$, that is $t^{-1}\alpha t = \alpha^\gamma$. It follows from here that $t^{-1}\alpha^x t = \alpha^{\gamma x}$ for any $x \in \mathbb{Z}$. Hence, $N \trianglelefteq G$.

It follows by a simple calculation that every element $g' \in G$ can be written as $t^p\alpha^q$ for some $p, q \in \mathbb{Z}$. That is $g'N = t^p\alpha^q N = t^p N$. Hence, $G/N \subseteq \mathbb{Z}$. Now suppose that there exist $\gamma \neq \beta \in \mathbb{Z}$ such that $t^\gamma N = t^\beta N$, that is $t^{\gamma-\beta} = \alpha^\tau$ for some $0 \leq \tau \leq n-1$. It follows from here that $t^{(\gamma-\beta)n} = \alpha^{\tau n} = 1$, which is contradiction with $\mathbb{Z}$ being an infinite cyclic group. Therefore, $G/N \cong \mathbb{Z}$. □

Corollary 2.2 *Let us consider the presentation of* $N\#_\varphi^f\mathbb{Z}$ *given in Theorem 2.1*

$$\langle \alpha, t\,;\, \alpha^n = 1,\ [t^i\alpha^k t^{-i}, t^j\alpha^l t^{-j}] = 1\ (k, l \in \mathbb{Z},\ (k, n) = (l, n) = 1) \rangle .$$

If $k, l = 1$, *then* $N\#_\alpha^f\mathbb{Z}$ *becomes Lamplighter group* $L = \mathbb{Z}_n \wr \mathbb{Z} = \oplus_{\mathbb{Z}}\mathbb{Z}_n \rtimes \mathbb{Z}$ *presented by*

$$L_n = \langle \alpha, t\ ;\ \alpha^n = 1,\ [t^i\alpha t^{-i}, t^j\alpha t^{-j}] = 1 \rangle ,$$

for all $i, j \in \mathbb{Z}$ *[14].*

3. A complete rewriting system for $N\#_\varphi^f\mathbb{Z}$

In this section, we obtain a complete rewriting system for the monoid presentation of $N\#_\varphi^f\mathbb{Z}$. To obtain a complete rewriting system, we order words in given alphabet in the deg-lex way by comparing two words first with their degrees (lengths), and then lexicographically when the lengths are equal. Since our complete rewriting systems depend on the lengths of words, we have the following main results accordingly as $m = 1$, $m = 2$, and $m \geq 3$ in the relator $ta_it^{-1} = a_{i+1}^m$ $(i \in \mathbb{Z},\ (m, n) = 1)$. The monoid presentation of $N\#_\varphi^f\mathbb{Z}$ is given as follows:

$$\langle a_i\ (i \in \mathbb{Z}), t, t^{-1}\ ;\ a_i^n = 1,\ a_ia_j = a_ja_i,\ ta_it^{-1} = a_{i+1}^m\ ((m, n) = 1),\ tt^{-1} = 1,\ t^{-1}t = 1 \rangle . \tag{3.1}$$

We note that $\overline{W}$ will denote the word which does not have the first generator of the word W. For example, let $W = a_1a_2a_3$. Then $\overline{W} = a_2a_3$. Additionally, the notations $(i) \cap (j)$ and $(i) \cup (j)$ will denote the intersection and inclusion overlapping words of left-hand side of relations (i) and (j), respectively.

Now we order the generators given in (3.1) as $a_i > a_j > t > t^{-1}$ $(i > j)$. This ordering will be acceptable for results given below. We have the following first result of this section.

Theorem 3.1 *A complete rewriting system for $m = 1$ given in presentation (3.1) consists of the following relations:*

$$
\begin{array}{lll}
(1) \quad a_i^n \to 1, & (2)\, a_i a_j \to a_j a_i \ (i > j), & (3)\, tt^{-1} \to 1, \\
(4) \quad t^{-1}t \to 1, & (5)\, a_i t^{-1} \to t^{-1} a_{i+1}, & (6)\, a_{i+1} t \to t a_i.
\end{array}
$$

Proof Since we have the ordering $a_i > a_j > t > t^{-1}\,(i > j)$ between generators, there are no infinite reduction steps for all overlapping words. Hence, the rewriting system is Noetherian. Now, to catch up the aim, we need to show that the confluent property holds. Thus, we have the following overlapping words and corresponding critical pairs, respectively.

$$
\begin{array}{ll}
(1) \cap (1) \ : a_i^{n+1},\ (a_i, a_i), & (1) \cap (2) \ : a_i^n a_j \ (i > j),\ (a_j, a_i^{n-1} a_j a_i), \\
(1) \cap (5) \ : a_i^n t^{-1},\ (t^{-1}, a_i^{n-1} t^{-1} a_{i+1}), & (1) \cap (6) \ : a_{i+1}^n t,\ (t, a_{i+1}^{n-1} t a_i), \\
(2) \cap (1) \ : a_i a_j^n \ (i > j),\ (a_j a_i a_j^{n-1}, a_i), & (2) \cap (2) \ : a_i a_j a_k \ (i > j > k),\ (a_j a_i a_k, a_i a_k a_j), \\
(2) \cap (5) \ : a_i a_j t^{-1} \ (i > j),\ (a_j a_i t^{-1}, a_i t^{-1} a_{j+1}), & (2) \cap (6) \ : a_i a_{j+1} t \ (i > j+1),\ (a_{j+1} a_i t, a_i t a_j), \\
(3) \cap (4) \ : tt^{-1}t,\ (t, t), & (4) \cap (3) \ : t^{-1}tt^{-1},\ (t^{-1}, t^{-1}), \\
(5) \cap (4) \ : a_i t^{-1} t,\ (t^{-1} a_{i+1} t, a_i), & (6) \cap (3) \ : a_{i+1} t t^{-1},\ (t a_i t^{-1}, a_{i+1}).
\end{array}
$$

In fact, all these above critical pairs are resolved by reduction steps. We show two of them as follows:

$$
(1) \cap (5) \ : a_i^n t^{-1}, \quad (t^{-1}, a_i^{n-1} t^{-1} a_{i+1}),
$$

$$
a_i^n t^{-1} \longrightarrow \begin{cases} t^{-1} \\ a_i^{n-1} t^{-1} a_{i+1} \to a_i^{n-2} t^{-1} a_{i+1}^2 \to \cdots \to t^{-1} a_{i+1}^n \to t^{-1}. \end{cases}
$$

$$
(6) \cap (3) \ : a_{i+1} t t^{-1}, \quad (t a_i t^{-1}, a_{i+1}),
$$

$$
a_{i+1} t t^{-1} \longrightarrow \begin{cases} t a_i t^{-1} \to t t^{-1} a_{i+1} \to a_{i+1} \\ a_{i+1}. \end{cases}
$$

After all above processes, we see that all critical pairs can be resolved. Thus, the rewriting system is complete. □

By Theorem 3.1, we have the following result.

