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On q- and h-deformations of 3d-superspaces

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Abstract: In this paper, we introduce nonstandard deformations of (1+2)- and (2+1)-superspaces via a contraction using standard deformations of them. This deformed superspaces are denoted by $\mathbb{A}_h^{1|2}$ and $\mathbb{A}_{h'}^{2|1}$, respectively. We find a two-parameter R-matrix satisfying quantum Yang–Baxter equation and thus obtain a new two-parameter nonstandard deformation of the supergroup $\mathrm{GL}(1|2)$. Finally, we get a new superalgebra derived from the Hopf superalgebra of functions on the quantum superspace $\mathbb{A}_{p,q}^{1|2}$.

Key words: Quantum superspace, Hopf superalgebra, quantum supergroup, quantum Lie superalgebra, super $\star$-algebra

1. Introduction

There are two distinct deformations for general Lie (super)groups as standard and nonstandard (or Jordanian). One of them is the well-known quantum (q-deformed) group and the other is the so-called Jordanian (h-deformed) one. Specially, quantum groups $\mathrm{GL}_q(2)$ [10] and $\mathrm{GL}_h(2)$ [9] have been obtained by deforming the coordinates of a plane to be noncommutative objects. In [1], the authors showed that the h-deformed group can be obtained from the q-deformed Lie group through a singular limit $q \to 1$ of a linear transformation. This method is known as the contraction procedure. Using this method, one- and two-parameter h-deformations of supergroup $\mathrm{GL}(1|1)$ were obtained in [7] and [2], respectively.

In this paper, we give some standard (as q-deformation) deformations of (1+2)-superspace using the Hopf superalgebra structure of $\mathcal{O}(\mathbb{A}^{1|2})$ and nonstandard (as h-deformation) deformations using standard deformations via a contraction. We also introduce an (h,h')-deformed supergroup acting on these two-parameter h-deformed superspaces. Finally, we define involutions on h-deformed superspaces and use the generators of (p,q)-deformed superalgebra $\mathcal{O}(\mathbb{A}_{p,q}^{1|2})$ to get a new Lie superalgebra.

Throughout the paper, we will fix a base field $\mathbb{K}$. The reader may consider it as the set of real numbers, $\mathbb{R}$, or the set of complex numbers, $\mathbb{C}$. We will denote by $\mathbb{G}$ the Grassmann numbers and by $\mathbb{K}'$ the set $\mathbb{K} \cup \mathbb{G}$.

2. On (p,q)-deformation of superspaces $\mathbb{A}^{1|2}$ and $\mathbb{A}^{2|1}$

In order to define superalgebras and Hopf superalgebras, some minor changes are made in familiar definitions. These are briefly mentioned in the following.

A supervector space $\mathcal{X}$ over a field $\mathbb{K}$ is a $\mathbb{Z}_2$-graded vector space $\mathcal{X}$ together with two subspaces $\mathcal{X}_0$ and $\mathcal{X}_1$ of $\mathcal{X}$ such that $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1$. If a space $\mathcal{X}$ is a superspace, then we denote by $\tau(a)$ the $\mathbb{Z}_2$-grade of

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the element $a \in \mathcal{X}$. If $\tau(a) = 0$, then we will call the element a even and if $\tau(a) = 1$, it is called odd.

If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a linear map of supervector spaces and it satisfies

$$\tau(f(v)) = \tau(f) + \tau(v) \pmod 2$$

for all $v \in \mathcal{X}$, then f is called a supervector space homomorphism.

A superalgebra (or $\mathbb{Z}_2$-graded algebra) $\mathcal{A}$ over $\mathbb{K}$ is a supervector space over $\mathbb{K}$ with a map $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ such that $\mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}$ for $i, j = 0, 1$. The superalgebra $\mathcal{A}$ is called supercommutative if

$$ab = (-1)^{\tau(a)\tau(b)} ba$$

for homogeneous elements $a, b \in \mathcal{A}$.

Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a map of definite degree of superalgebras. If it is a supervector space homomorphism and it obeys

$$f(ab) = (-1)^{\tau(a)\tau(f)} f(a) f(b), \quad \forall a, b \in \mathcal{A},$$

then f is called a superalgebra homomorphism.

2.1. The algebra of polynomials on the quantum superspace $\mathbb{A}_q^{1|2}$

Let $\mathbb{K}\langle X, \Theta_1, \Theta_2\rangle$ be a free algebra with unit generated by X, Θ_1, and Θ_2, where the coordinate X is even and the coordinates Θ_1 and Θ_2 are odd.

