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Peiffer pairings in multisimplicial groups and crossed n-cubes and applications for bisimplicial groups

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Abstract: We explore the Peiffer pairings within the Moore complex of multisimplicial groups, and as an application, we give a detailed construction of a crossed n- cube from an n-simplicial group in terms of these pairings. We also give explicit calculations of Peiffer pairings in the Moore bicomplex of a bisimplicial group to see the role of these pairings in the relationship between bisimplicial groups and crossed squares.

Key words: Bisimplicial groups, crossed modules, crossed squares, crossed complexes

1. Introduction

As algebraic models for homotopy 2-types, the notion of crossed module was introduced by Whitehead in [21]. Crossed squares as introduced by Guin-Walery and Loday [12], model connected 3-types. Crossed n-cubes (cf. [11]) model connected $(n+1)$-types, and they will be most interest to us in this paper. The observation that simplicial groups whose Moore complex is of length 1 are equivalent to Whitehead's crossed modules is well known. Conduché in [9] also proved that a 1-truncated bisimplicial group gives a crossed square.

The original motivation of Peiffer elements comes from the observation of Brown and Loday [5], about the normalisation of simplical groups. Let $\mathbf{G} = \{G_n\}$ be a simplicial group and let $\mathbf{N} = \{N_n\}$ be its Moore complex. Brown and Loday observed that if the group of 2-simplices G_2 is generated by degenerate elements, then $\partial_2(N_2) = [\ker d_0, \ker d_1]$. In this construction, N_0 acts on N_1 via congugation using so, namely ${}^{x}y = s_0xys_0x^{-1}$ for $x \in N_0$ and $y \in N_1$. One gets easily that $\partial_1 : N_1 \to N_0$ is a precrossed module. In such a context the element ${}^{\partial_1 x}yxy^{-1}x^{-1}$ is called the Peiffer commutator of x, y or more briefly a Peiffer element. An easy argument shows that the subgroup $[Kerd_0, Kerd_1]$ of N_1 is generated by elements of this form and it is thus entirely the Peiffer subgroup of N_1. If the length of Moore complex of the simplicial group $\mathbf{G}$ is 2, then $\partial_2(N_2) = [\ker d_0, \ker d_1] = \{1\}$ and then ${}^{\partial_1 x}yxy^{-1}x^{-1} = 1$ for all $x, y \in N_1$, and this is equivalent to $\partial_1 : N_1 \to N_0$ being a crossed module. In [18], Mutlu and Porter developed a variant of the Carrasco and Cegarra [6] pairing operators, that they called Peiffer pairings by denoting $F_{\alpha,\beta}$ and showed that these pairings give product of operators.

Mutlu and Porter [16, 17], using the ideas based on the works of Conduché [8], and Carrasco and Cegarra, [6], generalized Peiffer elements to higher dimensions by giving systematic maps of generating them and have

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investigated the images of the functions $F_{\alpha,\beta}$ in the Moore complex of a simplicial group. For further accounts of these functions in simplicial Lie algebras and simplicial commutative algebras see [1] and [3, 4], respectively.

The main aim of this paper is to define the functions $F_{\alpha,\beta}$ introduced by Mutlu and Porter in [16] for the Moore n-complex of an n-simplicial group, and as an application of these maps, to give the relationship between crossed n-cubes and n-simplicial groups. Porter in [19] has proven the relationship between n-types of simplicial groups and crossed n-cubes. Our result is closely related to the results of Porter. We prove in Theorem 3.3 that a 1-truncated n-simplicial group gives a crossed n-cube. In this relationship, it has been seen that the h-maps of a crossed n-cube are obtained by the images of the functions $F_{\alpha,\beta}$.

Subsequently, another interest of this paper is to give detailed calculations of the functions $F_{\alpha,\beta}$ in dimension 2, that is, in the Moore bicomplex of a bisimplicial group. Particularly, for dimension 2, in Proposition 4.4 of this work, we obtain the Conduché's result given in Proposition 3.2 of [9] between crossed squares and bisimplicial groups in terms of $F_{\alpha,\beta}$ functions. That is, we obtain that the h-map of a crossed square can be given by an image of the functions $F_{\alpha,\beta}$ in the Moore complex of a bisimplicial group. Furthermore, Carrasco and Cegarra gave the relation between simplicial groups and crossed complexes as a non-Abelian version of Dold–Kan theorem. Using the functions $F_{\alpha,\beta}$ in the Moore bicomplex of a bisimplicial group, in Theorem 4.15, we obtain a double crossed complex from a Moore bicomplex of a bisimplical group as a 2-dimensional version of Carrasco and Cegarra's result.

2. Simplicial groups

We refer the reader to Loday's and May's books (cf. [14, 15]), and Artin's and Mazur's [2] article and Duskin [10] for the basic properties of simplicial groups, bisimplicial groups, etc.

Let Δ be the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$. A simplicial group is a functor from the opposite category Δ^{op} to the category of groups **Grp**. That is, a simplicial group $\mathbf{G}$ consists of a family of groups G_n together with homomorphisms $d_i^n : G_n \to G_{n-1}$, $0 \leqslant i \leqslant n$, $(n \neq 0)$ and $s_j^n : G_n \to G_{n+1}$, $0 \leqslant j \leqslant n$, called face and degeneracy maps, satisfying the usual simplicial identities given in [15]. We will denote the category of simplicial groups by **SimpGrp**.

A chain complex of groups is a sequence of any length of groups and homomorphisms of groups;

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \longrightarrow \cdots$$

in which each composite $\partial_{n-1} \circ \partial_n$ is trivial. The chain complex is normal if each image $\partial_n(C_n)$ is a normal subgroup of C_{n-1}.

Given a simplicial group $\mathbf{G}$, the Moore complex $(\mathbf{NG}, \partial)$ of $\mathbf{G}$ is defined by

$$(NG)_n = \ker d_0^n \cap \ker d_1^n \cap \cdots \cap \ker d_{n-1}^n$$

with boundaries $\partial_n : NG_n \to NG_{n-1}$ induced from d_n^n by restriction. If $\mathbf{G}$ is a simplicial group then its Moore complex $\mathbf{NG}$ is a normal chain complex of groups.

We say that the Moore complex $\mathbf{NG}$ of a simplicial group $\mathbf{G}$ is of *length k* if $NG_n = \{1\}$ for all $n \geqslant k+1$.

2.1. Poset of surjective maps

In this section, we give the definition of the ordered set $S(n)$ from Mutlu and Porter [16, 18]. In the following we give from [16] hypercrossed complex pairings which are adapted by Carrasco and Cegarra's study (cf. [6]).

For the ordered set $[n] = \{0 < 1 < \ldots < n\}$, let $\alpha_i^n : [n+1] \to [n]$ be the increasing surjective map given by

$$\alpha_i^n(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

Let $S(n, n-l)$ be the set of all monotone increasing surjective maps form $[n]$ to $[n-l]$. This can be generated from the various α_i^n by composition. The composition of these generating maps satisfies the rule $\alpha_j \alpha_i = \alpha_{i-1}\alpha_j$ with $j < i$. This implies that every element $\alpha \in S(n, n-l)$ has a unique expression as $\alpha = \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_l}$ with $0 \leq i_1 < i_2 < \ldots < i_l \leq n$, where the indices i_k are the elements of $[n]$ at which $\{i_1, \ldots, i_l\} = \{i : \alpha(i) = \alpha(i+1)\}$. Thus, Mutlu and Porter (cf. [16]) have identified $S(n, n-l)$ with the set $\{(i_l, \ldots, i_1) : 0 \leq i_1 < i_2 < \ldots < i_l \leq n-1\}$. In particular the single element of $S(n,n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( ) denoted by $\emptyset_n$. Similarly the only element of $S(n,0)$ is $(n-1, n-2, \ldots, 0)$. For all $n \geq 0$, let

$$S(n) = \bigcup_{0 \leq l \leq n} S(n, n-l).$$

It has been defined in [16] that $\alpha = (i_l, \ldots, i_1) > \beta = (j_m, \ldots, j_1)$ in $S(n)$ if $i_1 = j_1, \ldots i_k = j_k$ but $i_{k+1} < j_{k+1}, (k \geq 0)$ or if $i_1 = j_1, \ldots i_m = j_m$ and $l > m$.

This makes $S(n)$ an ordered set. For instance, the orders of $S(2)$ and $S(3)$ and $S(4)$ are respectively:

$$\begin{aligned} S(2) &= \{\emptyset_2 < (1) < (0) < (1,0)\}, \\ S(3) &= \{\emptyset_3 < (2) < (1) < (2,1) < (0) < (2,0) < (1,0) < (2,1,0)\}, \\ S(4) &= \{\emptyset_4 < (3) < (2) < (3,2) < (1) < (3,1) < (2,1) < (3,2,1) < (0) \\ &\quad < (3,0) < (2,0) < (3,2,0) < (1,0) < (3,1,0) < (2,1,0) < (3,2,1,0)\}. \end{aligned}$$

If $\alpha, \beta \in S(n)$, Mutlu and Porter have defined $\alpha \cap \beta$ to be the set of indices which belong to both α and β. If $\alpha = (i_l, \ldots, i_1)$, then, it is called that α has length l and written $\#\alpha = l$.

2.2. Peiffer pairings in simplicial groups

From [16], we give the definition of the set $P(n)$ consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset$ and $\beta < \alpha$, with respect to lexicographic ordering in $S(n)$ where $\alpha = (i_r, \ldots, i_1), \beta = (j_s, \ldots, j_1) \in S(n)$. The pairings

$$\{F_{\alpha,\beta} : NG_{n-\#\alpha} \times NG_{n-\#\beta} \to NG_n : (\alpha, \beta) \in P(n), n \geqslant 0\}$$

are given as composites by the diagram

$$\begin{array}{ccc} NG_{n-\#\alpha} \times NG_{n-\#\beta} & \xrightarrow{F_{\alpha,\beta}} & NG_n \\ {\scriptstyle (s_\alpha, s_\beta)}\downarrow & & \uparrow{\scriptstyle p} \\ G_n \times G_n & \xrightarrow[{[-,-]}]{} & G_n \end{array}$$

where $s_\alpha = s_{i_r} \dots s_{i_1} : NG_{n-\#\alpha} \to G_n$, $s_\beta = s_{j_s} \dots s_{j_1} : NG_{n-\#\beta} \to G_n$, $p : G_n \to NG_n$ is defined by composite projections $p(x) = p_{n-1} \dots p_0(x)$, where $p_j(z) = zs_jd_j(z)^{-1}$ with $j = 0, 1, \dots, n-1$. The map $[-,-] : G_n \times G_n \to G_n$ is given by the commutator map. Thus,

$$F_{\alpha,\beta}(x_\alpha, y_\beta) = p[s_\alpha(x_\alpha), s_\beta(x_\beta)].$$

Definition 2.1 *([16]) Let N_n or more exactly N_n^G be the normal subgroup of G_n generated by elements of the form $F_{\alpha,\beta}(x_\alpha, y_\beta)$ where $x_\alpha \in NG_{n-\#\alpha}$ and $y_\beta \in NG_{n-\#\beta}$.*

This normal subgroup N_n^G depends functorially on G, but it is usually abbreviated N_n^G to N_n, when no change of group is involved. Mutlu and Porter in [16] illustrate this normal subgroup for $n = 2, 3, 4$. As a corollary they obtained the following equality

$$\partial_n(NG_n \cap D_n) = \partial_n(N_n^G \cap D_n)$$

where D_n is subgroup generated by the degenerate elements in dimension n.

Proposition 2.2 ([16]) Let $\mathbf{G}$ be a simplicial group. Then for $n \geqslant 2$ and $I, J \subseteq [n-1]$ with $I \cup J = [n-1]$,

$$[\bigcap_{i \in I} \ker d_i \ , \bigcap_{j \in J} \ker d_j] \subseteq \partial_n(NG_n \cap D_n).$$

Castiglioni and Ladra [7] gave a general proof over operads for these inclusions given by Mutlu and Porter in [16].

3. Multisimplicial groups

An n-simplicial group $\mathbf{G}_{\bullet_1 \bullet_2 \cdots \bullet_n}$ is a functor from the product category

$$\Delta^{op} \times \Delta^{op} \times \cdots \times \Delta^{op} = (\Delta^{op})^n$$

to the category of groups **Grp**, with the face and degeneracy maps denoted by respectively

$$d_{i_j}^j : G_{k_1,\dots,k_j,\dots,k_n} \longrightarrow G_{k_1,\dots,k_j-1,\dots,k_n}, \quad (\ 0 \leqslant i_j \leqslant k_j \ , \ 1 \leqslant j \leqslant n,)$$

and

$$s_{i_j}^j : G_{k_1,\dots,k_j,\dots,k_n} \longrightarrow G_{k_1,\dots,k_j+1,\dots,k_n}, \quad (\ 0 \leqslant i_j < k_j \ , \ 1 \leqslant j \leqslant n,),$$

where each j indicates the directions of n-simplicial group. The Moore n-complex (cf. [9]) of an n-simplicial group can be given by

$$NG_{k_1,k_2,\dots,k_n} = \bigcap_{(i_1,i_2,\dots,i_n)=(0,0,\dots,0)}^{(k_1-1,k_2-1,\dots,k_n-1)} \mathrm{Ker} d_{i_1}^1 \cap \mathrm{Ker} d_{i_2}^2 \cap \cdots \cap \mathrm{Ker} d_{i_n}^n$$

with the boundary homomorphism

$$\partial_{i_j}^j : NG_{k_1,\dots,k_j,\dots,k_n} \longrightarrow NG_{k_1,\dots,k_j-1,\dots,k_n}$$

induced by $d_{i_j}^j$. We denote the category of n-simplicial groups by $\mathbf{SimpGrp}^n$.