Corollary 3.2 *The normal form of a word u, representing an element of $N \#_\varphi^f \mathbb{Z}$, is $t^k a_{i_1}^{\epsilon_{i_1}} a_{i_2}^{\epsilon_{i_2}} \cdots a_{i_m}^{\epsilon_{i_m}}$, where $k \in \mathbb{Z}$, $0 \le \epsilon_{i_p} < n \ (1 \le p \le m)$ and $i_1 < i_2 < \cdots < i_m$.*

Theorem 3.3 *A complete rewriting system for $m = 2$ given in presentation (3.1) consists of the following* *relations:*

$$
\begin{array}{lll}
(1) \quad a_i^n \to 1, & (2)\ a_i a_j \to a_j a_i \ (i>j), & (3)\ tt^{-1} \to 1,\\
(4) \quad t^{-1}t \to 1, & (5)\ a_{i+1}^2 W t \to W t a_i, & (6)\ t^{-1} W a_{i+1}^2 \to a_i t^{-1} W,\\
(7) \quad W_1 t^{\epsilon} W_2 t^{-\epsilon} \to t^{\epsilon} W_2 t^{-\epsilon} W_1 \ (\epsilon = \pm 1), & (8)\ a_i^{\frac{n+1}{2}} \to t^{-1} a_{i+1} t, & (9)\ t a_i t^{-1} \to a_{i+1}^2,\\
(10) \quad a_{i+1} W_1 t W_2 a_i^{\frac{n-1}{2}} \to W_1 t W_2, & (11)\ a_i^{\frac{n-1}{2}} W_1 t^{-1} W_2 a_{i+1} \to W_1 t^{-1} W_2, & \\
(12) \quad a_{i+1}^2 W a_{j+1}^2 \to t a_i a_j t^{-1} W \ (j>i), & (13)\ a_{i+1} W_1 t W_2 a_i^{\frac{n-5}{2}} t W_3 a_{i-1} \to W_1 t W_2 t W_3, & \\
(14) \quad a_i W_1 t^{-1} a_{i+1}^{\frac{n-5}{2}} W_2 t^{-1} W_3 a_{i+2} \to W_1 t^{-1} W_2 t^{-1} W_3, & &
\end{array}
$$

where W_1, W_2, and W_3 are reduced words containing a_i $(i \in \mathbb{Z})$ and W is a reduced word generated by a_i and t.

Proof Noetherian property of the rewriting system can be seen easily. Now, to catch up the aim, we need to show that the confluent property holds. Thus, we have the following overlapping words and corresponding critical pairs, respectively.

$(1)\cap(1)\ : a_i^{n+1},\ (a_i, a_i),\quad (1)\cap(2)\ : a_i^n a_j\ (i>j),\ (a_j, a_i^{n-1} a_j a_i),\quad (1)\cap(5)\ : a_{i+1}^n W t,\ (Wt, a_{i+1}^{n-2} W t a_i),$

$(1)\cap(7)\ : a_i^n \overline{W_1} t^{\epsilon} W_2 t^{-\epsilon},\ (\overline{W_1} t^{\epsilon} W_2 t^{-\epsilon}, a_i^{n-1} t^{\epsilon} W_2 t^{-\epsilon} \overline{W_1}),$

$(1)\cup(8)\ : a_i^n,\ (1, a_i^{\frac{n-1}{2}} t^{-1} a_{i+1} t),\quad (1)\cap(10)\ : a_{i+1}^n W_1 t W_2 a_i^{\frac{n-1}{2}},\ (W_1 t W_2 a_i^{\frac{n-1}{2}}, a_{i+1}^{n-1} W_1 t W_2),$

$(1)\cap(11)\ : a_i^n W_1 t^{-1} W_2 a_{i+1},\quad (W_1 t^{-1} W_2 a_{i+1}, a_i^{\frac{n-1}{2}} W_1 t^{-1} W_2),$

$(1)\cap(12)\ : a_{i+1}^n W a_{j+1}^2\ (j>i),\ (W a_{j+1}^2, a_{i+1}^{n-2} t a_i a_j t^{-1} W),$

$(1)\cap(13)\ : a_{i+1}^n W_1 t W_2 a_i^{\frac{n-5}{2}} t W_3 a_{i-1},\ (W_1 t W_2 a_i^{\frac{n-5}{2}} t W_3 a_{i-1}, a_{i+1}^{n-1} W_1 t W_2 t W_3),$

$(1)\cap(14)\ : a_i^n W_1 t^{-1} a_{i+1}^{\frac{n-5}{2}} W_2 t^{-1} W_3 a_{i+2},\ (W_1 t^{-1} a_{i+1}^{\frac{n-5}{2}} W_2 t^{-1} W_3 a_{i+2}, a_i^{n-1} W_1 t^{-1} W_2 t^{-1} W_3),$

$(2)\cap(1)\ : a_i a_j^n\ (i>j),\ (a_j a_i a_j^{n-1}, a_i),\quad (2)\cap(2)\ : a_i a_j a_k\ (i>j>k),\ (a_j a_i a_k, a_i a_k a_j),$

$(2)\cap(5)\ : a_i a_{j+1}^2 W t\ (i>j+1),\ (a_{j+1} a_i a_{j+1} W t, a_i W t a_j),$

$(2)\cap(7)\ : a_i a_j \overline{W_1} t^{\epsilon} W_2 t^{-\epsilon}\ (i>j),\ (a_j a_i \overline{W_1} t^{\epsilon} W_2 t^{-\epsilon}, a_i t^{\epsilon} W_2 t^{-\epsilon} W_1),$

$(2)\cap(8)\ : a_i a_j^{\frac{n+1}{2}}\ (i>j),\ (a_j a_i a_j^{\frac{n-1}{2}}, a_i t^{-1} a_{j+1} t),$

$(2)\cap(10)\ : a_i a_{j+1} W_1 t W_2 a_j^{\frac{n-1}{2}}\ (i>j+1), (a_{j+1} a_i W_1 t W_2 a_j^{\frac{n-1}{2}}, a_i W_1 t W_2),$

$(2)\cap(11)\ : a_i a_j^{\frac{n-1}{2}} W_1 t^{-1} W_2 a_{j+1}\ (i>j), (a_j a_i a_j^{\frac{n-3}{2}} W_1 t^{-1} W_2 a_{j+1}, a_i W_1 t^{-1} W_2),$