Definition 2.1 *[11] Let I_q be the two-sided ideal of $\mathbb{K}\langle X, \Theta_1, \Theta_2\rangle$ generated by the elements $X\Theta_1 - q\Theta_1 X$, $X\Theta_2 - q\Theta_2 X$, $\Theta_1\Theta_2 + q^{-1}\Theta_2\Theta_1$, Θ_1^2, and Θ_2^2. The quantum superspace $\mathbb{A}_q^{1|2}$ with the function algebra*

$$\mathcal{O}(\mathbb{A}_q^{1|2}) = \mathbb{K}\langle X, \Theta_1, \Theta_2\rangle / I_q$$

is called $\mathbb{Z}_2$-graded quantum space (or quantum superspace).

This associative algebra over the complex number is known as the algebra of polynomials over quantum (1+2)-superspace. In accordance with the above definition, we have

$$X\Theta_i = q\Theta_i X, \quad \Theta_i\Theta_j = -q^{i-j}\Theta_j\Theta_i, \qquad (i, j = 1, 2) \tag{2.1}$$

where $q \in \mathbb{K} - \{0\}$.

Example 2.2 *If we consider the generators of the algebra $\mathcal{O}(\mathbb{A}_q^{1|2})$ as linear maps, then we can find the matrix representations of them. In fact, it can be seen that there exists a representation $\rho : \mathcal{O}(\mathbb{A}_q^{1|2}) \to M(3, \mathbb{K}')$ such that matrices*

$$\rho(X) = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^2 \end{pmatrix}, \quad \rho(\Theta_1) = \begin{pmatrix} 0 & 0 & \varepsilon_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(\Theta_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.2}$$

representing the coordinate functions satisfy relations (2.1) for all $\varepsilon_1, \varepsilon_2$.

Remark 2.3 *In the next section, we will assume that ε_1 and ε_2 are two Grassmann numbers.*

The following definition gives the product rule for tensor product of $\mathbb{Z}_2$-graded algebras.

Definition 2.4 *The product rule is defined by*

$$(a_1 \otimes a_2)(a_3 \otimes a_4) = (-1)^{\tau(a_2)\tau(a_3)}(a_1 a_3 \otimes a_2 a_4)$$

in the $\mathbb{Z}_2$-graded algebra $\mathcal{A} \otimes \mathcal{A}$, where $\mathcal{A}$ is the $\mathbb{Z}_2$-graded algebra and a_i's are homogeneous elements in $\mathcal{A}$.

A Hopf superalgebra is a supervector space $\mathcal{A}$ over $\mathbb{K}$ with two algebra homomorphisms $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, called the coproduct, $\epsilon : \mathcal{A} \to \mathbb{K}$, called the counit, and an algebra antihomomorphism $S : \mathcal{A} \to \mathcal{A}$, called the antipode, such that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta,$$

$$m \circ (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = m \circ (\mathrm{id} \otimes \epsilon) \circ \Delta,$$

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta,$$

and $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, $\epsilon(\mathbf{1}) = 1$, $S(\mathbf{1}) = \mathbf{1}$, where m is the multiplication map, id is the identity map and $\eta : \mathbb{K} \longrightarrow \mathcal{A}$.

Note. An element of a Hopf superalgebra $\mathcal{A}$ is expressed as a product on the generators and its antipode S is calculated with the property

$$S(ab) = (-1)^{\tau(a)\tau(b)} S(b)S(a), \qquad \forall a, b \in \mathcal{A}.$$

We denote the unital extension of $\mathcal{O}(\mathbb{A}_q^{1|2})$ by $\mathcal{F}(\mathbb{A}_q^{1|2})$ adding the unit and x^{-1}, the inverse of x, which obeys $xx^{-1} = \mathbf{1} = x^{-1}x$. The following theorem says that the superalgebra $\mathcal{F}(\mathbb{A}_q^{1|2})$ has a Hopf algebra structure [4]:

Theorem 2.5 *[4] The superalgebra $\mathcal{F}(\mathbb{A}_q^{1|2})$ is a Hopf superalgebra with the defining coproduct, counit, and antipode on the algebra $\mathcal{F}(\mathbb{A}_q^{1|2})$ as follows:*

(1) *The coproduct $\Delta : \mathcal{F}(\mathbb{A}_q^{1|2}) \longrightarrow \mathcal{F}(\mathbb{A}_q^{1|2}) \otimes \mathcal{F}(\mathbb{A}_q^{1|2})$ is defined by*

$$\Delta(X) = X \otimes X, \quad \Delta(\Theta_1) = \Theta_1 \otimes X + X \otimes \Theta_1, \quad \Delta(\Theta_2) = \Theta_2 \otimes X^2 + X^2 \otimes \Theta_2. \tag{2.3}$$