3.1. Peiffer pairings in n-simplicial groups

In this section, we define the functions $F_{\alpha,\beta}$ given in [16] for multisimplicial groups.

For $n,q \in \mathbb{N}$ with $q \leqslant n$ and for $\alpha \in S(n,q)$, the target of α is called $b(\alpha) : q = b(\alpha)$. Recall that the set $S(n)$ is partially ordered by the following relation $\alpha \leqslant \beta$ if, for $i \in [n]$, one has $\alpha(i) \geqslant \beta(i)$ where $[b(\alpha)]$ and $[b(\beta)]$ are considered as subsets of $\mathbb{N}$.

Given $n \neq 0, n \in \mathbb{N}$ and $\mathbf{n} = (k_1, k_2, \ldots, k_n) \in \mathbb{N}\times\mathbb{N}\times\ldots\times\mathbb{N} = \mathbb{N}^n$, let $S(\mathbf{n}) = S(k_1)\times S(k_2)\times\ldots\times S(k_n)$ with the product (partial) order.

Let $\alpha, \beta \in S(\mathbf{n})$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\,;\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ where $\alpha_i \in S(k_i)$ and $\beta_j \in S(k_j)$, $1 \leqslant i, j \leqslant n$.

The n-dimensional case of the functions $F_{\alpha,\beta}$ can be given as follows. The pairings that we will need

$$\{F_{\alpha,\beta} : NG_{\mathbf{n}-\#\alpha} \times NG_{\mathbf{n}-\#\beta} \longrightarrow NG_{\mathbf{n}} \;\; ; \; \alpha, \beta \in S(\mathbf{n})\}$$

are given as composites by the diagram

$$\begin{array}{ccc} NG_{k_1-\#\alpha_1,k_2-\#\alpha_2,\ldots,k_n-\#\alpha_n} \times NG_{k_1-\#\beta_1,k_2-\#\beta_2,\ldots,k_n-\#\beta_n} & \xrightarrow{F_{\alpha,\beta}} & NG_{k_1,k_2,\ldots,k_n} \\ \downarrow{\scriptstyle (s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_n}, s_{\beta_1}s_{\beta_2}\ldots s_{\beta_n})} & & \uparrow{\scriptstyle p} \\ G_{k_1,k_2,\ldots,k_n} \times G_{k_1,k_2,\ldots,k_n} & \xrightarrow[{[-,-]}]{} & G_{k_1,k_2,\ldots,k_n} \end{array}$$

where $s_\alpha : s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_n}$, for $1 \leqslant i \leqslant n$; $s_{\alpha_i} : s_{i_r}\cdots s_{i_1}$ for $\alpha_i = (i_r, \ldots, i_1) \in S(k_i)$ and similarly s_{β_i}, and p is defined by the composite projection

$$p = (p_{k_1-1}\ldots p_0)\,(p_{k_2-1}\ldots p_0)\ldots(p_{k_n-1}\ldots p_0)$$

where $p_j(x) = x^{-1}s_jd_j(x)$ in each simplicial directions, for any j, and $[-,-]$ is given by the commutator map.

3.2. Crossed n-cubes and n-simplicial groups

Crossed n-cubes were defined by Ellis and Steiner [11] for homotopy connected $(n+1)$-types. The following definition is equivalent to that given in [11]. In this section by use of the functions $F_{\alpha,\beta}$ for n-simplicial groups, we will construct a crossed n-cube structure from as n-simplicial groups.

Definition 3.1 *We denote by $\langle n\rangle$ the set $\{1,2,...,n\}$. A crossed n-cube, $\mathbf{M}$, is a family of groups, $\{M_A : A \subseteq \langle n\rangle\}$, together with homomorphisms, $\mu_i : M_A \to M_{A\setminus\{i\}}$, for $i \in \langle n\rangle$, $A \subseteq \langle n\rangle$, and functions, $h : M_A \times M_B \to M_{A\cup B}$, for all $A, B \subseteq \langle n\rangle$, such that if ${}^ab$ denotes $h(a,b)b$ for $a \in M_A$ and $b \in M_B$ with $A \subseteq B$, then for a, $a' \in M_A$, b, $b' \in M_B, c \in M_C$ and $i,j \in \langle n\rangle$, the following axioms hold:*

1. $\mu_i a = a$ if $i \notin A$
2. $\mu_i\mu_j a = \mu_j\mu_i a$
3. $\mu_i h(a,b) = h(\mu_i a, \mu_i b)$

4. $h(a,b) = h(\mu_i a, b) = h(a, \mu_i b)$ if $i \in A \cap B$
5. $h(a,a') = [a,a']$
6. $h(a,b) = h(b,a)^{-1}$
7. $h(a,b) = 1$ if $a = 1$ or $b = 1$
8. $h(aa',b) =^a h(a',b)h(a,b)$
9. $h(a,bb') = h(a,b)^b h(a,b')$
10. ${}^a h(h(a^{-1},b),c)^c h(h(c^{-1},a),b)^b h(h(b^{-1},c),a) = 1$
11. ${}^a h(b,c) = h({}^a b, {}^a c)$ if $A \subseteq B \cap C$.

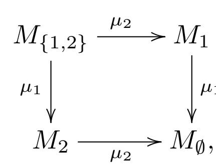
A morphism of crossed n-cubes

$$\mathbf{f} : \{\mathbf{M}_A\} \longrightarrow \{\mathbf{M}'_A\}$$

is a family of homomorphisms, $\{f_A : M_A \to M'_A \mid A \subseteq \langle n \rangle\}$, which commute with the maps, μ_{k_i}, and the h maps. We denote the category of crossed n-cubes by $\mathbf{Crs}^n$.

Example 3.2 (a) *For* $n = 1$, *a crossed 1-cube is the same as a crossed module* $M_1 \to M_\emptyset$.

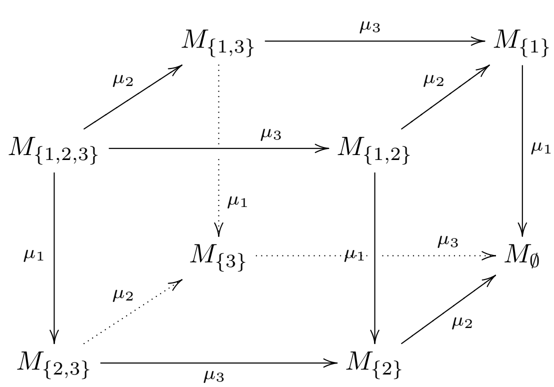
(b) *For* $n = 2$, *one has a crossed square*



where each μ_i *is a crossed module. The* h*-maps give actions and a pairing*

$$h : M_1 \times M_2 \to M_{\{1,2\}}.$$

(c) *For* $n = 3$, *one has a crossed 3-cube*



where each μ_i is a crossed module, and the h-maps give actions and the following pairings

$$\begin{array}{lcl}
h: M_1 \times M_2 \to M_{\{1,2\}} & , & h: M_1 \times M_3 \to M_{\{1,3\}} \\
h: M_2 \times M_3 \to M_{\{2,3\}} & , & h: M_{\{1,2\}} \times M_3 \to M_{\{1,2,3\}} \\
h: M_1 \times M_{\{2,3\}} \to M_{\{1,2,3\}} & , & h: M_{\{1,3\}} \times M_2 \to M_{\{1,2,3\}} \\
h: M_{\{2,3\}} \times M_{\{1,2\}} \to M_{\{1,2,3\}} & , & h: M_{\{1,2\}} \times M_{\{1,3\}} \to M_{\{1,2,3\}} \\
h: M_{\{2,3\}} \times M_{\{1,3\}} \to M_{\{1,2,3\}} & . &
\end{array}$$

We can give the main result of this section.

Theorem 3.3 *Let $\mathbf{G}_{\bullet_1\bullet_2\cdots\bullet_n}$ be an n-simplicial group with Moore n-complex $\mathbf{NG}_{\bullet_1\bullet_2\cdots\bullet_n}$, such that $NG_{\bullet_1\bullet_2\cdots\bullet_n} = \{1\}$ for any $\bullet_j \geqslant 2$, $(1 \leqslant j \leqslant n)$. Then this Moore n-complex has a crossed n-cube structure.*

Proof First, we define M_A for any subset $A \subset \langle n\rangle = \{1, 2, \ldots, n\}$ by

$$M_A = NG_{\underline{\epsilon}}$$

where $\underline{\epsilon} = (\epsilon_i | 1 \leq i \leq n)$ with $\epsilon_i = 1$ if $i \in A$ and 0 otherwise.

The map

$$\mu_i : M_A \longrightarrow M_{A-\{i\}}$$

is given by the face operator $d_1^{x_i} : NG_{\underline{\epsilon}(:\epsilon_i=1)} \longrightarrow NG_{\underline{\epsilon}(:\epsilon_i=0)}$, where x_i indicates the simplicial directions. For the subsets $B \subseteq A \subseteq < n >$, the structure morphism $\mu : M_A \to M_B$ is given by the simplicial structure, namely the operator $\prod_{i \in A \setminus B} d_1^i$.

For $A = \{i, i+1, \ldots j\}$ and $B = \{l, l+1, \ldots m\}$ where $1 \leq i, j, l, m \leq n$, we have

$$M_A = NG_{\underline{\epsilon}}$$

where $\underline{\epsilon} = (\epsilon_k : 1 \leq k \leq n)$ and for $i \leq k \leq j$, $\epsilon_k = 1$ and 0 otherwise and

$$M_B = NG_{\underline{\epsilon}}$$

where $\underline{\epsilon} = (\epsilon_k : 1 \leq k \leq n)$ and for $l \leq k \leq m$, $\epsilon_k = 1$ and 0 otherwise.

Let

$$X = \begin{cases}
(\underline{\epsilon} = (\epsilon_k : 1 \leq k \leq n)), \text{ for } i \leq k \leq m, \epsilon_k = 1, \text{ otherwise } 0, & if \quad i \leqslant l,\ j \leqslant m \\
(\underline{\epsilon} = (\epsilon_k : 1 \leq k \leq n)), \text{ for } i \leq k \leq j, \epsilon_k = 1, \text{ otherwise } 0, & if \quad i \leqslant l,\ j \geqslant m \\
(\underline{\epsilon} = (\epsilon_k : 1 \leq k \leq n)), \text{ for } l \leq k \leq m, \epsilon_k = 1, \text{ otherwise } 0, & if \quad i \geqslant l,\ j \leqslant m \\
(\underline{\epsilon} = (\epsilon_k : 1 \leq k \leq n)), \text{ for } l \leq k \leq j, \epsilon_k = 1, \text{ otherwise } 0, & if \quad i \geqslant l,\ j \geqslant m.
\end{cases}$$

The h maps $h : M_A \times M_B \to M_{A \cup B}$ are obtained from the commutative diagram

$$\begin{array}{ccc}
NG_{\underline{\epsilon}:(\epsilon_k=1:i\leq k\leq j)} \times NG_{\underline{\epsilon}:(\epsilon_k=1:l\leq k\leq m)} & \xrightarrow{F_{\alpha,\beta}} & NG_X \\
\Big\downarrow{\scriptstyle (s_\alpha, s_\beta)} & & \Big\uparrow{\scriptstyle p} \\
G_X \times G_X & \xrightarrow[{[-,-]}]{} & G_X
\end{array}$$

by composing of the maps $p,[-,-],(s_\alpha,s_\beta)$, for M_A, M_B as follows:

$$\begin{aligned} F_{\alpha,\beta}(x,y) &= p[-,-](s_\alpha,s_\beta)(x,y) \\ &= p[s_\alpha(x),s_\beta(y)] \\ &= \left[s_0^{x_i}s_0^{x_{i+1}}\cdots s_0^{x_j}(x),s_0^{x_l}s_0^{x_{l+1}}\cdots s_0^{x_m}(y)\right] \end{aligned}$$

where x_i and x_l indicate the simplicial directions and

$$\begin{aligned} \alpha &= (\underbrace{\emptyset,\emptyset,\ldots,\emptyset}_{(i-1)-times},\underbrace{(0),(0),\ldots,(0)}_{(j-i)-times},\underbrace{\emptyset,\emptyset,\ldots,\emptyset}_{(n-j)-times}) \\ \beta &= (\underbrace{\emptyset,\emptyset,\ldots,\emptyset}_{(l-1)-times},\underbrace{(0),(0),\ldots,(0)}_{(m-l)-times},\underbrace{\emptyset,\emptyset,\ldots,\emptyset}_{(n-l)-times}). \end{aligned}$$

For any subsets $A,\ B\subseteq<n>=\{1,2,\ldots,n\}$ and $M_A=N_{\underline{\varepsilon}}$ where $\underline{\varepsilon}=(\varepsilon_i|1\le i\le n)$ with $\varepsilon_i=1$ if $i\in A$ and $\varepsilon_i=0$ otherwise, and $M_B=N_{\underline{\underline{\varepsilon}}}$ where $\underline{\underline{\varepsilon}}=(\varepsilon_j|1\le j\le n)$ with $\varepsilon_j=1$ if $j\in A$ and $\varepsilon_j=0$ otherwise.