$(2)\cap(12)\ : a_i a_{j+1}^2 W a_{k+1}^2\ (i>j+1,\ \ k>j),\quad (a_{j+1} a_i a_{j+1} W a_{k+1}^2, a_i t a_j a_k t^{-1} W),$

$(2)\cap(13)\ : a_i a_{j+1} W_1 t W_2 a_j^{\frac{n-5}{2}} t W_3 a_{j-1}\ (i>j+1),\ \ (a_{j+1} a_i W_1 t W_2 a_j^{\frac{n-5}{2}} t W_3 a_{j-1}, a_i W_1 t W_2 t W_3),$

and

$(2)\cap(14)\ : a_i a_j W_1 t^{-1} a_{j+1}^{\frac{n-5}{2}} W_2 t^{-1} W_3 a_{j+2}\ (i>j),\ (a_j a_i W_1 t^{-1} a_{j+1}^{\frac{n-5}{2}} W_2 t^{-1} W_3 a_{j+2}, a_i W_1 t^{-1} W_2 t^{-1} W_3),$

$(3)\cap(4)\ : tt^{-1}t,\ (t,t),$ $(3)\cap(6)\ : tt^{-1}Wa_{i+1}^2,\ (Wa_{i+1}^2, ta_i t^{-1}W),$

$(4)\cap(3)\ : t^{-1}tt^{-1},\ (t^{-1},t^{-1}),$ $(4)\cap(9)\ : t^{-1}ta_i t^{-1},\ (a_i t^{-1}, t^{-1}a_{i+1}^2),$

$(5)\cap(3)\ : a_{i+1}^2 Wtt^{-1},\ (Wta_i t^{-1}, a_{i+1}^2 W),$ $(5)\cap(7)\ : a_{i+1}^2 WtW_2 t^{-1},\ (Wta_i W_2 t^{-1}, a_{i+1}^2 tW_2 t^{-1} W),$

$(5)\cap(9)\ : a_{i+1}^2 Wta_j t^{-1},\ (Wta_i a_j t^{-1}, a_{i+1}^2 Wa_{j+1}^2),$

$(6)\cap(1)\ : t^{-1}Wa_{i+1}^n,\ (a_i t^{-1} Wa_{i+1}^{n-2}, t^{-1}W),$

$(6)\cap(2)\ : t^{-1}Wa_{i+1}^2 a_j\ \ (i+1>j),\ (a_i t^{-1} Wa_j, t^{-1}Wa_{i+1}a_j a_{i+1}),$

$(6)\cap(5)\ : t^{-1}Wa_{i+1}^2 W_1 t,\ (a_i t^{-1} WW_1 t, t^{-1}WW_1 ta_i),$

$(6)\cap(8)\ : t^{-1}Wa_{i+1}^{\frac{n+1}{2}},\ (a_i t^{-1} Wa_{i+1}^{\frac{n-3}{2}}, t^{-1}Wt^{-1}a_{i+1}t),$

$(6)\cap(10)\ : t^{-1}Wa_{i+1}^2 W_1 tW_2 a_i^{\frac{n-1}{2}},\ (a_i t^{-1} WW_1 tW_2 a_i^{\frac{n-1}{2}}, t^{-1}Wa_{i+1}W_1 tW_2),$

$(6)\cap(11)\ : t^{-1}Wa_{i+1}^{\frac{n-1}{2}} W_1 t^{-1} W_2 a_{i+2},\ (a_i t^{-1} Wa_i^{\frac{n-5}{2}} W_1 t^{-1} W_2 a_{i+2}, t^{-1}WW_1 t^{-1} W_2),$

$(6)\cap(12)\ : t^{-1}Wa_{i+1}^2 W_1 a_{j+1}^2\ (j>i),\ (a_i t^{-1} WW_1 a_{j+1}^2, t^{-1}Wta_i a_j t^{-1} W_1),$

$(6)\cap(13)\ : t^{-1}Wa_{i+1}^2 W_1 tW_2 a_i^{\frac{n-5}{2}} tW_3 a_{i-1},\ (a_i t^{-1} WW_1 tW_2 a_i^{\frac{n-5}{2}} tW_3 a_{i-1}, t^{-1}Wa_{i+1}W_1 tW_2 tW_3),$

$(6)\cap(14)\ : t^{-1}Wa_i^2 W_1 t^{-1} a_{i+1}^{\frac{n-5}{2}} W_2 t^{-1} W_3 a_{i+2},$

$(a_{i-1} t^{-1} WW_1 t^{-1} a_{i+1}^{\frac{n-5}{2}} W_2 t^{-1} W_3 a_{i+2}, t^{-1}Wa_i W_1 t^{-1} W_2 t^{-1} W_3),$

$(7)\cap(3)\ : W_1 t^{-1} W_2 tt^{-1},\quad (t^{-1}W_2 tW_1 t^{-1}, W_1 t^{-1} W_2),$

$(7)\cap(4)\ : W_1 tW_2 t^{-1} t,\quad (tW_2 t^{-1} W_1 t, W_1 tW_2),$

$(7)\cap(6)\ : W_1 tW_2 t^{-1} Wa_{i+1}^2,\quad (tW_2 t^{-1} W_1 Wa_{i+1}^2, W_1 tW_2 a_i t^{-1} W),$

$(7)\cap(9)\ : W_1 t^{-1} W_2 ta_i t^{-1},\quad (t^{-1}W_2 tW_1 a_i t^{-1}, W_1 t^{-1} W_2 a_{i+1}^2),$

$(8)\cap(1)\ : a_i^n,\ (1, a_i^{\frac{n-1}{2}} t^{-1} a_{i+1} t),$ $(8)\cap(2)\ a_i^{\frac{n+1}{2}} a_j\ (i>j),\quad (t^{-1}a_{i+1}ta_j, a_i^{\frac{n-1}{2}} a_j a_i),$

$(8)\cap(5)\ : a_i^{\frac{n+1}{2}} Wt,\quad (t^{-1}a_{i+1}tWt, a_i^{\frac{n-3}{2}} Wta_{i-1}),$ $(8)\cap(8)\ : a_i^{\frac{n+3}{2}},\quad (t^{-1}a_{i+1}ta_i, a_i t^{-1} a_{i+1} t),$

$(8)\cap(10)\ : a_{i+1}^{\frac{n+1}{2}} W_1 tW_2 a_i^{\frac{n-1}{2}},\ (t^{-1}a_{i+1}tW_1 tW_2 a_i^{\frac{n-1}{2}}, a_{i+1}^{\frac{n-1}{2}} W_1 tW_2),$