(2) *The counit $\epsilon : \mathcal{F}(\mathbb{A}_q^{1|2}) \longrightarrow \mathbb{K}$ is given by*

$$\epsilon(X) = 1, \quad \epsilon(\Theta_i) = 0, \qquad (i = 1, 2).$$

(3) *The algebra $\mathcal{F}(\mathbb{A}_q^{1|2})$ admits a $\mathbb{K}$-algebra antihomomorphism (antipode) $S : \mathcal{F}(\mathbb{A}_q^{1|2}) \longrightarrow \mathcal{F}(\mathbb{A}_{q^{-1}}^{1|2})$ defined by*

$$S(X) = X^{-1}, \quad S(\Theta_1) = -X^{-1}\Theta_1 X^{-1}, \quad S(\Theta_2) = -X^{-2}\Theta_2 X^{-2}.$$

2.2. The algebra of polynomials on the quantum superspace $\mathbb{A}_{p,q}^{2|1}$

Let $\mathbb{K}\langle \Phi, Y_1, Y_2 \rangle$ be a free algebra with unit generated by Φ, Y_1 and Y_2, where $\tau(\Phi) = 1$ and $\tau(Y_1) = 0 = \tau(Y_2)$.

Definition 2.6 *[5] Let* $\Lambda(\mathbb{A}_q^{1|2})$ *be the algebra with the generators* Φ, Y_1, *and* Y_2 *satisfying the relations*

$$\Phi^2 = 0, \quad \Phi Y_1 = qp^{-1} Y_1 \Phi, \quad \Phi Y_2 = pq Y_2 \Phi, \quad Y_1 Y_2 = pq^{-1} Y_2 Y_1. \tag{2.4}$$

We call $\Lambda(\mathbb{A}_q^{1|2})$ *exterior algebra of the* $\mathbb{Z}_2$*-graded space* $\mathbb{A}_q^{1|2}$.

Remark 2.7 *The exterior algebra* $\Lambda(\mathbb{A}_q^{1|2})$ *of the superspace* $\mathbb{A}_q^{1|2}$ *can be thought of as a two-parameter deformation of the (2+1)-superspace* $\mathbb{A}^{2|1}$. *Thus, we denote this algebra by* $\mathcal{O}(\mathbb{A}_{p,q}^{2|1})$.

Example 2.8 *If we consider the generators of the algebra* $\mathcal{O}(\mathbb{A}_{p,q}^{2|1})$ *as linear maps, then we can find the matrix representations of them. In fact, it can be seen that there exists a representation* $\rho : \mathcal{O}(\mathbb{A}_{p,q}^{2|1}) \to M(3, \mathbb{K}')$ *such that matrices*

$$\rho(\Phi) = \begin{pmatrix} 0 & 0 & \epsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(Y_1) = \begin{pmatrix} q & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}, \quad \rho(Y_2) = \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

representing the coordinate functions satisfy relations (2.4) *for all* c, ε.

3. Two-parameter h-deformation of the superspaces

In this section, we introduce a two-parameter h-deformation of the superspace $\mathbb{A}^{1|2}$ (and its dual) from the (p,q)-deformation via a contraction similar to the method of [1].

We consider the q-deformed algebra of functions on the quantum superspace $\mathbb{A}_q^{1|2}$ generated by X, Θ_1, and Θ_2 with the relations (2.1) and we introduce new even coordinate x and odd coordinates θ_1, θ_2 with the change of basis in the coordinates of the q-superspace using the following g matrix:

$$\mathbf{X} = \begin{pmatrix} X \\ \Theta_1 \\ \Theta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tilde{h}' \\ 0 & 1 & 0 \\ \tilde{h} & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \theta_1 \\ \theta_2 \end{pmatrix} = g\,\mathbf{x}, \quad \tilde{h} = \frac{h}{q-1}, \quad \tilde{h}' = \frac{h'}{pq-1} \tag{3.1}$$

where h and h' ($h \neq 0 \neq h'$) are two new deformation parameters that will be replaced with q and p ($q \neq 1 \neq pq$) in the limits $q \to 1$ and $p \to 1$.

We now assume that the parameters h and h' are both Grassmann numbers ($h^2 = 0 = h'^2$, $hh' = -h'h$) and anticommute with θ_i for $i = 1, 2$. When the relations (2.1) are used, one gets

$$x\theta_1 = q\theta_1 x, \quad x\theta_2 = q\theta_2 x + hx^2, \quad \theta_2\theta_1 = -q\theta_1\theta_2, \quad \theta_1^2 = 0, \quad \theta_2^2 = -h\theta_2 x. \tag{3.2}$$

Note that the parameter h' does not enter the above relations. By taking the limit $q \to 1$, we obtain the following exchange relations, which define the h-superspace $\mathbb{A}_h^{1|2}$:

Definition 3.1 *[4] Let* $\mathcal{O}(\mathbb{A}_h^{1|2})$ *be the algebra with the generators* x, θ_1, *and* θ_2 *satisfying the relations*

$$x\theta_1 = \theta_1 x, \quad x\theta_2 = \theta_2 x + hx^2, \quad \theta_1\theta_2 = -\theta_2\theta_1, \quad \theta_1^2 = 0, \quad \theta_2^2 = -h\theta_2 x. \tag{3.3}$$

We call $\mathcal{O}(\mathbb{A}_h^{1|2})$ *the algebra of functions on the* $\mathbb{Z}_2$*-graded quantum space* $\mathbb{A}_h^{1|2}$.