The structure morphism $h:M_A\times M_B\to M_{A\cup B}$ is induced by the commutator on $G_{A\cup B}$ via the morphisms

$$\begin{aligned} s_{B\backslash(A\cap B)} &:= \prod_{i\in B\backslash(A\cap B)} s_0^i : G_A\to G_{A\cup B} \\ s_{A\backslash(A\cap B)} &:= \prod_{j\in A\backslash(A\cap B)} s_0^j : G_B\to G_{A\cup B}. \end{aligned}$$

Thus for $x\in M_A,\ y\in M_B$ the h-map is induced by the commutator

$$[s_{B\backslash(A\cap B)}(x),s_{A\backslash(A\cap B)}(y)]\in G_{A\cup B}.$$

Using the projection map $p:G_X\to NG_X$ given above, we obtain the h-map as follows: for $x\in M_A,\ y\in M_B$

$$h(x,y)=[p_0^{x_k}\ldots p_0^{x_i}(s_0^{x_i}\ldots s_0^{x_k})(x),p_0^{x_m}\ldots p_0^{x_j}(s_0^{x_j}\ldots s_0^{x_m})(y)]\in M_{A\cup B}$$

where for any $j,p_0^{x_j}(a)=as_0^{x_j}d_0^{x_j}(a)^{-1}$ for all $1\le i\le k\le n;\ i,\ldots,k\in A\backslash(A\cap B),\ 1\le j\le m\le n;\ j,\ldots,m\in B\backslash(A\cap B)$ and where $x_i,\ldots x_k,x_j\ldots x_n$ indicate the simplicial directions.

The action of $a\in M_A$ and $b\in M_B$ for $A\subseteq B\subseteq<n>$, can be given by

$${}^ab=(s_0^{x_i}\ldots s_0^{x_k})(a)b(s_0^{x_i}\ldots s_0^{x_k})(a)^{-1}$$

where $i,\ldots,k\in A\backslash B$.

From the definition of $\mu:M_A\to M_B$ given by the operator $\prod\limits_{i\in A\backslash B} d_1^i$, the axioms (1),(2) are immediate.

We show for this h-map the following equalities.

If $i\notin A,\ a\in M_A$ then $\mu_i=d_1^{x_i}s_0^{x_i}$. We obtain $\mu_i(a)=d_1^{x_i}s_0^{x_i}(a)=id(a)=a$ from the simplicial identities.

By the commutativity of the face and degeneracy maps in the simplicial directions, we obtain $\mu_i\mu_j=\mu_j\mu_i$.

For $M_A = NG_{\underline{\varepsilon}}$ where $\underline{\varepsilon} := (\varepsilon_i | 1 \leq i \leq n)$, $\varepsilon_i = 1$ if $i \in A$ and 0 otherwise, we obtain for the simplicial directions x_i, and for $\alpha = (\emptyset, \emptyset, \ldots, (0)_i, \emptyset, \ldots, \emptyset)$ and $\beta = (\emptyset, \emptyset, \ldots, (1)_i, \emptyset, \ldots, \emptyset)$

$$F_{\alpha,\beta}(x,y) = [s_0^{x_i}(x), s_1^{x_i}(y)][s_1^{x_i}(y), s_1^{x_i}(x)] \in NG_{\underline{\varepsilon}}$$

where $\underline{\varepsilon} := (\varepsilon_i | \varepsilon_i = 2, \text{ for } i \neq j, \ \varepsilon_j = 1 \text{ if } j \in A \text{ and } 0 \text{ otherwise})$ and since $NG_{\bullet_1 \bullet_2 \ldots \bullet_n} = \{1\}$ for $j \geq 2$, we obtain

$$d_2^{x_i}(F_{\alpha,\beta}(x,y)) = [s_0^{x_i} d_1^{x_i}(x), y][y, x] = 1$$

and then

$$h(\mu_i(x), y) = [s_0^{x_i} d_1^{x_i}(x), y] = [x, y] = h(x, y).$$

Let $\alpha = (\emptyset, \emptyset, \ldots, \emptyset)$, $\beta = (\emptyset, \emptyset, \ldots, \emptyset)$ and for $x, x' \in M_A$, we have $h(x, x') : M_A \times M_A \to M_A$,

$$h(x, x') \quad = \quad [x, x'].$$

For $x \in M_A$, $y \in M_B$, we have

$$\begin{aligned} h(x,y) &= [s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x), s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y)] \\ &= [s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y), s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x)]^{-1} \\ &= h(y,x)^{-1}. \end{aligned}$$

Similarly , if $x = 1$ or $y = 1$, we obtain

$$h(x,y) \quad = \quad [s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x), s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y)] = 1.$$

Furthermore we have for x, $x' \in M_A$, y, $y' \in M_B$

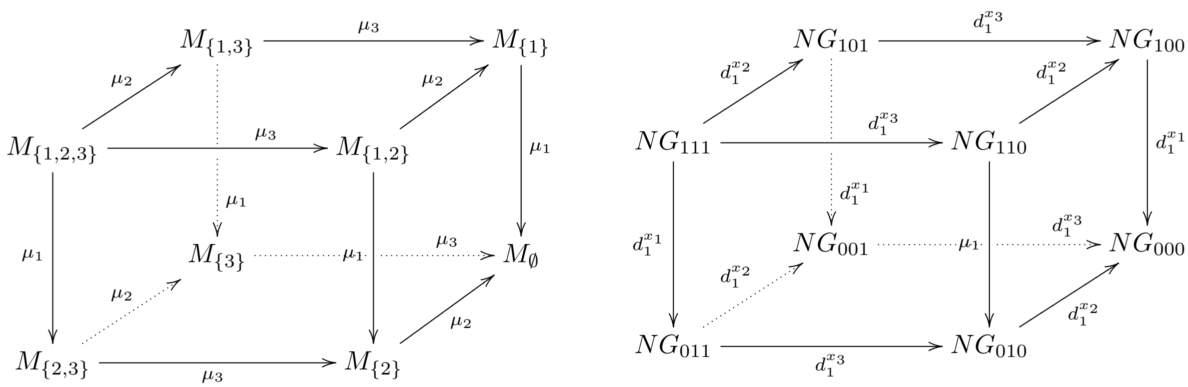
$$\begin{aligned} h(xx', y) &= [s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(xx'), s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y)] \\ &= [s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x) s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x'), s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y)] \\ &= s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x) s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x') s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y) s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x^{'-1}) \\ & \quad (s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y^{-1}) s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x^{-1}) s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{j}(x) s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y)) \\ & \quad s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x^{-1}) s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y^{-1}) \\ &= {}^{x}h(x', y) h(x, y) \end{aligned}$$

$$\begin{aligned} h(x, yy') &= [s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x), s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(yy')] \\ &= [s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x), s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y) s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y')] \\ &= s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x) s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y) (s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x^{-1}) s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y)^{-1} \\ & \quad s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_{i-1}}(y) s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x)) s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y') s_0^{x_i} s_0^{x_{i+1}} \cdots s_0^{x_j}(x^{-1}) \\ & \quad s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y')^{-1} s_0^{x_l} s_0^{x_{l+1}} \cdots s_0^{x_m}(y)^{-1} \\ &= h(x,y)^{y} h(x,y'). \end{aligned}$$

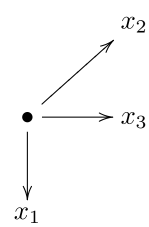
The remaining axioms can be shown similarly. □

We can illustrate this result for $n = 3$. A crossed 3-cube can be obtained from a 3-simplicial group as follows:

For $< n >= \{1, 2, 3\}$ we have the following diagrams



we show the simplicial directions by



The sets M_A can be given by

$$\begin{array}{lcl}
M_\emptyset = NG_{000} = G_{000} & , & M_{\{1\}} = NG_{100} = Kerd_0^{x_1} \\
M_{\{2\}} = NG_{010} = Kerd_0^{x_2} & , & M_{\{3\}} = NG_{001} = Kerd_0^{x_3} \\
M_{\{1,2\}} = NG_{110} = Kerd_0^{x_1} \cap Kerd_0^{x_2} & , & M_{\{2,3\}} = NG_{011} = Kerd_0^{x_2} \cap Kerd_0^{x_3} \\
M_{\{1,3\}} = NG_{101} = Kerd_0^{x_1} \cap Kerd_0^{x_3} & , & M_{\{1,2,3\}} = NG_{111} = Kerd_0^{x_1} \cap Kerd_0^{x_2} \cap Kerd_0^{x_3}.
\end{array}$$

The maps $\mu_i : M_A \to M_{A-\{i\}}$ are given in the above diagram.

The h-maps can be defined as follows:

$$h: \begin{array}{rl} NG_{100} \times NG_{010} & \longrightarrow NG_{110} \\ (x,y) & \longmapsto [s_0^{x_2}(x), s_0^{x_1}(y)] \end{array}, \quad h: \begin{array}{rl} NG_{100} \times NG_{001} & \longrightarrow NG_{101} \\ (x,y) & \longmapsto [s_0^{x_3}(x), s_0^{x_1}(y)] \end{array},$$

$$h: \begin{array}{rl} NG_{010} \times NG_{001} & \longrightarrow NG_{011} \\ (x,y) & \longmapsto [s_0^{x_3}(x), s_0^{x_2}(y)] \end{array}, \quad h: \begin{array}{rl} NG_{110} \times NG_{001} & \longrightarrow NG_{111} \\ (x,y) & \longmapsto [s_0^{x_3}(x), s_0^{x_1}s_0^{x_2}(y)] \end{array},$$

$$h: \begin{array}{rl} NG_{100} \times NG_{011} & \longrightarrow NG_{111} \\ (x,y) & \longmapsto [s_0^{x_2}s_0^{x_3}(x), s_0^{x_1}(y)] \end{array}, \quad h: \begin{array}{rl} NG_{101} \times NG_{010} & \longrightarrow NG_{111} \\ (x,y) & \longmapsto [s_0^{x_2}(x), s_0^{x_1}s_0^{x_3}(y)] \end{array},$$

$$\begin{array}{rll} h: & NG_{011}\times NG_{101} & \longrightarrow NG_{111} \\ & (x,y) & \longmapsto [s_0^{x_1}(x), s_0^{x_2}(y)] \end{array}, \quad \begin{array}{rll} h: & NG_{110}\times NG_{101} & \longrightarrow NG_{111} \\ & (x,y) & \longmapsto [s_0^{x_3}(x), s_0^{x_2}(y)] \end{array},$$

$$\begin{array}{rll} h: & NG_{1,1,0}\times NG_{0,1,1} & \longrightarrow NG_{1,1,1} \\ & (a,b) & \longmapsto [s_0^{x_3}(a), s_0^{x_1}(b)]. \end{array}$$

We can show the crossed 3-cubes axioms as follows:
1. Let $A=\{2,3\}$. Then if we have $i=1\notin A$, $\mu_i : M_A \to M_A$ is given by

$$\mu_i = \mu_1 = d_1^{x_1} s_0^{x_1}.$$

From the simplicial identities, we have $d_1^{x_1} s_0^{x_1} = id$. Therefore, for $i=1\notin A=\{2,3\}$ we obtain $\mu_i(a)=a$.

3. For the h-map given by

$$\begin{array}{rll} h: & NG_{1,1,0}\times NG_{0,1,1} & \longrightarrow NG_{1,1,1} \\ & (a,b) & \longmapsto [s_0^{x_3}(a), s_0^{x_1}(b)] \end{array}$$

we can write,

$$\begin{aligned} \mu_2 h(a,b) &= d_1^{x_2}[s_0^{x_3}(a), s_0^{x_1}(b)] \\ &= [s_0^{x_3} d_1^{x_2}(a), s_0^{x_1} d_1^{x_2}(b)] \ (\because \text{commutativity of simplicial directions}) \\ &= h(\mu_2 a, \mu_2 b) \end{aligned}$$

Similarly

$$\begin{aligned} \mu_3 h(a,b) &= d_1^{x_3}[s_0^{x_3}(a), s_0^{x_1}(b)] \\ &= [s_0^{x_3} d_1^{x_3}(a), s_0^{x_1} d_1^{x_3}(b)] \ (\because \text{commutativity of simplicial directions}) \\ &= h(\mu_3 a, \mu_3 b). \end{aligned}$$

By using similar calculations, this result for the other μ_i maps can be proven.