$(8)\cap(11)\ : a_i^{\frac{n+1}{2}} W_1 t^{-1} W_2 a_{i+1},\ (t^{-1}a_{i+1}tW_1 t^{-1} W_2 a_{i+1}, a_i W_1 t^{-1} W_2),$

$(8)\cap(12)\ : a_{i+1}^{\frac{n+1}{2}} Wa_{j+1}^2\ (j>i),\quad (t^{-1}a_{i+1}tWa_{j+1}^2, a_{i+1}^{\frac{n-1}{2}} ta_i a_j t^{-1} W),$

$(8)\cap(13)\ : a_{i+1}^{\frac{n+1}{2}} W_1 tW_2 a_i^{\frac{n-5}{2}} tW_3 a_{i-1},\ (t^{-1}a_{i+1}tW_1 tW_2 a_i^{\frac{n-5}{2}} tW_3 a_{i-1}, a_{i+1}^{\frac{n-1}{2}} W_1 tW_2 tW_3),$

$(8)\cap(14)\ : a_i^{\frac{n+1}{2}} W_1 t^{-1} a_{i+1}^{\frac{n-5}{2}} W_2 t^{-1} W_3 a_{i+2},\ (t^{-1}a_{i+1}tW_1 t^{-1} a_{i+1}^{\frac{n-5}{2}} W_2 t^{-1} W_3 a_{i+2}, a_i^{\frac{n-1}{2}} W_1 t^{-1} W_2 t^{-1} W_3),$

$(9)\cap(4)\ : ta_i t^{-1} t,\ (a_{i+1}^2 t,\ ta_i),$ $(9)\cap(6)\ : ta_i t^{-1} Wa_{j+1}^2,\ (a_{i+1}^2 Wa_{j+1}^2,\ ta_i a_j t^{-1} W),$

$(9)\cap(7)\ : ta_i t^{-1} W_2 t,\ (a_{i+1}^2 W_2 t,\ tt^{-1} W_2 ta_i),$

and

$$(9)\cap(14)\ : ta_it^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3a_{i+2},\ (a_{i+1}^{\frac{n-1}{2}}W_2t^{-1}W_3a_{i+2},\ tt^{-1}W_2t^{-1}W_3),$$

$$(10)\cap(1)\ : a_{i+1}W_1tW_2a_i^n,\ (W_1tW_2a_i^{\frac{n+1}{2}},\ a_{i+1}W_1tW_2),$$

$$(10)\cap(2)\ : a_{i+1}W_1tW_2a_i^{\frac{n-1}{2}}a_j\ (i>j),\ (W_1tW_2a_j,\ a_{i+1}W_1tW_2a_i^{\frac{n-1}{2}-1}a_ja_i),$$

$$(10)\cap(5)\ : a_{i+1}W_1tW_2a_i^{\frac{n-1}{2}}Wt,\ (W_1tW_2Wt,\ a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}Wta_{i-1}),$$

$$(10)\cap(7)\ : a_{i+1}W_1tW_2a_i^{\frac{n-1}{2}}\overline{W_3}t^{\epsilon}W_4t^{-\epsilon},\ (W_1tW_2\overline{W_3}t^{\epsilon}W_4t^{-\epsilon},\ a_{i+1}W_1tW_2a_i^{\frac{n-3}{2}}t^{\epsilon}W_4t^{-\epsilon}a_i\overline{W_3}),$$

$$(10)\cap(8)\ : a_{i+1}W_1tW_2a_i^{\frac{n+1}{2}},\ (W_1tW_2a_i,\ a_{i+1}W_1tW_2t^{-1}a_{i+1}t),$$

$$(10)\cap(10)\ : a_{i+1}W_1tW_2a_i^{\frac{n-1}{2}}W_3tW_4a_{i-1}^{\frac{n-1}{2}},\ (W_1tW_2W_3tW_4a_{i-1}^{\frac{n-1}{2}},\ a_{i+1}W_1tW_2a_i^{\frac{n-3}{2}}W_3tW_4),$$

$$(10)\cap(11)\ : a_{i+1}W_1tW_2a_i^{\frac{n-1}{2}}W_3t^{-1}W_4a_{i+1},\ (W_1tW_2W_3t^{-1}W_4a_{i+1},\ a_{i+1}W_1tW_2W_3t^{-1}W_4),$$

$$(10)\cap(12)\ : a_{i+1}W_1tW_2a_i^{\frac{n-1}{2}}Wa_{j+1}^2\ (j>i),\ (W_1tW_2Wa_{j+1}^2,\ a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}ta_{i-1}a_jt^{-1}W),$$

$$(10)\cap(13)\ : a_{i+1}W_1tW_2a_i^{\frac{n-1}{2}}W_3tW_4a_{i-1}^{\frac{n-5}{2}}tW_5a_{i-2},$$

$$(W_1tW_2W_3tW_4a_{i-1}^{\frac{n-5}{2}}tW_5a_{i-2},\ a_{i+1}W_1tW_2a_i^{\frac{n-3}{2}}W_3tW_4tW_5),$$

$$(10)\cap(14)\ : a_{i+1}W_1tW_2a_i^{\frac{n-1}{2}}W_3t^{-1}a_{i+1}^{\frac{n-5}{2}}W_4t^{-1}W_5a_{i+2},$$

$$(W_1tW_2W_3t^{-1}W_4a_{i+1}^{\frac{n-5}{2}}W_4t^{-1}W_5a_{i+2},\ a_{i+1}W_1tW_2a_i^{\frac{n-3}{2}}W_3t^{-1}W_4t^{-1}W_5),$$

$$(11)\cap(1)\ : a_i^{\frac{n-1}{2}}W_1t^{-1}W_2a_{i+1}^n,\ (W_1t^{-1}W_2a_{i+1}^{n-1},\ a_i^{\frac{n-1}{2}}W_1t^{-1}W_2),$$

$$(11)\cap(2)\ : a_i^{\frac{n-1}{2}}W_1t^{-1}W_2a_{i+1}a_j\ (i+1>j),\ (W_1t^{-1}W_2a_j,\ a_i^{\frac{n-1}{2}}W_1t^{-1}W_2a_ja_{i+1}),$$

$$(11)\cap(5)\ : a_i^{\frac{n-1}{2}}W_1t^{-1}W_2a_{i+1}^2Wt,\ (W_1t^{-1}W_2a_{i+1}Wt,\ a_i^{\frac{n-1}{2}}W_1t^{-1}W_2Wta_i),$$