Example 3.2 *Let us assume that ε_1 and ε_2 are two Grassmann numbers. If the g matrix in (3.1) is used, the matrix representation in (2.2) takes the following form:*

$$\rho(x) = q\begin{pmatrix} 1-\tilde{h}\tilde{h}' & 0 & 0 \\ 0 & 1-\tilde{h}\tilde{h}' & -q^{-1}\tilde{h}'\varepsilon_2 \\ 0 & 0 & q(1-\tilde{h}\tilde{h}') \end{pmatrix}, \quad \rho(\theta_1) = \begin{pmatrix} 0 & 0 & \varepsilon_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(\theta_2) = -\begin{pmatrix} q\tilde{h} & 0 & 0 \\ 0 & q\tilde{h} & -(1+\tilde{h}\tilde{h}')\varepsilon_2 \\ 0 & 0 & q^2\tilde{h} \end{pmatrix}. \tag{3.4}$$

These matrices satisfy the relations (3.2), for all ε_1 and ε_2.

Proof Existing claims come from the fact that ρ is an algebra homomorphism. □

In the case of dual (exterior) h'-superspace, we use the transformation

$$\hat{\mathbf{X}} = g\hat{\mathbf{x}} \tag{3.5}$$

with the components φ, y_1, and y_2 of $\hat{\mathbf{x}}$. The definition is given below.

Definition 3.3 *Let $\mathcal{O}(\mathbb{A}_{h'}^{2|1}) := \Lambda(\mathbb{A}_h^{1|2})$ be the algebra with the generators φ, y_1, and y_2 satisfying the relations*

$$\varphi y_1 = y_1\varphi, \quad \varphi y_2 = y_2\varphi + h' y_2^2, \quad y_1 y_2 = y_2 y_1, \quad \varphi^2 = h' y_2 \varphi \tag{3.6}$$

where $\tau(\varphi) = 1$ and $\tau(y_1) = 0 = \tau(y_2)$. We call $\Lambda(\mathbb{A}_h^{1|2})$ the quantum exterior algebra of the $\mathbb{Z}_2$-graded quantum space $\mathbb{A}_h^{1|2}$.

Remark 3.4 *The parameter h does not enter the relations (3.6). The exterior algebra $\Lambda(\mathbb{A}_h^{1|2})$ of the superspace $\mathbb{A}_h^{1|2}$ can be thought of as an h'-deformation of the (2+1)-superspace $\mathbb{A}^{2|1}$.*

4. An R-matrix and its properties

The relations in (2.1) can be written in a compact form as follows:

$$p\,\mathbf{X}\otimes\mathbf{X} = \hat{R}_{p,q}\,\mathbf{X}\otimes\mathbf{X} \tag{4.1}$$

with an R-matrix given by [6]

$$\hat{R}_{p,q} = \begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p-1 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & pq & 0 & 0 \\ 0 & pq^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -pq^{-1} & 0 \\ 0 & 0 & q^{-1} & 0 & 0 & 0 & p-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q & 0 & p-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

where $p, q \in \mathbb{K} - \{0\}$. This matrix satisfies the graded braid equation and the matrix $R_{p,q} = P\hat{R}_{p,q}$ satisfies the graded Yang–Baxter equation where P is the super permutation matrix.

It can be considered that a change of basis in the quantum superspaces leads to a two-parameter R-matrix. The corresponding R-matrix can be obtained as

$$\hat{R}_{h,h'} = \lim_{(p,q)\to(1,1)} \left[(g\otimes g)^{-1}\hat{R}_{p,q}(g\otimes g)\right]$$

where it is assumed that $\otimes$ is graded. As a result, we obtain the following R-matrix

$$\hat{R}_{h,h'} = \begin{pmatrix} 1+hh' & 0 & h' & 0 & 0 & 0 & -h' & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ h & 0 & hh' & 0 & 0 & 0 & 1 & 0 & -h' \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -h & 0 & 1 & 0 & 0 & 0 & hh' & 0 & -h' \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -h & 0 & 0 & 0 & -h & 0 & hh'-1 \end{pmatrix}.$$

The equation in (4.1) with the new R-matrix $\hat{R}_{h,h'}$ takes the form

$$\mathbf{x}\otimes\mathbf{x} = \hat{R}_{h,h'}\,\mathbf{x}\otimes\mathbf{x},$$

that is, the relations (3.3) are equivalent to this equation.

The R-matrix $\hat{R}_{h,h'}$ has some interesting properties. Some of them are listed below, where sometimes we write $\hat{R} = \hat{R}_{h,h'}$ for simplicity.