4. For example, for $x,y\in NG_{110}$, we obtain

$$F_{((0),\emptyset,\emptyset)((1),\emptyset,\emptyset)}(x,y) = [s_0^{x_1}(x), s_1^{x_1}(y)][s_1^{x_1}(y), s_1^{x_1}(x)] \in NG_{110}$$

Since $NG_{210}=\{1\}$, we obtain

$$d_2^{x_1}(F_{\alpha,\beta}(x,y)) = [s_0^{x_1} d_1^{x_1}(x), y][y,x] = 1 \in NG_{110}.$$

Thus we obtain

$$h(\mu_1(x), y) = [s_0^{x_1} d_1^{x_1}(x), y] = [x,y] \in NG_{110}.$$

and for $x,y\in NG_{111}$,

$$F_{(\emptyset,\emptyset,(0))(\emptyset,\emptyset,(1))}(x,y) = [s_0^{x_3}(x), s_1^{x_3}(y)][s_1^{x_3}(y), s_1^{x_3}(x)] \in NG_{112}$$

Since $NG_{112} = \{1\}$, we obtain

$$d_2^{x_3}(F_{\alpha,\beta}(x,y)) = [s_0^{x_3} d_1^{x_3}(x), y][y,x] = 1 \in NG_{111}.$$

Thus we obtain

$$h(\mu_1(x), y) = [s_0^{x_3} d_1^{x_3}(x), y] = [x,y] = h(x,y).$$

5. Let $\alpha = (\emptyset,\emptyset,\emptyset)$, $\beta = (\emptyset,\emptyset,\emptyset)$ and for $a, a' \in M_A$, we have $h : M_A \times M_A \to M_A$,

$$h(a,a') \quad = \quad [a,a'].$$

6. For the map $h : NG_{1,1,0} \times NG_{0,1,1} \to NG_{1,1,1}$, we have

$$h(a,b) = [s_0^{x_3}(a), s_0^{x_1}(b)] = [s_0^{x_1}(b), s_0^{x_3}(a)]^{-1} = h(b,a)^{-1}.$$

7. For example for $h : NG_{1,1,0} \times NG_{0,1,1} \to NG_{1,1,1}$ if $a = 1$, we obtain

$$h(1,b) = [s_0^{x_3}(1), s_0^{x_1}(b)] = [1, s_0^{x_1}(b)] = 1.$$

8. For the map $h : NG_{1,1,0} \times NG_{0,1,1} \to NG_{1,1,1}$, we obtain

$$\begin{aligned}
h(aa',b) &= [s_0^{x_3}(aa'), s_0^{x_1}(b)] \\
&= s_0^{x_3}(a)s_0^{x_3}(a')s_0^{x_1}(b)s_0^{x_3}(a')^{-1}s_0^{x_3}(a)^{-1}s_0^{x_1}(b)^{-1} \\
&= s_0^{x_3}(a)s_0^{x_3}(a')s_0^{x_1}(b)s_0^{x_3}(a')^{-1}(s_0^{x_1}(b)^{-1}s_0^{x_1}(b))s_0^{x_3}(a)^{-1}s_0^{x_1}(b)^{-1} \\
&= s_0^{x_3}(a)[s_0^{x_3}(a'), s_0^{x_1}(b)]s_0^{x_3}(a)^{-1}[s_0^{x_3}(a), s_0^{x_1}(b)] \\
&= {}^{a}h(a',b)h(a,b).
\end{aligned}$$

9. For the map $h : NG_{1,1,0} \times NG_{0,1,1} \to NG_{1,1,1}$, we obtain

$$\begin{aligned}
h(a,bb') &= [s_0^{x_3}(a), s_0^{x_1}(bb')] \\
&= s_0^{x_3}(a)s_0^{x_1}(b)s_0^{x_1}(b')s_0^{x_3}(a)^{-1}s_0^{x_1}(b')^{-1}s_0^{x_1}(b)^{-1} \\
&= s_0^{x_3}(a)s_0^{x_1}(b)s_0^{x_3}(a)^{-1}s_0^{x_1}(b)^{-1}s_0^{x_1}(b)s_0^{x_3}(a)s_0^{x_1}(b') \\
&\quad\ s_0^{x_3}(a)^{-1}s_0^{x_1}(b)^{-1}s_0^{x_1}(b')^{-1} \\
&= h(a,b)s_0^{x_1}(b)s_0^{x_3}(a)s_0^{x_1}(b')s_0^{x_3}(a)^{-1}s_0^{x_1}(b')^{-1}s_0^{x_1}(b)^{-1} \\
&= h(a,b)s_0^{x_1}(b)h(a,b')s_0^{x_1}(b)^{-1} \\
&= h(a,b)\,{}^{b}h(a,b').
\end{aligned}$$

10. We must show that

$${}^{a}h(h(a^{-1},b),c)\,{}^{c}h(h(c^{-1},a),b)\,{}^{b}h(h(b^{-1},c),a) = 1.$$

We calculate that for $a \in NG_{1,0,0}$, $b \in NG_{0,1,0}$, $c \in NG_{0,1,1}$,

$$\begin{aligned}
{}^{a}h(h(a^{-1},b),c) &= s_0^{x_3}s_0^{x_2}(a)h(h(a^{-1},b),c)s_0^{x_3}s_0^{x_2}(a)^{-1} \\
&= s_0^{x_3}s_0^{x_2}(a)[s_0^{x_3}[s_0^{x_2}(a)^{-1}, s_0^{x_1}(b)]s_0^{x_1}(c)]s_0^{x_3}s_0^{x_2}(a)^{-1} \\
&= s_0^{x_3}s_0^{x_1}(b)s_0^{x_3}s_0^{x_2}(a)s_0^{x_3}s_0^{x_1}(b)^{-1}s_0^{x_1}(c)s_0^{x_3}s_0^{x_1}(b) \\
&\quad\ s_0^{x_3}s_0^{x_2}(a)^{-1}s_0^{x_3}s_0^{x_1}(b)^{-1}s_0^{x_3}s_0^{x_2}(a)s_0^{x_1}(c)s_0^{x_3}s_0^{x_2}(a)^{-1}
\end{aligned}$$

and,

$$\begin{aligned} {}^{c}h(h(c^{-1},a),b) &= s_0^{x_3}s_0^{x_2}(a)s_0^{x_1}(c)^{-1}s_0^{x_3}s_0^{x_2}(a)^{-1}s_0^{x_3}s_0^{x_1}(b)s_0^{x_3}s_0^{x_2}(a) \\ &\quad s_0^{x_1}(c)^{-1}s_0^{x_3}s_0^{x_2}(a)^{-1}s_0^{x_1}(c)s_0^{x_3}s_0^{x_1}(b)^{-1}s_0^{x_1}(c)^{-1} \end{aligned}$$

and,

$$\begin{aligned} {}^{b}h(h(b^{-1},c),a) &= s_0^{x_1}(c)s_0^{x_3}s_0^{x_1}(b)s_0^{x_1}(c)^{-1}s_0^{x_3}s_0^{x_2}(a)s_0^{x_1}(c)s_0^{x_3}s_0^{x_1}(b)^{-1} \\ &\quad s_0^{x_1}(c)^{-1}s_0^{x_3}s_0^{x_1}(b)s_0^{x_3}s_0^{x_2}(a)^{-1}s_0^{x_3}s_0^{x_1}(b)^{-1} \end{aligned}$$

thus we obtain

$${}^{a}h(h(a^{-1},b),c){}^{c}h(h(c^{-1},a),b){}^{b}h(h(b^{-1},c),a) = 1.$$

11. For $A = \{1\}$, $B = \{1,2\}$, $C = \{2,3\}$ and $A \subseteq B \cap C$ then,

$$\begin{aligned} {}^{a}h(b,c) &= s_0^{x_3}s_0^{x_1}(a)h(b,c)s_0^{x_3}s_0^{x_1}(a)^{-1} \\ &= s_0^{x_3}s_0^{x_1}(a)[s_0^{x_3}(b),s_0^{x_1}(c)]s_0^{x_3}s_0^{x_1}(a)^{-1} \\ &= [s_0^{x_3}(s_0^{x_1}(a)bs_0^{x_1}(a)^{-1}),s_0^{x_1}(a)(s_0^{x_3}(a)cs_0^{x_3}(a)^{-1})] \ (\because \text{simplicial directions}) \\ &= [s_0^{x_3}({}^{a}b),s_0^{x_1}({}^{a}c)] \\ &= h({}^{a}b,{}^{a}c). \end{aligned}$$

4. Applications in low dimensions of $F_{\alpha,\beta}$ functions

In the following subsections, we give applications of the functions $F_{\alpha,\beta}$ in dimensions 1 and 2. In particular, in dimension 1, we obtain Mutlu and Porter's result about the relation between crossed modules and simplicial groups with Moore complex of length 1 and in dimension 2, we obtain Conduché's result about the relation between crossed squares and bisimplicial groups with Moore bicomplex of length 1.

4.1. 1-dimensional case

In Theorem 3.3, if we take $n = 1$, then we obtain a simplicial group, $\mathbf{G}_\bullet$ and we obtain the following well-known result (cf. [8, 9, 16]).

Proposition 4.1 *([16]) Let $\mathbf{G}_\bullet$ be a simplicial group and $\mathbf{NG}_\bullet$ its Moore complex. Suppose $N(G)_n = \{1\}$ for $n \geqslant 2$. Then the morphism $\partial_1 : N(G)_1 \to N(G)_0$ is a crossed module where $N(G)_0$ acts on $N(G)_1$ by conjugacy via the degeneracy map s_0.*

In the proof of this proposition, the role of $F_{\alpha,\beta}$ functions in the simplicial group whose Moore complex of length 1 can be summarised as follows. We know from [16] that for $x, y \in NG_1 = \ker d_0$, $F_{(0)(1)}(x,y) = [s_0x, s_1y][s_1y, s_1x] \in NG_2 \cap D_2 = N_2 \cap D_2$. Now we explain how we are using the hypothesis that NG vanishes if the simplicial degree is > 1. Since the Moore complex of the simplicial group G is of length 1, we have $NG_2 \cap D_2 = \{1\}$ and then we obtain $\partial_2(NG_2 \cap D_2) = \{1\}$, thus

$$\begin{aligned} d_2F_{(0)(1)}(x,y) &= d_2([s_0x,s_1y][s_1y,s_1x]) \\ &= [s_0d_1x,y][y,x] \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} {}^{\partial_1(x)}y &= s_0 d_1 x y s_0 d_1 x^{-1} \\ &= x y x^{-1} \end{aligned}$$

for $x, y \in NG_1$.

4.2. 2-dimensional case

A 2-simplicial group or *a bisimplicial group* $\mathbf{G}_{\bullet,\bullet}$ is a functor from the product category $\Delta^{op} \times \Delta^{op}$ to the category of groups **Grp**, with the face and degeneracy maps given by

$$\begin{aligned} &d_i^h : G_{p,q} \to G_{p-1,q} \\ &s_i^h : G_{p,q} \to G_{p+1,q} \quad 0 \le i \le p \\ &d_j^v : G_{p,q} \to G_{p,q-1} \\ &s_j^v : G_{p,q} \to G_{p,q+1} \quad 0 \le j \le q \end{aligned}$$

such that the maps d_i^h, s_i^h commute with d_j^v, s_j^v and that d_i^h, s_i^h (resp. d_j^v, s_j^v) satisfy the usual simplicial identities.

We think of d_j^v, s_j^v as the vertical operators and d_i^h, s_i^h as the horizontal operators. If $\mathbf{G}_{\bullet,\bullet}$ is a bisimplicial group, it is convenient to think of an element of $G_{p,q}$ as a product of a p-simplex and a q-simplex.

The Moore bicomplex of a bisimplicial group $\mathbf{G}_{\bullet,\bullet}$ is defined by

$$NG_{n,m} = \bigcap_{(i,j)=(0,0)}^{(n-1,m-1)} \mathrm{Ker} d_i^h \cap \mathrm{Ker} d_j^v$$

with the boundary homomorphisms

$$\partial_i^h : NG_{n,m} \longrightarrow NG_{n-1,m}$$

and

$$\partial_j^v : NG_{n,m} \longrightarrow NG_{n,m-1}$$

induced by the face maps d_i^h and d_j^v.

This Moore bicomplex is illustrated by the following diagram.

$$\begin{array}{ccccccc} & & \downarrow & \vdots & \downarrow & \vdots & \downarrow \\ \cdots & \longrightarrow & NG_{2,2} & \longrightarrow & NG_{1,2} & \xrightarrow[\partial_1^h]{} & NG_{0,2} \\ & & \downarrow & & \downarrow{\scriptstyle \partial_2^v} & & \downarrow{\scriptstyle \partial_2^v} \\ \cdots & \longrightarrow & NG_{2,1} & \longrightarrow & NG_{1,1} & \xrightarrow[\partial_1^h]{} & NG_{0,1} \\ & & \downarrow & & \downarrow{\scriptstyle \partial_1^v} & & \downarrow{\scriptstyle \partial_1^v} \\ \cdots & \longrightarrow & NG_{2,0} & \xrightarrow[\partial_2^h]{} & NG_{1,0} & \xrightarrow[\partial_1^h]{} & NG_{0,0}. \end{array}$$

Now we can give the functions $F_{\underline{\alpha},\underline{\beta}}$ in the Moore (bi)complex of a bisimplicial group.

Given $\underline{n} = (k_1, k_2) \in \mathbb{N}\times\mathbb{N}$, let $S(\underline{n}) = S(k_1)\times S(k_2)$ with the product (partial) order. Let $\underline{\alpha}$, $\underline{\beta} \in S(\underline{n})$ and $\underline{\alpha} = (\alpha_1, \alpha_2)$; $\underline{\beta} = (\beta_1, \beta_2)$ where $\alpha_i, \beta_i \in S(k_i)$ for $1 \leqslant i \leqslant 2$. The pairings that we will need

$$\left\{F_{\underline{\alpha},\underline{\beta}} : NG_{\underline{n}-\#\underline{\alpha}} \times NG_{\underline{n}-\#\underline{\beta}} \longrightarrow NG_{\underline{n}} \;\; ; \; \underline{\alpha}, \underline{\beta} \in S(\underline{n}), \underline{\alpha} \neq \underline{\beta}\right\}$$

are given by composing of the maps in the following diagram

$$\begin{array}{ccc}
NG_{k_1-\#\alpha_1,k_2-\#\alpha_2} \times NG_{k_1-\#\beta_1,k_2-\#\beta_2} & \xrightarrow{F_{\underline{\alpha},\underline{\beta}}} & NG_{k_1,k_2} \\
\Big\downarrow {\scriptstyle (s_{\underline{\alpha}},s_{\underline{\beta}})} & & \Big\uparrow {\scriptstyle p} \\
G_{k_1,k_2} \times G_{k_1,k_2} & \xrightarrow[\mu]{} & G_{k_1,k_2}
\end{array}$$

where $s_{\underline{\alpha}} : s^h_{\alpha_1} s^v_{\alpha_2}$, and where $s^h_{\alpha_1} = s^h_{i_r} \dots s^h_{i_1}$ for $\alpha_1 = (i_r, \dots, i_1) \in S(k_1)$ and similarly $s_{\underline{\beta}} = s^h_{\beta_1} s^v_{\beta_2}$, $s^v_{\beta_2} = s^v_{j_m} \dots s^v_{j_1}$ for $\beta_1 = (j_m, \dots, j_1) \in S(k_2)$ and where

$$p : G_{k_1,k_2} \to NG_{k_1,k_2}$$

is defined by the composite projection

$$p = \left(p^h_{k_1-1} ... p^h_0\right)\left(p^v_{k_2-1} ... p^v_0\right)$$

where $p_j(x) = x^{-1} s_j d_j(x)$ in both horizontal and vertical directions and μ is given by the commutator map.