$$(11)\cap(6)\ : a_i^{\frac{n-1}{2}}W_1t^{-1}W_2a_{i+1}^2,\ (W_1t^{-1}W_2a_{i+1},\ a_i^{\frac{n-1}{2}}W_1a_it^{-1}W_2),$$

$$(11)\cap(7)\ : a_i^{\frac{n-1}{2}}W_1t^{-1}W_2a_{i+1}tW_3t^{-1}\ (W_1t^{-1}W_2tW_3t^{-1},\ a_i^{\frac{n-1}{2}}W_1t^{-1}tW_3t^{-1}W_2a_{i+1}),$$

$$(11)\cap(8)\ : a_i^{\frac{n-1}{2}}W_1t^{-1}W_2a_{i+1}^{\frac{n+1}{2}},\ (W_1t^{-1}W_2a_{i+1}^{\frac{n-1}{2}},\ a_i^{\frac{n-1}{2}}W_1t^{-1}W_2t^{-1}a_{i+2}t),$$

$$(11)\cap(10)\ : a_i^{\frac{n-1}{2}}W_1t^{-1}W_2a_{i+1}W_3tW_4a_i^{\frac{n-1}{2}},\ (W_1t^{-1}W_2W_3tW_4a_i^{\frac{n-1}{2}},\ a_i^{\frac{n-1}{2}}W_1t^{-1}W_2W_3tW_4),$$

$$(11)\cap(11)\ : a_i^{\frac{n-1}{2}}W_1t^{-1}W_2a_{i+1}^{\frac{n-1}{2}}W_3t^{-1}W_4a_{i+2},\ (W_1t^{-1}W_2a_{i+1}^{\frac{n-3}{2}}W_3t^{-1}W_4a_{i+2},\ a_i^{\frac{n-1}{2}}W_1t^{-1}W_2W_3t^{-1}W_4),$$

$$(11)\cap(12)\ : a_i^{\frac{n-1}{2}}W_1t^{-1}W_2a_{i+1}^2Wa_{j+1}^2\ (j>i),\ (W_1t^{-1}W_2a_{i+1}Wa_{j+1}^2,\ a_i^{\frac{n-1}{2}}W_1t^{-1}W_2ta_ia_jt^{-1}W),$$

$$(11)\cap(13)\ : a_i^{\frac{n-1}{2}}W_1t^{-1}W_2a_{i+1}W_3tW_4a_i^{\frac{n-5}{2}}tW_5a_{i-1},$$

$$(W_1t^{-1}W_2W_3tW_4a_i^{\frac{n-5}{2}}tW_5a_{i-1},\ a_i^{\frac{n-1}{2}}W_1t^{-1}W_2W_3tW_4tW_5),$$

$$(11)\cap(14)\ : a_i^{\frac{n-1}{2}}W_1t^{-1}W_2a_{i+1}W_3t^{-1}a_{i+2}^{\frac{n-5}{2}}W_4t^{-1}W_5a_{i+3},$$

$$(W_1t^{-1}W_2W_3t^{-1}a_{i+2}^{\frac{n-5}{2}}W_4t^{-1}W_5a_{i+3},\ a_i^{\frac{n-1}{2}}W_1t^{-1}W_2W_3t^{-1}W_4t^{-1}W_5),$$

and

$$
\begin{aligned}
(12)\cap(1) &: a_{i+1}^2 W a_{j+1}^n \ (j>i),\ (ta_ia_jt^{-1}Wa_{j+1}^{n-2}, a_{i+1}^2W),\\
(12)\cap(2) &: a_{i+1}^2 W a_{j+1}^2 a_k \ (j>i,\ j+1>k),\ (ta_ia_jt^{-1}Wa_k, a_{i+1}^2Wa_{j+1}a_ka_{j+1}),\\
(12)\cap(5) &: a_{i+1}^2 W a_{j+1}^2 W_1 t \ (j>i),\ (ta_ia_jt^{-1}WW_1t, a_{i+1}^2WW_1ta_j),\\
(12)\cap(7) &: a_{i+1}^2 W a_{j+1}^2 \overline{W_1} t W_2 t^{-1} \ (j>i),\ (ta_ia_jt^{-1}W\overline{W_1}tW_2t^{-1}, a_{i+1}^2Wa_{j+1}tW_2t^{-1}a_{j+1}\overline{W_1}),\\
(12)\cap(7) &: a_{i+1}^2 W a_{j+1}^2 \overline{W_1} t^{-1} W_2 t \ (j>i),\ (ta_ia_jt^{-1}W\overline{W_1}t^{-1}W_2t, a_{i+1}^2Wa_{j+1}t^{-1}W_2ta_{j+1}\overline{W_1}),\\
(12)\cap(8) &: a_{i+1}^2 W a_{j+1}^{\frac{n+1}{2}} \ (j>i),\ (ta_ia_jt^{-1}Wa_{j+1}^{\frac{n-3}{2}}, a_{i+1}^2Wt^{-1}a_{j+2}t),\\
(12)\cap(10) &: a_{i+1}^2 W a_{j+1}^2 W_1 t W_2 a_j^{\frac{n-1}{2}} \ (j>i),\ (ta_ia_jt^{-1}WW_1tW_2a_j^{\frac{n-1}{2}}, a_{i+1}^2Wa_{j+1}W_1tW_2),\\
(12)\cap(11) &: a_{i+1}^2 W a_{j+1}^{\frac{n-1}{2}} W_1 t^{-1} W_2 a_{j+2} \ (j>i),\ (ta_ia_jt^{-1}Wa_{j+1}^{\frac{n-5}{2}}W_1t^{-1}W_2a_{j+2}, a_{i+1}^2WW_1t^{-1}W_2),\\
(12)\cap(12) &: a_{i+1}^2 W_1 a_{j+1}^2 W_2 a_{k+1}^2 \ (k>j>i),\ (ta_ia_jt^{-1}W_1W_2a_{k+1}^2, a_{i+1}^2W_1ta_ja_kt^{-1}W_2),\\
(12)\cap(13) &: a_{i+1}^2 W a_{j+1}^2 W_1 t W_2 a_j^{\frac{n-5}{2}} t W_3 a_{j-1} \ (j>i),\\
&\quad (ta_ia_jt^{-1}WW_1tW_2a_j^{\frac{n-5}{2}}tW_3a_{j-1},\ a_{i+1}^2Wa_{j+1}W_1tW_2tW_3),\\
(12)\cap(14) &: a_{i+1}^2 W a_{j+1}^2 W_1 t^{-1} a_{j+2}^{\frac{n-5}{2}} W_2 t^{-1} W_3 a_{j+3} \ (j>i),\\
&\quad (ta_ia_jt^{-1}WW_1t^{-1}a_{j+2}^{\frac{n-5}{2}}W_2t^{-1}W_3a_{j+3}, a_{i+1}^2Wa_{j+1}W_1t^{-1}W_2t^{-1}W_3),\\
(13)\cap(1) &: a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3a_{i-1}^n,\ (W_1tW_2tW_3a_{i-1}^{n-1}, a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3),\\
(13)\cap(2) &: a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3a_{i-1}a_j \ (i-1>j),\ (W_1tW_2tW_3a_j, a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3a_ja_{i-1}),\\
(13)\cap(5) &: a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3a_{i-1}^2Wt,\ (W_1tW_2tW_3a_{i-1}Wt, a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3Wta_{i-2}),\\
(13)\cap(7) &: a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3a_{i-1}\overline{W_4}tW_5t^{-1},\\
&\quad (W_1tW_2tW_3\overline{W_4}tW_5t^{-1},\ a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3tW_5t^{-1}a_{i-1}\overline{W_4}),\\
(13)\cap(7) &: a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3a_{i-1}\overline{W_4}t^{-1}W_5t,\\
&\quad (W_1tW_2tW_3\overline{W_4}t^{-1}W_5t,\ a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3t^{-1}W_5ta_{i-1}\overline{W_4}),\\
(13)\cap(8) &: a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3a_{i-1}^{\frac{n+1}{2}},\ (W_1tW_2tW_3a_{i-1}^{\frac{n-1}{2}}, a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3t^{-1}a_it),\\
(13)\cap(10) &: a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3a_{i-1}W_4tW_5a_{i-2}^{\frac{n-1}{2}},\\
&\quad (W_1tW_2tW_3W_4tW_5a_{i-2}^{\frac{n-1}{2}}, a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3W_4tW_5),\\
(13)\cap(11) &: a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3a_{i-1}^{\frac{n-1}{2}}W_4t^{-1}W_5a_i,\\
&\quad (W_1tW_2tW_3a_{i-1}^{\frac{n-3}{2}}W_4t^{-1}W_5a_i,\ a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3W_4t^{-1}W_5),\\
(13)\cap(12) &: a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3a_{i-1}^2Wa_{j+1}^2 \ (j>i),\\
&\quad (W_1tW_2tW_3a_{i-1}Wa_{j+1}^2,\ a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3ta_{i-2}a_jt^{-1}W),
\end{aligned}
$$