1. The matrix $\hat{R}_{h,h'}$ satisfies the graded braid equation $\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$, where $\hat{R}_{12} = \hat{R}\otimes I_3$ and $\hat{R}_{12} = I_3\otimes\hat{R}$.

2. The matrix $R_{h,h'} = P\hat{R}_{h,h'}$ satisfies the graded Yang–Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, where R_{13} acts both on the first and third spaces.

3. The matrix $\hat{R}_{h,h'}$ holds $\hat{R}^2_{h,h'} = I_9$; thus, it has two eigenvalues ± 1.

4. If we set $hh' = 0$, then the matrix $R_{h,h'}$ can be decomposed in the form

$$R_{h,h'} = R(h)R(h')$$

where

$$R(h) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -h & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ h & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & h & 0 & 0 & 0 & h & 0 & 1 \end{pmatrix}, \quad R(h') = R^{\mathrm{st}}(h)|_{h=h'}.$$

It can be checked that these matrices satisfy the graded Yang–Baxter equation.

5. If $P_{\pm}$ are the projections onto the eigenspaces ± 1 of $\hat{R}_{h,h'}$, then we have

$$\hat{R}_{h,h'} = P_+ - P_-.$$

Let $\mathcal{O}(\mathbb{A}^{1|2})$ and $\mathcal{O}(\mathbb{A}^{2|1})$ be the quotients of algebras generated by x, θ_1, θ_2 and φ, y_1, y_2 modulo the two-sided ideals generated by KerP_- and KerP_+, respectively. Then $\mathcal{O}(\mathbb{A}^{1|2})$ and $\mathcal{O}(\mathbb{A}^{2|1})$ are isomorphic to $\mathcal{O}(\mathbb{A}_h^{1|2})$ with defining relations (3.3) and $\mathcal{O}(\mathbb{A}_{h'}^{2|1})$ with defining relations (3.6), respectively. That is, we can write

$$P_- \, \mathbf{x} \otimes \mathbf{x} = 0 \quad \text{and} \quad (-1)^{\tau(\hat{\mathbf{x}})} \, P_+ \, \hat{\mathbf{x}} \otimes \hat{\mathbf{x}} = 0.$$

5. The quantum superbialgebra $\mathcal{O}(\mathrm{M}_{h,h'}(1|2))$

Let T be a 3x3 matrix in $\mathbb{Z}_2$-graded space given by

$$T = \begin{pmatrix} a & \alpha & \beta \\ \gamma & b & c \\ \delta & d & e \end{pmatrix} = (t_{ij})$$

where a, b, c, d, e are even and α, β, γ, and δ are odd. The coordinate ring of such matrices over a field $\mathbb{K}$ is simply the polynomial ring in nine variables, that is $\mathcal{O}(\mathrm{M}(1|2)) = \mathbb{K}[a, b, c, d, e, \alpha, \beta, \gamma, \delta]$.

In this section, we will assume that the matrix entries of T belong to a free superalgebra and define a two-parameter h-analogue of $\mathcal{O}(\mathrm{M}(1|2))$. To do so, let x, θ_1, θ_2 be elements of the superalgebra $\mathcal{O}(\mathbb{A}_h^{1|2})$ subject to the relations (3.3) and φ, y_1, y_2 be elements of $\mathcal{O}(\mathbb{A}_{h'}^{2|1})$ subject to the relations (3.6), and t_{ij} be nine generators which supercommute with the elements of $\mathcal{O}(\mathbb{A}_h^{1|2})$ and $\mathcal{O}(\mathbb{A}_{h'}^{2|1})$. It is well known that the supermatrix T defines the linear transformations $T : \mathbb{A}_h^{1|2} \longrightarrow \mathbb{A}_h^{1|2}$ and $T : \mathbb{A}_{h'}^{2|1} \longrightarrow \mathbb{A}_{h'}^{2|1}$. Let $\mathbf{x} = (x, \theta_1, \theta_2)^t$ and $\hat{\mathbf{x}} = (\varphi, y_1, y_2)^t$. Thus, we can give the following theorem.

Theorem 5.1 *Under the above hypotheses, the following conditions are equivalent:*

(i) $T\mathbf{x} = \mathbf{x}' \in \mathbb{A}_h^{1|2}$ *and* $T\hat{\mathbf{x}} = \hat{\mathbf{x}}' \in \mathbb{A}_{h'}^{2|1}$,