For $\underline{\alpha} = (\alpha_1, \alpha_2), \underline{\beta} = (\beta_1, \beta_2) \in S(k_1) \times S(k_2)$, we obtain

$$\begin{aligned}
F_{\underline{\alpha},\underline{\beta}}(x, y) &= p[-,-](s_{\underline{\alpha}}, s_{\underline{\beta}})(x, y) \\
&= p[-,-](s^h_{\alpha_1} s^v_{\alpha_2}(x), s^h_{\beta_1} s^v_{\beta_2}(y)) \\
&= p[s^h_{\alpha_1} s^v_{\alpha_2}(x), s^h_{\beta_1} s^v_{\beta_2}(y)]
\end{aligned}$$

where $x \in NG_{k_1-\#\alpha_1,k_2-\#\alpha_2}$ and $y \in NG_{k_1-\#\beta_1,k_2-\#\beta_2}$.

4.3. Calculations of the functions $F_{\alpha,\beta}$ in low dimensions for bisimplical groups

For $0 \leqslant k_1, k_2 \leqslant 2$, we consider the sets $S(k_1) \times S(k_2)$. We shall calculate the images of the functions $F_{\underline{\alpha},\underline{\beta}}$ for all $\underline{\alpha}, \underline{\beta} \in S(k_1) \times S(k_2)$.

First, consider $(n, m) = (0, 1)$ or $(n, m) = (1, 0)$. We get the $F_{\underline{\alpha},\underline{\beta}}$ functions whose codomain is $NG_{0,1}$ or $NG_{1,0}$ respectively. Let $(n, m) = (0, 1)$. We get

$$S(n, m) = S(0) \times S(1) = \{(\emptyset, \emptyset), (\emptyset, (0))\}.$$

For $\underline{\alpha} = (\emptyset, \emptyset)$ and $\underline{\beta} = (\emptyset, (0))$, the function

$$F_{(\emptyset,\emptyset),(\emptyset,(0))} : NG_{0,1} \times NG_{0,0} \longrightarrow NG_{0,1}$$

can be given as follows:

$$\begin{aligned}F_{(\emptyset,\emptyset),(\emptyset,(0))}(x,y) &= p[-,-](s^h_\emptyset s^v_\emptyset(x), s^h_\emptyset s^v_{(0)}(y))\\ &= p^v_0[id(x), s^v_0(y)] \ (\because s^h_\emptyset = s^v_\emptyset = id)\\ &= [x, s^v_0 y]s^v_0 d^v_0[s^v_0 y, x]\\ &= [x, s^v_0 y][s^v_0 y, s^v_0 d^v_0(x)]\\ &= [x, s^v_0 y][s^v_0 y, 1] \ ((\because x \in \ker d^v_0 = NG_{0,1})\\ &= [x, s^v_0 y]\end{aligned}$$

for $x \in NG_{0,1}$ and $y \in NG_{0,0}$.

Suppose now that $(n,m) = (1,0)$. We have $S(1) \times S(0) = \{(\emptyset,\emptyset),((0),\emptyset)\}$. For $\underline{\alpha} = (\emptyset,\emptyset)$ and $\underline{\beta} = ((0),\emptyset)$, the function

$$F_{(\emptyset,\emptyset),((0),\emptyset)} : NG_{1,0} \times NG_{0,0} \longrightarrow NG_{1,0}$$

is defined by

$$\begin{aligned}F_{(\emptyset,\emptyset),((0),\emptyset)}(x,y) &= p[-,-](s^h_\emptyset s^v_\emptyset(x), s^h_0 s^v_\emptyset(y))\\ &= p^h_0[id(x), s^h_0(y)]\\ &= [x, s^h_0 y]s^h_0 d^h_0[s^h_0 y, x]\\ &= [x, s^h_0 y][s^h_0 y, s^h_0 d^h_0(x)]\\ &= [x, s^h_0 y][s^h_0 y, 1] \ ((\because x \in \ker d^h_0 = NG_{1,0})\\ &= [x, s^h_0 y]\end{aligned}$$

for all $x \in NG_{1,0}$ and $y \in NG_{00}$.

We give the calculations of other functions in Appendix.

Definition 4.2 *Let $\mathbf{G}_{\bullet,\bullet}$ be a bisimplicial group and $n,m > 1$, and $D_{n,m}$ the subgroup in $G_{n,m}$ generated by degenerate elements. Let $N^G_{n,m}$ be the normal subgroup of $G_{n,m}$ generated by elements of the form*

$$F_{\underline{\alpha},\underline{\beta}}(x,y) \ \textit{with} \ \ \underline{\alpha} = (\alpha_1,\alpha_2), \underline{\beta} = (\beta_1,\beta_2) \in S(n) \times S(m)$$

where $x \in NG_{n-\#\alpha_1, m-\#\alpha_2}$ and $y \in NG_{n-\#\beta_1, m-\#\beta_2}$.

Considering the equalities given in Definition 2.1 of Section 2, for bisimplicial groups, we can write the following equalities

$$\partial^h_n(NG_{n,m} \cap D_{n,m}) = \partial^h_n(N^G_{n,m} \cap D_{n,m}),$$

and

$$\partial^v_m(NG_{n,m} \cap D_{n,m}) = \partial^v_m(N^G_{n,m} \cap D_{n,m})$$

in each direction.

As we stated in Proposition 2.2. that Mutlu and Porter obtained the following inclusion

$$[\bigcap_{i\in I}\ker d_i \ , \bigcap_{j\in J}\ker d_j] \subseteq \partial_n(NG_n \cap D_n).$$

for $n \geqslant 2$ and $I, J \subseteq [n-1]$ with $I \cup J = [n-1]$, in the Moore complex of a simplicial group $\mathbf{G}$.

The following result is corresponding to 2-dimensional version of this result given in [16].

Theorem 4.3 *Let $\mathbf{G}_{\bullet,\bullet}$ be a bisimplicial group. Then for $n \geqslant 1, m \geqslant 2$ and $I, J \subseteq [m-1]$ with $I \cup J = [m-1]$, there is the inclusion*

$$[K_I \cap K_H, K_J \cap K_H] \subseteq \partial^v_m\left(NG_{n,m} \cap D_{n,m}\right)$$

where

$$K_H = \bigcap_{i=0}^{n-1}\ker d^h_i$$

and

$$K_I = \bigcap_{i\in I}\ker d^v_i \quad \textit{and} \quad K_J = \bigcap_{j\in J}\ker d^v_j.$$

Similarly, for $n \geqslant 2, m \geqslant 1$ and $I', J' \subseteq [n-1]$ with $I' \cup J' = [n-1]$, there is the inclusion

$$[K_{I'} \cap K_V, K_{J'} \cap K_V] \subseteq \partial^h_n\left(NG_{n,m} \cap D_{n,m}\right)$$

where

$$K_V = \bigcap_{i=0}^{m-1}\ker d^v_i$$

and

$$K_{I'} = \bigcap_{i\in I'}\ker d^h_i \quad \textit{and} \quad K_{J'} = \bigcap_{j\in J'}\ker d^h_j.$$

Proof We know that from the results of [16], there are already the following inclusions in both horizontal and vertical directions: if m is constant in the horizontal direction, for $n \geqslant 2$, we have

$$[K_{I'}, K_{J'}] \subseteq \partial^h_n\left(NG_{n,m} \cap D_{n,m}\right)$$

and if n is constant in the vertical direction, then for $m \geqslant 2$, we have

$$[K_I, K_J] \subseteq \partial^v_m\left(NG_{n,m} \cap D_{n,m}\right).$$

The result can be seen easily by using these inclusions. □

Mutlu and Porter for a simplicial group $\mathbf{G}$ investigated the images of the functions $F_{\alpha,\beta}$ for $n = 2, 3, 4$. Thus they obtained the following equalities for $n = 2$ and $n = 3$:

$$\partial_2(NG_2 \cap D_2) = [\ker d_0, \ker d_1]$$

and

$$\begin{aligned}\partial_3(NG_3 \cap D_3) &= [\ker d_2, \ker d_0 \cap \ker d_1][\ker d_1, \ker d_0 \cap \ker d_2][\ker d_0, \ker d_1 \cap \ker d_2] \\ &\quad [\ker d_0 \cap \ker d_2, \ker d_0 \cap \ker d_1][\ker d_1 \cap \ker d_2, \ker d_0 \cap \ker d_1] \\ &\quad [\ker d_1 \cap \ker d_2, \ker d_0 \cap \ker d_2].\end{aligned}$$

For bisimplical groups, we now investigate the images of these functions in the Moore bicomplex in low dimensions. The images of these functions under the boundary homomorphisms ∂_n^h and ∂_m^v are as follows:

For $NG_{1,2}$, take $x, y \in NG_{1,1} = \ker d_0^h \cap \ker d_0^v$, we obtain

$$\begin{aligned}\partial_2^v(F_{(\emptyset,(0)),(\emptyset,(1))})(x,y) &= d_2^v\left([s_0^v(x), s_1^v(y)][s_1^v y, s_1^v x]\right) \\ &= [s_0^v d_1^v x, y][y, x],\end{aligned}$$

where $[s_0^v d_1^v x, y][y, x] \in [\ker d_0^v, \ker d_1^v]$ from [16]. Further we obtain

$$d_0^h([s_0^v d_1^v x, y][y, x]) = 1,$$

hence $[s_0^v d_1^v x, y][y, x] \in [\ker d_0^v \cap \ker d_0^h, \ker d_1^v \cap \ker d_0^h]$.

Similarly for $y \in NG_{0,2}$ and $x \in NG_{1,1}$, from $NG_{1,1} \times NG_{0,2}$ to $NG_{1,2}$ we obtain

$$\begin{aligned}\partial_2^v(F_{(\emptyset,(1)),((0),\emptyset)})(x,y) &= d_2^v[s_1^v(x), s_0^h(y)] \\ &= [x, d_2^v s_0^h(y)].\end{aligned}$$

Since $d_0^h[x, d_2^v s_0^h(y)] = 1$ we have $[x, d_2^v s_0^h(y)] \in \ker d_0^h$. Furthermore, $[x, d_2^v s_0^h(y)] \in [\ker d_0^v, \ker d_1^v]$.

By a similar way one can show that the images of other generating elements are in

$$[\ker d_0^v \cap \ker d_0^h, \ker d_1^v \cap \ker d_0^h].$$

We have the following equality

$$\partial_2^v (NG_{1,2} \cap D_{1,2}) = [\ker d_0^v \cap \ker d_0^h, \ker d_1^v \cap \ker d_0^h].$$

For $NG_{2,1}$ take $x, y \in NG_{1,1}$. We obtain

$$\partial_2^h(F_{(((0),\emptyset),((1),\emptyset))})(x,y) = [s_0^h d_1^h(x), y][y, x]$$

where $[s_0^h d_1^h(x), y][y, x] \in [\ker d_0^h, \ker d_1^h]$ and

$$d_0^v([s_0^h d_1^h(x), y][y, x]) = 1,$$

hence

$$[s_0^h d_1^h(x), y][y, x] \in \ker d_0^v \cap [\ker d_0^h, \ker d_1^h].$$

We obtain the following equality

$$\partial_2^h (NG_{2,1} \cap D_{2,1}) = [\ker d_0^h \cap \ker d_0^v, \ker d_1^h \cap \ker d_0^v].$$

For $(n,m)=(2,1)$ and $(n,m)=(1,2)$, we can summarize these situations in the following diagram.

$\underline{\alpha}$	$\underline{\beta}$	I'	J'	V
$((0),\emptyset)$	$((1),\emptyset)$	$\{0\}$	$\{1\}$	$\{0\}$
$\underline{\alpha}$	$\underline{\beta}$	I	J	H
$(\emptyset,(0))$	$(\emptyset,(1))$	$\{0\}$	$\{1\}$	$\{0\}$

For $NG_{2,2}$ take $x,y\in NG_{2,1}=\ker d_0^h\cap\ker d_1^h\cap\ker d_0^v$, we obtain

$$\begin{aligned}\partial_2^v(F_{((\emptyset,(0)),(\emptyset,(1))})(x,y) &= d_2^v\left([s_0^v(x),s_1^v(y)][s_1^v(y),s_1^v(x)]]\right)\\ &= [s_0^vd_1^vx,y][y,x]\end{aligned}$$

where $[s_0^vd_1^vx,y][y,x]\in[\ker d_0^v,\ker d_1^v]$ from [16]. Further we obtain

$$d_0^h([s_0^vd_1^vx,y][y,x])=1,$$

hence $[s_0^vd_1^vx,y][y,x]\in[\ker d_0^v\cap\ker d_0^h,\ker d_1^v\cap\ker d_0^h]$.

Similarly for $x\in NG_{1,2}$ and $y\in NG_{2,1}$, from $NG_{1,2}\times NG_{2,1}$ to $NG_{2,2}$ we obtain

$$\begin{aligned}\partial_2^v(F_{((1),\emptyset),(\emptyset,(0))}(x,y)) &= d_2^v[s_1^h(x),s_0^v(y)]\\ &= [d_2^vs_1^h(x),d_2^vs_0^v(y)].\end{aligned}$$

For $d_0^h[x,d_2^vs_0^h(y)]=1$ we have $[x,d_2^vs_0^h(y)]\in\ker d_0^h$. Furthermore, $[x,d_2^vs_0^h(y)]\in[\ker d_0^v,\ker d_1^v]$.
By a similar way, one can show that the images of other generating elements are in

$$[\ker d_0^v\cap\ker d_0^h,\ker d_1^v\cap\ker d_0^h].$$

We have the following equality

$$\partial_2^v\,(NG_{2,2}\cap D_{2,2})=[\ker d_0^v\cap\ker d_0^h,\ker d_1^v\cap\ker d_0^h].$$

We can summarize this in the following diagram for $(n,m)=(2,2)$.