and

$$(13)\cap(13)\ : a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3a_{i-1}W_4tW_5a_{i-2}^{\frac{n-5}{2}}tW_6a_{i-3},$$
$$(W_1tW_2tW_3W_4tW_5a_{i-2}^{\frac{n-5}{2}}tW_6a_{i-3},\ a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3W_4tW_5tW_6),$$
$$(13)\cap(14)\ : a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3a_{i-1}W_4t^{-1}a_i^{\frac{n-5}{2}}W_5t^{-1}W_6a_{i+1},$$
$$(W_1tW_2tW_3W_4t^{-1}a_i^{\frac{n-5}{2}}W_5t^{-1}W_6a_{i+1},\ a_{i+1}W_1tW_2a_i^{\frac{n-5}{2}}tW_3W_4t^{-1}W_5t^{-1}W_6),$$
$$(14)\cap(1)\ : a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3a_{i+2}^{n},\ (W_1t^{-1}W_2t^{-1}W_3a_{i+2}^{n-1}, a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3),$$
$$(14)\cap(2)\ : a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3a_{i+2}a_j\ (i+2>j),\ (W_1t^{-1}W_2t^{-1}W_3a_j, a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3a_ja_{i+2}),$$
$$(14)\cap(5)\ : a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3a_{i+2}^2Wt,\ (W_1t^{-1}W_2t^{-1}W_3a_{i+2}Wt, a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3Wta_{i+1}),$$
$$(14)\cap(7)\ : a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3a_{i+2}\overline{W_4}t^{\epsilon}W_5t^{-\epsilon},$$
$$(W_1t^{-1}W_2t^{-1}W_3\overline{W_4}t^{\epsilon}W_5t^{-\epsilon},\ a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3t^{\epsilon}W_5t^{-\epsilon}a_{i+2}\overline{W_4}),$$
$$(14)\cap(8)\ : a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3a_{i+2}^{\frac{n+1}{2}},\ (W_1t^{-1}W_2t^{-1}W_3a_{i+2}^{\frac{n-1}{2}},\ a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3t^{-1}a_{i+3}t),$$
$$(14)\cap(10)\ : a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3a_{i+2}W_4tW_5a_{i+1}^{\frac{n-1}{2}},$$
$$(W_1t^{-1}W_2t^{-1}W_3W_4tW_5a_{i+1}^{\frac{n-1}{2}},\ a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3W_4tW_5),$$
$$(14)\cap(11)\ : a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3a_{i+2}^{\frac{n-1}{2}}W_4t^{-1}W_5a_{i+3},$$
$$(W_1t^{-1}W_2t^{-1}W_3a_{i+2}^{\frac{n-3}{2}}W_4t^{-1}W_5a_{i+3},\ a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3W_4t^{-1}W_5),$$
$$(14)\cap(12)\ : a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3a_{i+2}^2Wa_{j+1}^2\ (j>i),$$
$$(W_1t^{-1}W_2t^{-1}W_3a_{i+2}Wa_{j+1}^2,\ a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3ta_{i+1}a_jt^{-1}W),$$
$$(14)\cap(13)\ : a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3a_{i+2}W_4tW_5a_{i+1}^{\frac{n-5}{2}}tW_6a_i,$$
$$(W_1t^{-1}W_2t^{-1}W_3W_4tW_5a_{i+1}^{\frac{n-5}{2}}tW_6a_i,\ a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3W_4tW_5tW_6),$$
$$(14)\cap(14)\ : a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3a_{i+2}W_4t^{-1}a_{i+3}^{\frac{n-5}{2}}W_5t^{-1}W_6a_{i+4},$$
$$(W_1t^{-1}W_2t^{-1}W_3W_4t^{-1}a_{i+3}^{\frac{n-5}{2}}W_5t^{-1}W_6a_{i+4},\ a_iW_1t^{-1}a_{i+1}^{\frac{n-5}{2}}W_2t^{-1}W_3W_4t^{-1}W_5t^{-1}W_6).$$

In fact, all these above critical pairs are resolved by reduction steps. We show some of them as follows.