(ii) the relations are satisfied

$$
\begin{aligned}
&a\alpha = (1+hh')\alpha a - h'(\alpha\delta + da), \quad a\beta = \beta a + h'(a^2 - ea - \beta\delta) - h\beta^2,\\
&a\gamma = (1+hh')\gamma a + h(\gamma\beta - ca), \quad ac = ca - hc\beta - h'\gamma a + hh'\gamma\beta,\\
&a\delta = \delta a + h(a^2 - ea + \delta\beta) + h'\delta^2, \quad ad = da + h\alpha a + h'd\delta - hh'\alpha\delta,\\
&ae = ea + h\beta(a-e) + h'(e-a)\delta, \quad \alpha\beta = -(1+hh')\beta\alpha + h'(\beta d + e\alpha),\\
&\alpha\gamma = -\gamma\alpha, \quad \alpha c = c\alpha, \quad \alpha\delta = -\delta\alpha - ha\alpha + h'\delta d - hh'ad,\\
&\alpha d = d\alpha + h'd^2, \quad \alpha e = e\alpha + h\beta\alpha + h'ed - hh'd\beta,\\
&\beta\gamma = -\gamma\beta + hc\beta - h'\gamma a - hh'ca, \quad \beta c = (1-hh')c\beta - h'(\gamma\beta + ca),\\
&\beta\delta = -\delta\beta + (h\beta + h'\delta)(e-a), \quad \beta d = d\beta + h\alpha\beta + h'de - hh'e\alpha,\\
&\beta e = e\beta + h'(e^2 - ea - \delta\beta) - h\beta^2, \quad \gamma c = c\gamma + hc^2,\\
&\gamma\delta = -(1+hh')\delta\gamma + h(e\gamma + \delta c), \quad \gamma d = d\gamma,\\
&\gamma e = e\gamma + hec - h'\delta\gamma - hh'c\delta, \quad c\delta = \delta c - hec - h'\delta\gamma - hh'\gamma e,\\
&cd = dc, \quad ce = (1-hh')ec + h'(e\gamma - \delta c), \ \delta d = (1-hh')d\delta + h(\alpha\delta - da),\\
&\delta e = e\delta + h(e^2 - ea + \beta\delta) + h'\delta^2, \quad de = (1-hh')ed + h(\beta d - e\alpha),\\
&\alpha^2 = h'\alpha d, \quad \beta^2 = h'\beta(e-a), \quad \gamma^2 = h\gamma c, \quad \delta^2 = h\delta(e-a),\\
&b\,t_{ij} = t_{ij}b, \quad a(h\beta + h'\delta) = (h\beta + h'\delta)a, \quad e(h\beta + h'\delta) = (h\beta + h'\delta)e.
\end{aligned} \tag{5.1}
$$

Proof A direct verification shows that the relations (5.1) respect the ideals defining $\mathbb{A}_h^{1|2}$ and $\mathbb{A}_{h'}^{2|1}$. □

Standard FRT construction [8], namely, the relations (5.1), is obtained via the matrix $\hat{R}_{h,h'}$ given in Section 4:

Theorem 5.2 *A 3x3-matrix T is a $\mathbb{Z}_2$-graded quantum supermatrix if and only if*

$$\hat{R}_{h,h'}T_1T_2 = T_1T_2\hat{R}_{h,h'}$$

where $T_1 = T \otimes I_3$ and $T_2 = PT_1P$.

Definition 5.3 *The superalgebra $\mathcal{O}(\mathrm{M}_{h,h'}(1|2))$ is the quotient of the free algebra $\mathbb{K}\{a,b,c,d,e,\alpha,\beta,\gamma,\delta\}$ by the two-sided ideal $J_{h,h'}$ generated by the relations (5.1) of* Theorem 5.1*.*

Remark 5.4 *The quantum matrix space $\mathrm{M}_{p,q}(1|2)$ is obtained in [6]. It is clear that a change of basis in the quantum superspace leads to the similarity transformation $T = g^{-1}T'g$, where $T' \in \mathrm{M}_{p,q}(1|2)$. Therefore, the entries of the transformed quantum matrix T fulfill the commutation relations (5.1) of the matrix elements of the matrix T in $\mathrm{M}(1|2)$.*

Theorem 5.5 *The superalgebra $\mathcal{O}(\mathrm{M}_{h,h'}(1|2))$ with the following two algebra homomorphisms of superalgebras*
(1) the coproduct $\Delta : \mathcal{O}(\mathrm{M}_{h,h'}(1|2)) \longrightarrow \mathcal{O}(\mathrm{M}_{h,h'}(1|2)) \otimes \mathcal{O}(\mathrm{M}_{h,h'}(1|2))$ determined by $\Delta(t_{ij}) = \sum_{k=1}^{3} t_{ik} \otimes t_{kj}$,

(2) *the counit* $\epsilon : \mathcal{O}(\mathrm{M}_{h,h'}(1|2)) \longrightarrow \mathbb{K}$ *determined by* $\epsilon(t_{ij}) = \delta_{ij}$
becomes a super bialgebra.