$\underline{\alpha}$	$\underline{\beta}$	I'	J'	V
$((0),\emptyset)$	$((1),\emptyset)$	$\{0\}$	$\{1\}$	$\{0,1\}$
$\underline{\alpha}$	$\underline{\beta}$	I	J	H
$(\emptyset,(0))$	$(\emptyset,(1))$	$\{0\}$	$\{1\}$	$\{0,1\}$

Using the calculation method given above, we obtained the following equalities in low dimensions for bisimplicial groups.

$$\begin{aligned}
&\partial_2^v\,(NG_{0,2}\cap D_{0,2})=[\ker d_0^v,\ker d_1^v],\\
&\partial_2^v\,(NG_{1,2}\cap D_{1,2})=[\ker d_0^v\cap\ker d_0^h,\ker d_1^v\cap\ker d_0^h],\\
&\partial_2^v\,(NG_{2,2}\cap D_{2,2})=[\ker d_0^v\cap\ker d_0^h\cap\ker d_1^h,\ker d_1^v\cap\ker d_0^h\cap\ker d_1^h],\\
&\partial_2^h\,(NG_{2,0}\cap D_{2,0})=[\ker d_0^h,\ker d_1^h],\\
&\partial_2^h\,(NG_{2,1}\cap D_{2,1})=[\ker d_0^h\cap\ker d_0^v,\ker d_1^h\cap\ker d_0^v]\ ,\\
&\partial_2^h\,(NG_{2,2}\cap D_{2,2})=[\ker d_0^h\cap\ker d_0^v\cap\ker d_1^v,\ker d_1^h\cap\ker d_0^v\cap\ker d_1^v].
\end{aligned}$$

4.4. Crossed squares and bisimplicial groups

In Theorem 3.3, we give the general relationship between crossed n-cubes and n-simplicial groups by using the $F_{\alpha,\beta}$ functions in the Moore complex of an n-simplicial group. In this section, to see the role of these functions in dimension 2, we give the following proposition as an application of these functions. This result was initially proven by Conduché in [9] without using $F_{\alpha,\beta}$ functions. In this result, to see the role of these functions in the structure, we reprove this result and we see that the h-map of the crossed square given by Conduché (cf. [9]) can be given by the function $F_{(\emptyset,(0)),((0),\emptyset)} : NG_{01} \times NG_{10} \to NG_{11}$ in the Moore bicomplex of a bisimplicial group.

First, we recall the definition of a crossed square from [13].

A crossed square of groups is a commutative square of group morphisms

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\nu]{} & P \end{array}$$

with an action of P on every other group and a map $h : M \times N \to L$ such that

1. The maps λ and λ' are P-equivariant and $\nu, \mu,\ \mu \circ \lambda$ and $\nu \circ \lambda'$ are crossed modules,
2. $\lambda \circ h(x, y) = x^{\nu(y)} x^{-1},\ \lambda' \circ h(x, y) = ({}^{\mu(x)}y)y^{-1}$,
3. $h(\lambda(z), y) = z^{\nu(y)} z^{-1},\ h(x, \lambda'(z)) = ({}^{\mu(x)}z)z^{-1}$,
4. $h(xx', y) ={}^{\mu(x)} h(x', y)h(x, y),\ \ h(x, yy') = h(x, y)^{\nu(y)} h(x, y')$,
5. $h({}^t x, {}^t y) ={}^t h(x, y)$

for $x, x' \in M,\ y, y' \in N, z \in L$ and $t \in P$.

Proposition 4.4 *Let $\mathbf{G}_{\bullet,\bullet}$ be a bisimplicial group and $\mathbf{NG}_{\bullet,\bullet}$ its Moore bicomplex. Suppose $NG_{k_1,k_2} = \{1\}$ for any $k_1 \geqslant 2$ or $k_2 \geqslant 2$. Then the diagram*

$$\begin{array}{ccc} NG_{1,1} & \xrightarrow{\partial_1^h} & NG_{0,1} \\ {\scriptstyle \partial_1^v}\downarrow & & \downarrow{\scriptstyle \partial_1^v} \\ NG_{1,0} & \xrightarrow[\partial_1^h]{} & NG_{0,0} \end{array}$$

is a crossed square. $NG_{0,0}$ acts on other groups via the degeneracies s_0^h and s_0^v.

The h-map is given by the map $F_{(\emptyset,(0)),((0),\emptyset)}(x, y)$, namely,

$$\begin{aligned} h : NG_{0,1} \times NG_{1,0} &\to NG_{1,1} \\ (x, y) &\mapsto h(x, y) = F_{(\emptyset,(0)),((0),\emptyset)}(x, y) \end{aligned}$$

for $x \in NG_{0,1},\ y \in NG_{1,0}$ where $(\emptyset, (0)), ((0), \emptyset) \in S(1) \times S(1)$ and

$$F_{(\emptyset,(0)),((0),\emptyset)}(x, y) = [s_0^h(x), s_0^v(y)].$$

Proof It is straightforward from the direct calculations of $F_{\alpha,\beta}$. □

4.5. Double crossed complexes and bisimplicial groups

Carrasco and Cegarra [6] have defined hypercrossed complexes and proved the non-Abelian version of the Dold–Kan theorem. They gave a functor from the category of simplicial groups to that of crossed complexes. Arvasi, using the image of the $F_{\alpha,\beta}$ function in the Moore complex of a simplicial group have reconstructed this functor from simplicial groups to crossed complex. The following result can be found in [6].

Recall that a crossed complex is a sequence of groups

$$C: \qquad \cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

(i) (C_1, C_0, ∂_1) is a crossed module, i.e. if $x, y \in C_1$, then ${}^{\partial_1 x}y = xyx^{-1}$;

(ii) for $i > 1$, C_i is an C_0-module (Abelian) on which $\partial_1 C_1$ operates trivially and each ∂_i is an operator morphism; and

(iii) for $i \geq 1$, $\partial_{i+1}\partial_i = 0$.

Morphisms of crossed complexes are defined in the obvious way.

Theorem 4.5 *Let* $\mathbf{G}$ *be a simplicial group. Then defining*

$$C_n = \frac{NG_n}{(NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1})}$$

with $\partial_n(\overline{z}) = \overline{d_n(z)}$ *gives a crossed complex* $C(\mathbf{G})$ *of groups.*

In this section, we obtain a double crossed complex of groups from the Moore bicomplex of a bisimplicial group via the $F_{\alpha,\beta}$ functions. By considering crossed complexes of group(oid)s internal to the category of crossed complexes of group(oid)s, Tonks [20], introduced the notion of double crossed complexes of group(oid)s. By the same way, the notion of a double crossed complex of groups has been given by considering it internal to the category of crossed complexes of groups.

Definition 4.6 ([20])*A double crossed complex of groups consists of*

(i) a collection of groups $L_{i,j}$ *for* $i, j \geq 0$,

(ii) the actions

$$L_{0,j} \times L_{k,j} \xrightarrow{\alpha^h} L_{k,j} \qquad\qquad L_{i,0} \times L_{i,k} \xrightarrow{\alpha^v} L_{i,k}$$

for $i, j \geq 0$, $k \geq 1$,

(iii) horizontal and vertical boundary maps

$$L_{i,j} \xrightarrow{\delta_i^h} L_{i-1,j} \qquad\qquad L_{j,i} \xrightarrow{\delta_i^v} L_{j,i-1}$$

for $i \geq 1$, $j \geq 0$.

These data are such that

(i) for each $j \geq 0$ *the horizontal structure* $((L_{i,j})_{i\geq 0}, \alpha^h, (\delta_i^h)_{i\geq 1})$ *defines a crossed complex of groups,*

(ii) for each $i \geq 0$ *the vertical structure* $((L_{i,j})_{j\geq 0}, \alpha^v, (\delta_j^v)_{j\geq 1})$ *defines a crossed complex of groups,*

(iii) the horizontal structure maps commute with the vertical structure maps. That is,

(a) *the function* δ^h *defines complex morphisms between the vertical crossed complex, and similarly* δ^v *between the horizontal ones,*

(b) *the horizontal and vertical actions satisfy an interchange law. That is, if the expressions* $\alpha^v(\alpha^h(a,b), \alpha^h(c,d))$ *and* $\alpha^h(\alpha^v(a,c), \alpha^v(b,d))$ *are both defined, then they are equal.*

A double crossed complex of groups may be represented diagrammatically as follows:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \xrightarrow[\delta_3^h]{} & L_{2,2} & \longrightarrow & L_{1,2} & \xrightarrow[\delta_1^h]{} & L_{0,2} \\
 & & \downarrow & & \downarrow{\scriptstyle \delta_2^v} & & \downarrow{\scriptstyle \delta_2^v} \\
\cdots & \xrightarrow[\delta_3^h]{} & L_{2,1} & \longrightarrow & L_{1,1} & \xrightarrow[\delta_1^h]{} & L_{0,1} \\
 & & \downarrow & & \downarrow{\scriptstyle \delta_1^v} & & \downarrow{\scriptstyle \delta_1^v} \\
\cdots & \xrightarrow[\delta_3^h]{} & L_{2,0} & \xrightarrow[\delta_2^h]{} & L_{1,0} & \xrightarrow[\delta_1^h]{} & L_{0,0}.
\end{array}$$

Now, we can give the relationship between the Moore bicomplex of a bisimplicial group and a double crossed complex by using the $F_{\alpha,\beta}$ functions.

Lemma 4.7 *The subgroup* $(NG_{n,m} \cap D_{n,m})d_{n+1}^v(NG_{n+1,m} \cap D_{n+1,m})d_{n+1}^h(NG_{n,m+1} \cap D_{n,m+1})$ *is a normal subgroup in* $NG_{n,m}$*.*

Let $\mathbf{G}_{\bullet,\bullet}$ be a bisimplicial group. Using the following lemmas, we will obtain a double crossed complex from $\mathbf{G}_{\bullet,\bullet}$. Define

$$L_{n,m}(\mathbf{G}_{\bullet,\bullet}) = \frac{NG_{n,m}}{(NG_{n,m} \cap D_{n,m})d_{n+1}^v(NG_{n+1,m} \cap D_{n+1,m})d_{n+1}^h(NG_{n,m+1} \cap D_{n,m+1})}.$$

Lemma 4.8 *Let* $x, y \in NG_{n,m}$ *for* $n \geq 2$*, then* $[x,y] = a^{-1}d_{n+1}^v w$*, where*

$a = F^n_{(\emptyset,n-2),(\emptyset,n-1)}(d_n^v x, d_n^v y) = [s_{n-2}^v d_n^v x, s_{n-1}^v d_n^v y][s_{n-1}^v d_n^v y, s_{n-1}^v d_n^v x]$ *and*

$w = F^{n+1}_{(\emptyset,n-2),(\emptyset,n-1)}(x,y) = [s_{n-2}^v x, s_{n-1}^v y][s_{n-1}^v y, s_{n-1}^v x][s_n^v x, s_n^v y].$

Similarly, $x, y \in NG_{n,m}$ *for* $m \geq 2$*, then* $[x,y] = b^{-1}d_{m+1}^h v$*, where*

$b = F^m_{(m-2,\emptyset),(m-1,\emptyset)}(d_m^h x, d_m^h y) = [s_{m-2}^h d_m^h x, s_{m-1}^h d_m^h y][s_{m-1}^h d_m^h y, s_{m-1}^h d_m^h x]$ *and*

$v = F^{m+1}_{(m-2,\emptyset),(m-1,\emptyset)}(x,y) = [s_{m-2}^h x, s_{m-1}^h y][s_{m-1}^h y, s_{m-1}^h x][s_m^h x, s_m^h y].$

Corollary 4.9 *For* $n \geq 2$ *and* $m \geq 2$*,* $L_{n,m}(\mathbf{G}_{\bullet,\bullet})$ *is abelian.*

Lemma 4.10 *For* $n \geq 2$, $x \in NG_{n-1,m}$ *and* $y \in NG_{n,m}$,

$$[s^v_{n-1}x, y] \in (NG_{n,m} \cap D_{n,m})d^v_{n+1}(NG_{n+1,m} \cap D_{n+1,m}).$$

Similarly, if for $m \geq 2$, $x \in NG_{n,m-1}$ *and* $y \in NG_{n,m}$, *then*

$$[s^h_{m-1}x, y] \in (NG_{n,m} \cap D_{n,m})d^h_{m+1}(NG_{n,m+1} \cap D_{n,m+1}).$$

The importance of this result is that the natural actions of $NG_{l,m}$ on $NG_{n,m}$ are by conjugations via degeneracies. In particular we choose the action

$$^{\overline{x}}\overline{y} = \overline{s_l^{(n-l)}(x)ys_l^{(n-l)}(x)^{-1}}$$

where the $(n-l)$ superfix denotes an iterated application of the map. Thus if $n \geq 2$ then $NG_{n-1,m}$ acts trivially on $NG_{n,m}$ $\overline{[s^v_{n-1}x, y]} = 1$.