$$(5)\cap(9)\ : a_{i+1}^2Wta_jt^{-1},\ (Wta_ia_jt^{-1}, a_{i+1}^2Wa_{j+1}^2),$$

$$a_{i+1}^2Wta_jt^{-1} \longrightarrow \begin{cases} \bullet\ Wta_ia_jt^{-1} \to ta_ia_jt^{-1}W \\ \bullet\ a_{i+1}^2Wa_{j+1}^2 \to ta_ia_jt^{-1}W. \end{cases}$$

$(11) \cap (12)\ : a_i^{\frac{n-1}{2}} W_1 t^{-1} W_2 a_{i+1}^2 W a_{j+1}^2\ (j > i),\quad (W_1 t^{-1} W_2 a_{i+1} W a_{j+1}^2,\ a_i^{\frac{n-1}{2}} W_1 t^{-1} W_2 t a_i a_j t^{-1} W),$

$$a_i^{\frac{n-1}{2}} W_1 t^{-1} W_2 a_{i+1}^2 W a_{j+1}^2 \longrightarrow \begin{cases} \bullet\ W_1 \underbrace{t^{-1} W_2 a_{i+1} W a_{j+1}^2} \to W_1 a_j t^{-1} W_2 a_{i+1} W \\ \bullet\ \underbrace{a_i^{\frac{n-1}{2}} W_1 t^{-1} W_2 t}\, a_i a_j t^{-1} W \to t^{-1} W_2 t \underbrace{a_i^{\frac{n+1}{2}}} W_1 a_j t^{-1} W \\ \qquad \to t^{-1} W_2 \underbrace{t t^{-1}} a_{i+1} t W_1 a_j t^{-1} W \\ \qquad \to t^{-1} \underbrace{W_2 a_{i+1} t W_1 a_j t^{-1}} W \\ \qquad \to W_1 a_j t^{-1} W_2 a_{i+1} W. \end{cases}$$

After all above processes, we see that all critical pairs can be resolved. Thus, the rewriting system is complete. □

Theorem 3.4 *A complete rewriting system for $m \geq 3$ given in presentation (3.1) consists of the following relations:*

$$\begin{aligned} &(1)\quad a_i^n \to 1,\quad (2)\ a_i a_j \to a_j a_i\ (i > j),\quad (3)\ t t^{-1} \to 1,\quad (4)\ t^{-1} t \to 1,\quad (5)\ a_{i+1}^m \to t a_i t^{-1}\ ((m,n) = 1),\\ &(6)\quad a_j t^{-1} a_i^k W t \to t^{-1} a_i^k W t a_j,\qquad (7)\ a_i t W a_j^k t^{-1} \to t W a_j^k t^{-1} a_i\quad ,\\ &(8)\quad a_r t^{-1} a_i^k W_1 t W_2 a_j \to a_j a_r t^{-1} a_i^k W_1 t W_2\quad (r > j), \end{aligned}$$

where $0 \leq k < n$ ($k \in \mathbb{Z}$), W_1, and W_2 are reduced words containing a_i ($i \in \mathbb{Z}$), and W is reduced word generated by a_i and t.

Proof Noetherian property of the rewriting system can be seen easily. Now, we need to show that the confluent property holds. To do that we have the following overlapping words and corresponding critical pairs, respectively.

$(1) \cap (1)\ : a_i^{n+1},\ (a_i, a_i),\qquad (1) \cap (2)\ : a_i^n a_j\ (i > j),\ (a_j, a_i^{n-1} a_j a_i),$

$(1) \cap (6)\ : a_j^n t^{-1} a_i^k W t,\ (t^{-1} a_i^k W t, a_j^{n-1} t^{-1} a_i^k W t a_j),$

$(1) \cap (7)\ : a_i^n t W a_j^k t^{-1},\ (t W a_j^k t^{-1}, a_i^{n-1} t W a_j^k t^{-1} a_i),$

$(1) \cap (8)\ : a_r^n t^{-1} a_i^k W_1 t W_2 a_j\ (r > j),\ (t^{-1} a_i^k W_1 t W_2 a_j, a_r^{n-1} a_j a_r t^{-1} a_i^k W_1 t W_2),$

$(2) \cap (1)\ : a_i a_j^n\ (i > j),\ (a_j a_i a_j^{n-1}, a_i),\qquad (2) \cap (2)\ : a_i a_j a_k\ (i > j > k),\ (a_j a_i a_k, a_i a_k a_j),$

$(2) \cap (5)\ : a_i a_{j+1}^m\ (i > j+1),\ (a_{j+1} a_i a_{j+1}^{m-1}, a_i t a_j t^{-1}),$

$(2) \cap (6)\ : a_r a_j t^{-1} a_i^k W t\ (r > j),\ (a_j a_r t^{-1} a_i^k W t, a_r t^{-1} a_i^k W t a_j),$

$(2) \cap (7)\ : a_r a_i t W a_j^k t^{-1}\ (r > i),\ (a_i a_r t W a_j^k t^{-1}, a_r t W a_j^k t^{-1} a_i),$

$(2) \cap (8)\ : a_s a_r t^{-1} a_i^k W_1 t W_2 a_j\ (s > r > j), (a_r a_s t^{-1} a_i^k W_1 t W_2 a_j, a_s a_j a_r t^{-1} a_i^k W_1 t W_2),$

$(3) \cap (4)\ : t t^{-1} t,\ (t, t),\qquad (4) \cap (3)\ : t^{-1} t t^{-1},\ (t^{-1}, t^{-1}),$

and

$$(5)\cup(1)\ : a_i^n,\ (ta_{i-1}t^{-1}a_i^{n-m},1), \qquad (5)\cap(2)\ : a_i^m a_j\ (i>j),\ (ta_{i-1}t^{-1}a_j, a_i^{m-1}a_ja_i),$$

$$(5)\cap(5)\ : a_i^{m+1},\ (ta_{i-1}t^{-1}a_i, a_ita_{i-1}t^{-1}),$$

$$(5)\cap(6)\ : a_j^m t^{-1}a_i^kWt,\ (ta_{j-1}t^{-1}t^{-1}a_i^kWt, a_j^{m-1}t^{-1}a_i^kWta_j),$$

$$(5)\cap(7)\ : a_i^m tWa_j^kt^{-1},\ (ta_{i-1}t^{-1}tWa_j^kt^{-1}, a_i^{m-1}tWa_j^kt^{-1}a_i),$$

$$(5)\cap(8)\ : a_r^m t^{-1}a_i^kW_1tW_2a_j\ (r>j),\ (ta_{r-1}t^{-1}t^{-1}a_i^kW_1tW_2a_j, a_r^{m-1}a_ja_rt^{-1}a_i^kW_1tW_2),$$