Proof It can be easily checked the properties of the costructures hold:
(i) The coproduct Δ is coassociative in the sense of

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

where id denotes the identity map on $\mathrm{M}_{h,h'}(1|2)$ and $\Delta(ab) = \Delta(a)\Delta(b)$, $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$.
(ii) The counit ϵ has the property

$$m \circ (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = m \circ (\mathrm{id} \otimes \epsilon) \circ \Delta$$

where m stands for the algebra product and $\epsilon(ab) = \epsilon(a)\epsilon(b)$, $\epsilon(\mathbf{1}) = 1$. □

It is well known that $\mathcal{O}(\mathbb{A}^{1|2})$ is comodule algebra over the bialgebra $\mathcal{O}(\mathrm{M}(1|2))$. The following theorem gives a quantum version of this fact.

Theorem 5.6 *There exist algebra homomorphisms*

$$\delta_L : \mathcal{O}(\mathbb{A}_h^{1|2}) \longrightarrow \mathcal{O}(\mathrm{M}_{h,h'}(1|2)) \otimes \mathcal{O}(\mathbb{A}_h^{1|2}), \quad \delta_L(x_i) = \sum_{k=1}^{3} t_{ik} \otimes x_k,$$

$$\tilde{\delta}_L : \mathcal{O}(\mathbb{A}_{h'}^{2|1}) \longrightarrow \mathcal{O}(\mathrm{M}_{h,h'}(1|2)) \otimes \mathcal{O}(\mathbb{A}_{h'}^{2|1}), \quad \tilde{\delta}_L(\hat{x}_i) = \sum_{k=1}^{3} t_{ik} \otimes \hat{x}_k$$

where $x_i \in \{x, \theta_1, \theta_2\}$ *and* $\hat{x}_i \in \{\varphi, y_1, y_2\}$.

Proof Using the relations (3.3) and (3.6) together with (5.1), it is enough to check that

$$\delta_L(x\theta_1 - \theta_1 x) = \delta_L(x)\delta_L(\theta_1) - \delta_L(\theta_1)\delta_L(x) = 0,$$

etc., in $\mathcal{O}(\mathrm{M}_{h,h'}(1|2)) \otimes \mathcal{O}(\mathbb{A}_h^{1|2})$. To see that δ_L defines a comodule structure we check that

$$(\Delta \otimes \mathrm{id}) \circ \delta_L = (\mathrm{id} \otimes \delta_L) \circ \delta_L, \quad m \circ (\epsilon \otimes \mathrm{id}) \circ \delta_L = \mathrm{id}.$$

□

A quantum supergroup (Hopf superalgebra) can be regarded as a generalization of the notion of a supergroup. It is defined by

$$\mathcal{O}(\mathrm{GL}_{h,h'}(1|2)) = \mathcal{O}(\mathrm{M}_{h,h'}(1|2))[t]/(t\,\mathrm{sdet}_{h,h'} - 1).$$

This case is also inviting to generalize the corresponding notions of differential geometry [12]. A differential calculus on $\mathcal{O}(\mathrm{GL}_{h,h'}(1|2))$ will be discussed in the next work.

6. A Lie superalgebra derived from $\mathcal{F}(\mathbb{A}_{p,q}^{1|2})$

It is known that an element of a Lie group can be represented by exponential of an element of its Lie algebra. In [3], by virtue of this fact, using the generators of the superalgebra $\mathcal{F}(\mathbb{A}_q^{1|1})$, a new superalgebra is obtained from this algebra. In this section, we will obtain a new superalgebra from $\mathcal{F}(\mathbb{A}_{p,q}^{1|2})$. Thus, let us begin with the definition of $\mathcal{F}(\mathbb{A}_{p,q}^{1|2})$ which is an extension to two parameters of $\mathcal{F}(\mathbb{A}_{q}^{1|2})$.

Definition 6.1 *Let $I_{p,q}$ be the two-sided ideal of $\mathbb{K}\langle X,\Theta_1,\Theta_2\rangle$ generated by the elements $X\Theta_1 - q\Theta_1 X$, $X\Theta_2 - p\Theta_2 X$, $\Theta_1\Theta_2 + pq^{-2}\Theta_2\Theta_1$, Θ_1^2, and Θ_2^2. The quantum superspace $\mathbb{A}_{p,q}^{1|2}$ with the function algebra*

$$\mathcal{O}(\mathbb{A}_{p,q}^{1|2}) = \mathbb{K}\langle X,\Theta_1,\Theta_2\rangle / I_{p,q}$$

is called quantum superspace.

In accordance with this definition, we have

$$X\Theta_1 = q\Theta_1 X, \quad X\Theta_2 = p\Theta_2 X, \quad \Theta_1\Theta_2 = -pq^{-2}\Theta_2\Theta_1, \quad \Theta_i^2 = 0 \tag{6.1}$$

where $p,q \in \mathbb{K}-\{0\}$.