Lemma 4.11 *For each* n, m *the maps* $\partial^v_n : L_{n,m}(\mathbf{G}_{\bullet,\bullet}) \to L_{n-1,m}(\mathbf{G}_{\bullet,\bullet})$ *and* $\partial^h_m : L_{n,m}(\mathbf{G}_{\bullet,\bullet}) \to L_{n,m-1}(\mathbf{G}_{\bullet,\bullet})$ *are crossed modules.*

Lemma 4.12 *For* $x \in NG_{n-i+1,m}$, $y \in NG_{n,m}$ *and* $1 \leq k < i$,

$$s_n^{v(k)}s_{n-i}^{v(i-k-1)}xys_n^{v(k)}s_{n-i}^{v(i-k-1)}x^{-1} \equiv s_n^{v(k-1)}s_{n-i}^{v(i-k)}xys_n^{v(k-1)}s_{n-i}^{v(i-k)}x^{-1}$$

$$mod(NG_{n,m} \cap D_{n,m})d^v_{n+1}(NG_{n+1,m} \cap D_{n+1,m}).$$

Proof Writing $s_n^{v(k)}$ for $(s^v_{n+k-1}, s^v_{n+k-2}, ..., s^v_n)$ and $s_{n-i}^{v(i-k)}$ for $(s^v_{n-k-1}, s^v_{n-k-2}, ..., s^v_{n-i})$, we consider the element $[s_n^{v(k)}s_{n-i}^{v(i-k)}x, s^v_k y] \in G_{n+1,m}$ the mapping

$$p_l : G_{n+1,m} \to NG_{n+1,m} \subset G_{n+1,m} \subset \text{ with } 0 \leq l \leq n$$

given by

$$p_l(z) = zs^v_l d^v_l z^{-1}.$$

We also note that the particular case of $F_{\alpha,\beta}$ for $\alpha = (n, n-i; m)$, $\beta = (k, m)$ is

$$\begin{aligned} F_{\alpha,\beta}(x,y) &= F_{(n,n-i;m),(k,m)}(x,y) \\ & \quad p_n p_{n-1}...p_0[s_n^{v(k)}s_{n-i}^{v(i-k)}x, s^v_k y] \in NG_{n+1,m} \cap D_{n+1,m}. \end{aligned}$$

We will prove that $d^v_{n+1}(F_{(n,n-i;m),(k,m)}(x,y))$ is basically the difference between the two elements of this lemma.

Indeed by putting

$$F_{(n,n-i;m),(k,m)}(x,y) = z_{(k)(i)}(x,y) = [s_n^{v(k)}s_{n-i}^{v(i-k)}x, s^v_k y]$$

and for $\alpha = (n, n-i; m)$ and $\beta = (k, m)$ with any j, $0 \leq j \leq n+1$, we obtain

$$d_j^v z_{(k)(i)}(x,y) = \begin{cases} 1 & \text{if} \quad k > j \\ [s_{n-1}^{v(k)} s_{n-i}^{v(i-k-1)} x, y] & \text{if} \quad k = j \\ [s_n^{v(k-1)} s_{n-i}^{v(i-k)} x, y] & \text{if} \quad k = j-1 \\ 1 & \text{if} \quad 1 < j-i-k+1 \\ 1 & \text{if} \quad j > i+1 \end{cases}$$

and $d_{n+1}^v z_{(k)(i)}(x,y) = z_{(k-1),(i-1)}(x, d_{n+1}^v y)$.

This gives

$$p_n p_{n-1} ... p_0(z_{(k)(i)}(x,y)) = p_n ... p_{i+k}(z_{(k)(i)}(x,y))$$

since the operators p_l for $l > i+1$ are trivial. We also note that

$$p_n ... p_{i+k}(z_{(k)(i)}(x,y)) = p_n ... p_{k+1}(z_{(k)(i)}(x,y)).$$

Now if $w \in G_{n+1,,m}$, then

$$\begin{aligned} d_{n+1}^v p_n(w) &= d_{n+1}^v w d_n^v w^{-1} \\ d_{n+1}^v p_n p_{n-1}(w) &= d_{n+1}^v p_{n-1} w \left(d_n^v p_{n-1} w\right)^{-1} \end{aligned} \tag{4.1}$$

and so on. It follows that

$$d_{n+1}^v p_n ... p_{k+1}(z_{(k)(i)}(x,y)) = p_n \dots p_k(z_{(k-1)(i-1)}(x, d_n^v y))(d_n^v p_{n-1} \dots p_{k+1}(z_{(k)(i)}(x,y)))^{-1}.$$

The first of these two terms is in $NG_{n,m} \cap D_{n,m}$ and hence we only check the second one. From (4.1), we get

$$d_{n+1}^v p_{n-1} ... p_{k+1}(z_{(k)(i)}(x,y)) \quad = \quad d_n^v p_{n-2} ... p_{k+1}(w)(d_{n-1}^v p_{n-2} ... p_{k+1}(w))^{-1}$$

and this implies

$$d_l^v p_{l+1} ... p_{k+1}(z_{(k)(i)}(x,y))$$

and others of form

$$d_{l-1}^v p_{l+1} ... p_{k+1}(z_{(k)(i)}(x,y)).$$

If $j < k-1$, $a \in G_{n,m}$

$$d_j^v p_k(a) = d_j^v(a)(s_{k-1}^v d_{k-1}^v d_j^v(a)^{-1}) = p_{k-1} d_j^v(a),$$

so any term of the form $d_{l-1}^v p_{l+1} ... p_{k+1}(z_{(k)(i)}(x,y))$ can be written as

$$p_{l-1} \cdots p_k(d_{l-1}^v(z_{(k)(i)}(x,y)))$$

and so is trivial if $l > 1$. Hence the only term is $d_k^v p_k(z_{(k)(i)}(x,y))$ and so

$$\begin{aligned} d_k^v p_k(z_{(k)(i)}(x,y)) &= d_k^v [s_n^{v(k)} s_{n-i}^{v(i-k)} x, s_k^v y] d_k^v s_k^v d_k^v [s_k^v y, s_n^{v(k)} s_{n-i}^{v(i-k)} x] \\ &= s_n^{v(k)} s_{n-i}^{v(i-k-1)} xy s_n^{v(k)} s_{n-i}^{v(i-k-1)} x^{-1} s_n^{v(k-1)} s_{n-i}^{v(i-k)} xy^{-1} s_n^{v(k-1)} s_{n-i}^{v(i-k)} x^{-1} \end{aligned}$$

i.e. the difference of the two terms in the statement of the lemma. Putting

$$b = s_n^{v(k)} s_{n-i}^{v(i-k-1)} xy s_n^{v(k)} s_{n-i}^{v(i-k-1)} x^{-1} s_n^{v(k-1)} s_{n-i}^{v(i-k)} xy^{-1} s_n^{v(k-1)} s_{n-i}^{v(i-k)} x^{-1}$$

It then follows that

$$d_{n+1}^v(F_{(n,n-i;m),(k,m)}(x,y)) = p_n...p_k(z_{(k-1)(i-1)}(x,d_n^v y))b^{-1}.$$

Having $p_n...p_k(z_{(k-1)(i-1)}(x,d_n^v y)) \in NG_{n,m}\cap D_{n,m}$ and $w\in NG_{n+1,m}\cap D_{n+1,m}$ implies that

$$\begin{aligned} s_n^{v(k-1)}s_{n-i}^{v(i-k)}xys_n^{v(k-1)}s_{n-i}^{v(i-k)}x^{-1} \quad &\equiv \quad s_n^{v(k)}s_{n-i}^{v(i-k-1)}xys_n^{v(k)}s_{n-i}^{v(i-k-1)}x^{-1} \\ &\quad \mathrm{mod}(NG_{n,m}\cap D_{n,m})d_{n+1}^v(NG_{n+1,m}\cap D_{n+1,m}). \end{aligned}$$

□

Lemma 4.13 *If* $n\geq 1,\ x\in NG_{n-i,m}$ *and* $y\in NG_{n,m}$ *then*

$$s_n^{v(i+1)}d_{n-i}^v xys_n^{v(i+1)}d_{n-i}^v x^{-1}\equiv s_{n-i}^{v(i)}xys_{n-1}^{v(i)}x^{-1}\ \ mod(NG_{n,m}\cap D_{n,m})d_{n+1}^v(NG_{n+1,m}\cap D_{n+1,m}).$$

Proof Take the element

$$t=s_{n-i}^{v(i+1)}xs_n^v ys_n^v s_{n-i}^{v(i+1)}x^{-1}s_n^v ys_n^v s_{n-i}^{v(i)}xs_n^v ys_n^v s_{n-i}^{v(i)}x^{-1}.$$

This is $F_{(\alpha;m),(n,m)}(x,y)=p_n...p_0[s_\alpha^{v(i+1)}x,s_n^v y]$ i.e. where $s_\alpha^{v(i+1)}=s_n^v s_{n-1}^v\cdots s_{n-i}^v$. It is readily checked that $0\leq i\leq n-1$, for $d_i^v(t)=1$ for $i\geq 0$ and $d_{n+1}^v(t)$ is the difference between the elements mentioned in the statement of the lemma. □

Lemma 4.14 *If* $n\geq 2,\ x\in NG_{1,m}$ *and* $y\in NG_{n,m}$ *then*

$$[s_{n-1}^v\cdots s_1^v x,y]\equiv 1\ \ mod(NG_{n,m}\cap D_{n,m})d_{n+1}^v(NG_{n+1,m}\cap D_{n+1,m}).$$

Proof Consider

$$\begin{aligned} u \quad &= \quad [s_n^v y,s_n^v\cdots s_1^v x][s_{n-1}^v y,s_{n-1}^v ys_n^v\cdots s_1^v x]^{-1} \\ &\quad \prod_{i=2}^{n}[s_{n-i}^v y,s_n^v\cdots s_1^v x]^{(-1)^i} \end{aligned}$$

u is easily checked to be in $NG_{n+1,m}\cap D_{n+1,m}$.

$$\begin{aligned} d_{n+1}^v(u) \quad &= \quad [y,s_{n-1}^v\cdots s_1^v x][s_{n-1}^v d_n^v y,s_{n-1}^v\cdots s_1^v x]^{-1} \\ &\quad [s_{n-2}^v d_n^v y,s_{n-1}^v\cdots s_1^v x]\prod_{i=3}^{n}[s_{n-i}^v d_n^v y,s_{n-1}^v\cdots s_1^v x]^{(-1)^i} \end{aligned}$$

Writing

$$\begin{aligned} t \quad &= \quad [s_{n-1}^v d_n^v y,s_{n-1}^v\cdots s_1^v x]^{-1}[s_{n-2}^v d_n^v y,s_{n-1}^v\cdots s_1^v x] \\ &\quad \prod_{i=3}^{n}[s_{n-i}^v d_n^v y,s_{n-1}^v\cdots s_1^v x]^{(-1)^i} \end{aligned}$$

it is readily checked that $t\in NG_{n,m}$ and is as required. □

Thus, we can give the main result of this section as follows.

Theorem 4.15 *Let* $(\mathbf{G}_{\bullet,\bullet})$ *be a bisimplicial group. Then the construction*

$$L_{n,m}(\mathbf{G}_{\bullet,\bullet}) = \frac{NG_{n,m}}{(NG_{n,m} \cap D_{n,m})d^v_{n+1}(NG_{n+1,m} \cap D_{n+1,m})d^h_{n+1}(NG_{n,m+1} \cap D_{n,m+1})}$$

with $\partial^v(\overline{x}) = \overline{d^v_n(x)}$ *,* $\partial^h(\overline{x}) = \overline{d^h_n(x)}$ *gives a double crossed complex.*

Proof It is straightforward from the lemmas given above. □

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1. Appendix

For $(n,m)=(1,1)$, consider the set

$$S(1)\times S(1)=\{(\emptyset,\emptyset),(\emptyset,(0)),((0),\emptyset),((0),(0))\}.$$

1. Take $\underline{\alpha}=(\emptyset,\emptyset)$ and $\underline{\beta}=(\emptyset,(0))$. In this case, the function $F_{\underline{\alpha},\underline{\beta}}$ becomes from $NG_{1,1}\times NG_{1,0}$ to $NG_{1,1}$. This map can be defined for any $x\in NG_{1,1}$ and $y\in NG_{1,0}$ by

$$F_{(\emptyset,\emptyset),(\emptyset,(0))}(x,y)=[x,s_0^v(y)].$$

2. Take $\underline{\alpha}=(\emptyset,\emptyset)$ and $\underline{\beta}=((0),\emptyset)$. In this case, the function $F_{\underline{\alpha},\underline{\beta}}$ becomes from $NG_{1,1}\times NG_{0,1}$ to $NG_{1,1}$. This map can be defined by

$$F_{(\emptyset,\emptyset),((0),\emptyset)}(x,y)=[x,s_0^h(y)].$$

for any $x\in NG_{1,1}$ and $y\in NG_{0,1}$.

3. For $\underline{\alpha}=(\emptyset,\emptyset)$ and $\underline{\beta}=((0),(0))$. The map

$$F_{(\emptyset,\emptyset),((0),(0))}:NG_{1,1}\times NG_{0,0}\to NG_{1,1}$$

is defined by

$$F_{(\emptyset,\emptyset),((0),(0))}(x,y)=[x,(s_0^v s_0^h(y))]$$

for all $x\in NG_{1,1}$ and $y\in NG_{00}$.