$$(6)\cap(3)\ : a_jt^{-1}a_i^kWtt^{-1},\ (a_jt^{-1}a_i^kW, t^{-1}a_i^kWta_jt^{-1}),$$

$$(6)\cap(7)\ : a_jt^{-1}a_i^{k_1}tWa_r^{k_2}t^{-1}, (t^{-1}a_i^{k_1}tWa_ja_r^{k_2}t^{-1}, a_jt^{-1}tWa_r^{k_2}t^{-1}Wa_i^{k_1}),$$

$$(6)\cup(8)\ : a_rt^{-1}a_i^kW_1tW_2a_j\ (r>j),\ (t^{-1}a_i^kW_1ta_rW_2a_j, a_ja_rt^{-1}a_i^kW_1tW_2),$$

$$(7)\cap(4)\ : a_itWa_j^kt^{-1}t,\ (tWa_j^kt^{-1}ta_i, a_itWa_j^k),$$

$$(7)\cap(6)\ : a_itW_1a_j^{k_1}t^{-1}a_r^{k_2}W_2t, (tW_1a_j^{k_1}t^{-1}a_ia_r^{k_2}W_2t, a_itW_1t^{-1}a_r^{k_2}W_2ta_j^{k_1}),$$

$$(7)\cap(8)\ : a_itWa_j^{k_1}t^{-1}a_r^{k_2}W_1tW_2a_s\ (j>s,\ r>i), \quad (tWa_j^{k_1}t^{-1}a_ia_r^{k_2}W_1tW_2a_s, a_itWa_sa_j^{k_1}t^{-1}a_r^{k_2}W_1tW_2),$$

$$(8)\cap(1)\ : a_rt^{-1}a_i^kW_1tW_2a_j^n\ (r>j), (a_ja_rt^{-1}a_i^kW_1tW_2a_j^{n-1}, a_rt^{-1}a_i^kW_1tW_2),$$

$$(8)\cap(2)\ : a_rt^{-1}a_i^kW_1tW_2a_ja_s\ (r>j>s),\ (a_ja_rt^{-1}a_i^kW_1tW_2a_s, a_rt^{-1}a_i^kW_1tW_2a_sa_j),$$

$$(8)\cap(5)\ : a_rt^{-1}a_i^kW_1tW_2a_j^m\ (r>j), (a_ja_rt^{-1}a_i^kW_1tW_2a_j^{m-1}, a_rt^{-1}a_i^kW_1tW_2ta_{j-1}t^{-1}),$$

$$(8)\cap(6)\ : a_rt^{-1}a_i^{k_1}W_1tW_2a_jt^{-1}a_s^{k_2}W_3t\ (r>j),\ (a_ja_rt^{-1}a_i^{k_1}W_1tW_2t^{-1}a_s^{k_2}W_3t, a_rt^{-1}a_i^{k_1}W_1tW_2t^{-1}a_s^{k_2}W_3ta_j),$$

$$(8)\cap(7)\ : a_rt^{-1}a_i^{k_1}W_1tW_2a_jtW_3a_s^{k_2}t^{-1}\ (r>j),\ (a_jart^{-1}a_i^{k_1}W_1tW_2tW_3a_s^{k_2}t^{-1}, a_rt^{-1}a_i^{k_1}W_1tW_2tW_3a_s^{k_2}t^{-1}a_j),$$

$$(8)\cap(8)\ : a_rt^{-1}a_i^{k_1}W_1tW_2a_jt^{-1}a_l^{k_2}W_3tW_4a_s\ (r>j>s),$$

$$(a_ja_rt^{-1}a_i^{k_1}W_1tW_2t^{-1}a_l^{k_2}W_3tW_4a_s, a_rt^{-1}a_i^{k_1}W_1tW_2a_sa_jt^{-1}a_l^{k_2}W_3tW_4).$$

In fact, all these above critical pairs are resolved by reduction steps. We show some of them.

$$(5)\cap(2)\ : a_i^ma_j\ (i>j),\ (ta_{i-1}t^{-1}a_j, a_i^{m-1}a_ja_i),$$

$$a_i^ma_j \longrightarrow \begin{cases} ta_{i-1}t^{-1}a_j \\ a_i^{m-1}a_ja_i \to a_ja_i^m \to a_jta_{i-1}t^{-1} \to ta_{i-1}t^{-1}a_j. \end{cases}$$

$$(7)\cap(6)\ : a_itW_1a_j^{k_1}t^{-1}a_r^{k_2}W_2t\ (r>i), (tW_1a_j^{k_1}t^{-1}a_ia_r^{k_2}W_2t, a_itW_1t^{-1}a_r^{k_2}W_2ta_j^{k_1}),$$

$$a_itW_1a_j^{k_1}t^{-1}a_r^{k_2}W_2t \longrightarrow \begin{cases} tW_1a_j^{k_1}t^{-1}a_ia_r^{k_2}W_2t \to tW_1t^{-1}a_ia_r^{k_2}W_2ta_j^{k_1} \\ a_itW_1t^{-1}a_r^{k_2}W_2ta_j^{k_1} \to tW_1t^{-1}a_ia_r^{k_2}W_2ta_j^{k_1}. \end{cases}$$

After all above processes, we see that all critical pairs can be resolved. Thus, the rewriting system is complete. □

By considering the right sides of the relations given in Theorems 3.3 and 3.4, we have the following result for both theorems:

Corollary 3.5 *The normal form of a word* u *, representing an element of* $N\#_{\varphi}^{f}\mathbb{Z}$ *, is* $t^{k_1}W_1t^{k_2}W_2t^{k_3}W_3\cdots t^{k_q}W_q$ *, where* $k_i \in \mathbb{Z}\,(1 \leq i \leq q)$ *and* $W_i := a_{i_1}a_{i_2}\cdots a_{i_m}(1 \leq i \leq q,\ i_1 < i_2 < \cdots < i_m)$ *,* $W_it^{k_{i+1}}W_{i+1}t^{k_{i+2}}\ (1 \leq i \leq q-2)$ *and* $W_it^{k_{i+1}}W_{i+1}t^{k_{i+2}}W_{i+2}\ (1 \leq i \leq q-2)$ *are irreducible words in* $N\#_{\varphi}^{f}\mathbb{Z}$ *.*

By Corollaries 3.2 and 3.5, we have the following result.

Corollary 3.6 *The word problem for the group* $N\#_{\varphi}^{f}\mathbb{Z}$ *is solvable.*

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