4. For $\underline{\alpha}=((0),\emptyset)$ and $\underline{\beta}=(\emptyset,(0))$. The map

$$F_{((0),\emptyset),(\emptyset,(0))}:NG_{0,1}\times NG_{1,0}\to NG_{1,1}$$

can be calculated for any $x\in NG_{0,1}$ and $y\in NG_{1,0}$ by

$$\begin{aligned}F_{((0),\emptyset),(\emptyset,(0))}(x,y)=&p[-,-](s_0^h(x),s_0^v(y))\\ =&p_0^h p_0^v(s_0^h(x)s_0^v(y)s_0^h(x)^{-1}s_0^v(y)^{-1})\\ =&p_0^h((s_0^v(y)s_0^h(x)s_0^v(y)^{-1}s_0^h(x)^{-1})\\ &s_0^v d_0^v(s_0^h(x)s_0^v(y)s_0^h(x)^{-1}s_0^v(y)^{-1}))\\ =&p_0^h(s_0^v(y)s_0^h(x)s_0^v(y)^{-1}s_0^h(x)^{-1})(s_0^v d_0^v\\ &s_0^h(x)s_0^v d_0^v s_0^v(y)s_0^v d_0^v s_0^h(x)^{-1}s_0^v d_0^v s_0^v(y)^{-1})\\ =&p_0^h(s_0^v(y)s_0^h(x)s_0^v(y)^{-1}s_0^h(x)^{-1})\\ =&(s_0^h(x)s_0^v(y)s_0^h(x)^{-1}s_0^v(y)^{-1})\\ =&[s_0^h(x),s_0^v(y)].\end{aligned}$$

5. For $\underline{\alpha} = ((0), \emptyset)$ and $\underline{\beta} = ((0), (0))$. The map

$$F_{((0),\emptyset),((0),(0))} : NG_{0,1} \times NG_{0,0} \to NG_{1,1}$$

can be calculated for any $x \in NG_{0,1}$ and $y \in NG_{0,0}$ by

$$\begin{aligned} F_{((0),\emptyset),((0),(0))}(x,y) &= p[-,-](s_{\underline{\alpha}}, s_{\underline{\beta}})(x,y) \\ &= p_0^h p_0^v [s_0^h(x), s_0^h s_0^v(y)] \\ &= 1 \end{aligned}$$

6. Similarly for $\underline{\alpha} = (\emptyset, (0))$ and $\underline{\beta} = ((0), (0))$, the map

$$F_{(\emptyset,(0)),((0),(0))} : NG_{1,0} \times NG_{0,0} \to NG_{1,1}$$

is the identity as given in the previous step.

By taking $(n,m) = (0,2)$ and $(2,0)$, we calculate the possible non identity maps with codomain $NG_{0,2}$ and $NG_{2,0}$ respectively.

First $(n,m) = (0,2)$. Consider the set

$$S(0) \times S(2) = \{(\emptyset, \emptyset), (\emptyset, (0)), (\emptyset, (1)), (\emptyset, (1,0))\}.$$

We try to find the functions $F_{\underline{\alpha},\underline{\beta}}$ with codomain $NG_{0,2}$. In this case the only nonidentity map $F_{\underline{\alpha},\underline{\beta}}$ can be defined by choosing $\underline{\alpha} = (\emptyset, (0))$ and $\underline{\beta} = (\emptyset, (1))$. This is a map from $NG_{0,1} \times NG_{0,1}$ to $NG_{0,2}$. This map is calculated as follows. For $x, y \in NG_{0,1}$, we obtain

$$\begin{aligned} F_{(\emptyset,(0)),(\emptyset,(1))}(x,y) &= p[-,-](s_{\underline{\alpha}}, s_{\underline{\beta}})(x,y) \\ &= p_1^v p_0^v [s_0^v(x), s_1^v(y)] \\ &= [s_0^v x, s_1^v y][s_1^v y, s_1^h x] \in NG_{0,2}. \end{aligned}$$

Now suppose $(n,m) = (2,0)$. From the set

$$S(2) \times S(0) = \{(\emptyset, \emptyset), ((0), \emptyset), ((1), \emptyset), ((1,0), \emptyset)\}$$

we can choose $\underline{\alpha} = ((0), \emptyset)$ and $\underline{\beta} = ((1), \emptyset)$. This map is from $NG_{1,0} \times NG_{1,0}$ to $NG_{2,0}$. This map can be given by for $x, y \in NG_{1,0}$

$$\begin{aligned} F_{((0),\emptyset),((1),\emptyset)}(x,y) &= p[-,-](s_{\underline{\alpha}}, s_{\underline{\beta}})(x,y) \\ &= p_1^h p_0^h [s_0^h(x), s_1^h(y)] \\ &= [s_0^h(x), s_1^h(y)][s_1^h(y), s_1^h(x)] \in NG_{2,0}. \end{aligned}$$

Now, by taking $(n,m) = (1,2)$ and $(2,1)$, we shall define the possible nonidentity maps $F_{\underline{\alpha},\underline{\beta}}$ whose codomain $NG_{1,2}$ and $NG_{2,1}$ respectively.

First suppose that $(n,m)=(1,2)$. We set

$$S(1)\times S(2)=\{(\emptyset,\emptyset),(\emptyset,(1)),(\emptyset,(0)),(\emptyset,(1,0)),((0),\emptyset),((0),(1)),((0),(0)),((0),(1,0))\}.$$

In the following calculations, by taking appropriate $\underline{\alpha},\underline{\beta}$ from the set $S(1)\times S(2)$, we shall give all the non identity maps whose codomain $NG_{1,2}$. To obtain these maps, we can choose the possible $\underline{\alpha},\underline{\beta}$ from the set $S(1)\times S(2)$ as follows:

1. $(\underline{\alpha},\underline{\beta})=((\emptyset,(0)),(\emptyset,(1)))$
2. $(\underline{\alpha},\underline{\beta})=((\emptyset,(1)),((0),\emptyset))$
3. $(\underline{\alpha},\underline{\beta})=((\emptyset,(0)),((0),\emptyset))$
4. $(\underline{\alpha},\underline{\beta})=(((0),(1)),(\emptyset,(0)))$
5. $(\underline{\alpha},\underline{\beta})=(((0),(0)),(\emptyset,(1)))$.

Now we give the functions $F_{\underline{\alpha},\underline{\beta}}$ for these pairings $(\underline{\alpha},\underline{\beta})$.

1. $F_{(\emptyset,(0)),(\emptyset,(1))}:NG_{1,1}\times NG_{1,1}\longrightarrow NG_{1,2}$ can be given by

$$\begin{aligned}F_{(\emptyset,(0)),(\emptyset,(1))}(x,y)=&p[-,-](s_{\underline{\alpha}},s_{\underline{\beta}})(x,y)\\=&p_1^vp_0^vp_0^h[s_0^vx,s_1^vy]\\=&[s_0^v(x),s_1^v(y)][s_1^v(y),s_1^v(x)]\in NG_{1,2}\end{aligned}$$

for $x,y\in NG_{1,1}$.

2. The map $F_{(\emptyset,(1)),((0),\emptyset)}:NG_{1,1}\times NG_{0,2}\longrightarrow NG_{1,2}$ is given by

$$F_{(\emptyset,(1)),((0),\emptyset)}(x,a)=[s_1^v(x),s_0^h(a)]\in NG_{1,2}$$

for $x\in NG_{1,1}$ and $a\in NG_{0,2}$.

3. For $\underline{\alpha}=(\emptyset,(0)),\underline{\beta}=((0),\emptyset),x\in NG_{1,1}$ and $a\in NG_{0,2}$, we have the following map

$$\begin{aligned}F_{(\emptyset,(0)),((0),\emptyset)}:\quad NG_{1,1}\times NG_{0,2}&\longrightarrow NG_{1,2}\\(x,a)&\longmapsto[s_0^v(x),s_0^h(a)]\in NG_{1,2}.\end{aligned}$$

4. For $\underline{\alpha}=((0),(1)),\underline{\beta}=(\emptyset,(0))$, $x\in NG_{0,1}$ and $y\in NG_{1,1}$, we get the following map

$$\begin{aligned}F_{(0),(1)),(\emptyset,(0))}:\quad NG_{0,1}\times NG_{1,1}&\longrightarrow NG_{1,2}\\(x,y)&\longmapsto[s_0^hs_1^v(x),s_0^v(y)]\in NG_{1,2}.\end{aligned}$$

5. For $\underline{\alpha}=((0),(0)),\underline{\beta}=(\emptyset,(1)),x\in NG_{0,1}$ and $y\in NG_{1,1}$, we get the following map

$$\begin{aligned}F_{(0),(0)),(\emptyset,(1))}:\quad NG_{0,1}\times NG_{1,1}&\longrightarrow NG_{1,2}\\(x,y)&\longmapsto[s_0^hs_0^v(x),s_1^v(y)]\in NG_{1,2}.\end{aligned}$$

Now suppose that $(n,m)=(2,1)$. We consider the set $S(2)\times S(1)$. By choosing appropriate $\underline{\alpha},\underline{\beta}$ from the set $S(2)\times S(1)$, we can calculate similarly all the nonidentity maps with codomain $NG_{2,1}$. To obtain these maps, we take the possible $\underline{\alpha},\underline{\beta}$ as follows:

1. $(\underline{\alpha},\underline{\beta}) = (((0),\emptyset),((1),\emptyset,))$ 2. $(\underline{\alpha},\underline{\beta}) = (((1),\emptyset),(\emptyset,(0)))$
3. $(\underline{\alpha},\underline{\beta}) = (((0),\emptyset),(\emptyset,(0)))$ 4. $(\underline{\alpha},\underline{\beta}) = (((1),(0)),((0),\emptyset))$
5. $(\underline{\alpha},\underline{\beta}) = (((0),(0)),((1),\emptyset))$.

For these $(\underline{\alpha},\underline{\beta})$, the corresponding $F_{\underline{\alpha},\underline{\beta}}$ functions can be calculated as follows.

1. For $\underline{\alpha} = ((0),\emptyset)$ and $\underline{\beta} = ((1),\emptyset)$, we obtain the map

$$\begin{aligned} F_{((0),\emptyset),((1),\emptyset)}: \quad NG_{1,1}\times NG_{1,1} &\longrightarrow NG_{2,1} \\ (x,y) &\longmapsto [s_0^h(x),s_1^h(y)][s_1^h(y),s_1^h(x)] \in NG_{2,1}. \end{aligned}$$

2. For $\underline{\alpha} = ((1),\emptyset), \underline{\beta} = (\emptyset,(0))$, we get the map

$$\begin{aligned} F_{((1),\emptyset),(\emptyset,(0))}: \quad NG_{1,1}\times NG_{2,0} &\longrightarrow NG_{2,1} \\ (x,a) &\longmapsto [s_1^h(x),s_0^v(a)] \in NG_{2,1}. \end{aligned}$$

3. For $\underline{\alpha} = ((0),\emptyset), \underline{\beta} = (\emptyset,(0))$, we get the map

$$\begin{aligned} F_{((0),\emptyset),(\emptyset,(0))}: \quad NG_{1,1}\times NG_{2,0} &\longrightarrow NG_{2,1} \\ (x,a) &\longmapsto [s_0^h(x),s_0^v(a)] \in NG_{2,1}. \end{aligned}$$

4. For $\underline{\alpha} = ((1),(0)), \underline{\beta} = ((0),\emptyset)$, we get the following map

$$\begin{aligned} F_{((1),(0)),((0),\emptyset)}: \quad NG_{1,0}\times NG_{1,1} &\longrightarrow NG_{2,1} \\ (x,y) &\longmapsto [s_1^h s_0^v(x),s_0^h(y)] \in NG_{2,1}. \end{aligned}$$

5. For $\underline{\alpha} = ((0),(0)), \underline{\beta} = ((1),\emptyset)$, we get the following map

$$\begin{aligned} F_{((0),(0)),((1),\emptyset)}: \quad NG_{1,0}\times NG_{1,1} &\longrightarrow NG_{2,1} \\ (x,y) &\longmapsto [s_0^h s_0^v(x),s_1^h(y)] \in NG_{2,1}. \end{aligned}$$

Let $(n,m) = (2,2)$. By choosing appropriate $\underline{\alpha},\underline{\beta}$ from the set $S(2)\times S(2)$, we can calculate the nonidentity maps with codomain $NG_{2,2}$. The possible $\underline{\alpha},\underline{\beta}$ are given as follows:

1. $(\underline{\alpha},\underline{\beta}) = (((0),\emptyset),((1),\emptyset))$ 2. $(\underline{\alpha},\underline{\beta}) = (((1),\emptyset),(\emptyset,(0)))$
3. $(\underline{\alpha},\underline{\beta}) = (((0),\emptyset),(\emptyset,(0)))$ 4. $(\underline{\alpha},\underline{\beta}) = (((1),(0)),((0),\emptyset))$
5. $(\underline{\alpha},\underline{\beta}) = (((0),(0)),((1),\emptyset))$, 6. $(\underline{\alpha},\underline{\beta}) = ((\emptyset,(0)),(\emptyset,(1)))$.

For these $(\underline{\alpha},\underline{\beta})$, the corresponding $F_{\underline{\alpha},\underline{\beta}}$ functions can be calculated similarly. These functions are

$F_{((0),\emptyset),((1),\emptyset)}(x,y) = [s_0^h x, s_1^h y][s_1^h y, s_1^h x]$,
$F_{(\emptyset,(0)),(\emptyset,(1))}(x,y) = [s_0^v(x),s_1^v(y)][s_1^v(y),s_1^v(x)]$,
$F_{((1),\emptyset),(\emptyset,(0))}(x,y) = [s_1^h(x),s_0^v(y)]$,
$F_{((0),\emptyset),(\emptyset,(0))}(x,y) = [s_0^h(x),s_0^v(y)]$,
$F_{((0),(1)),(\emptyset,(0))}(x,y) = [s_0^v s_1^h(x),s_0^v(y)]$,
$F_{((0),(0)),(\emptyset,(1))}(x,y) = [s_1^v s_0^h(x),s_1^v(y)]$.