Example 6.2 *If we consider the generators of the algebra $\mathcal{O}(\mathbb{A}_{p,q}^{1|2})$ as linear maps, then we can find the matrix representations of them. In fact, it can be seen that there exists a representation $\rho : \mathcal{O}(\mathbb{A}_{p,q}^{1|2}) \to M(3,\mathbb{K}')$ such that matrices*

$$\rho(X) = \begin{pmatrix} q & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & pq \end{pmatrix}, \quad \rho(\Theta_1) = \begin{pmatrix} 0 & 0 & \varepsilon_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(\Theta_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 \\ 0 & 0 & 0 \end{pmatrix}$$

representing the coordinate functions satisfy relations (6.1) for all $\varepsilon_1, \varepsilon_2$.

Let $\mathbb{K}\langle u,\xi_1,\xi_2\rangle$ be a free algebra generated by u, ξ_1, ξ_2, where $\tau(u) = 0$, $\tau(\xi_1) = 1 = \tau(\xi_2)$. Let $\mathcal{L}$ be the quotient of the free algebra $\mathbb{K}\langle u,\xi_1,\xi_2\rangle$ by the two-sided ideal J_0 generated by the elements $u\xi_k - \xi_k u$, $\xi_1\xi_2 + \xi_2\xi_1$, ξ_k^2 for $k = 1,2$.

Now, we will show that the Hopf superalgebra of Theorem 2.5 can be embedded into the enveloping superalgebra of a Lie superalgebra, with Lie structure and a deformed coproduct. Thus, let us define the generators of the algebra $\mathcal{F}(\mathbb{A}_{p,q}^{1|2})$ as

$$X := e^u, \quad \Theta_k := e^{ku}\xi_k,$$

for $k = 1,2$. The first equality implies that the generator X is invertible. Then, by direct calculations we can prove the following lemma.

Lemma 6.3 *The generators u, ξ_1, ξ_2 have the following commutation relations (Lie (anti-)brackets), for $j,k = 1,2$*

$$[u,\xi_k] = \mathbf{i}\,\hbar_k\,\xi_k, \quad [\xi_j,\xi_k]_+ = 0, \tag{6.2}$$

where $q = e^{\mathbf{i}\,\hbar_1}$, $p = e^{\mathbf{i}\,\hbar_2}$ with $\mathbf{i} = \sqrt{-1}$ and $\hbar_1,\hbar_2 \in \mathbb{R}$.

We denote the algebra for which the generators obey the relations (6.2) by $\mathcal{L}_{\hbar_1,\hbar_2} := \mathcal{L}(\mathbb{A}_{p,q}^{1|2})$. Let $U(\mathcal{L}_{\hbar_1,\hbar_2})$ be the algebra defined by (6.2). The Hopf superalgebra structure of $U(\mathcal{L}_{\hbar_1,\hbar_2})$ can be read off from Theorem 2.5:

Theorem 6.4 *The superalgebra $U(\mathcal{L}_{\hbar_1,\hbar_2})$ is a Hopf superalgebra with coproduct, counit, and antipode on the algebra $\mathcal{L}_{\hbar_1,\hbar_2}$ defined by*

$$\Delta(u_i) = u_i \otimes \mathbf{1} + \mathbf{1} \otimes u_i, \qquad \epsilon(u_i) = 0, \qquad S(u_i) = -u_i.$$

for $u_i \in \{u, \xi_1, \xi_2\}$.

Example 6.5 *There exists a Lie algebra homomorphism μ from $\mathcal{L}_{\hbar_1,\hbar_2}$ into $\mathrm{M}(3, \mathbb{K}')$.*

Proof We see that there exists an algebra homomorphism ρ from $\mathcal{F}(\mathbb{A}_{p,q}^{1|2})$ into $\mathrm{M}(3, \mathbb{K}')$ such that the relations (6.1) hold. As a consequence of this fact, there exists a Lie algebra homomorphism μ from $\mathcal{L}_{\hbar_1,\hbar_2}$ into $\mathrm{M}(3, \mathbb{K}')$. The action of μ on the generators of $\mathcal{L}_{\hbar_1,\hbar_2}$ is of the form

$$\mu(u) = \begin{pmatrix} \mathbf{i}\hbar_2 & 0 & 0 \\ 0 & \mathbf{i}\hbar_1 & 0 \\ 0 & 0 & \mathbf{i}(\hbar_1 + \hbar_2) \end{pmatrix}, \quad \mu(\xi_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^{-\mathbf{i}(\hbar_1+\hbar_2)}\varepsilon_1 & 0 & 0 \end{pmatrix}, \quad \mu(\xi_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e^{-2\mathbf{i}(\hbar_1+\hbar_2)}\varepsilon_2 & 0 \end{pmatrix} \tag{6.3}$$

where ε_1 and ε_2 are two Grassmann numbers. To see that the relations (6.2) are preserved under the action of μ, we use the fact that

$$\mu[a, b] = [\mu(a), \mu(b)],$$

for all $a, b \in \mathcal{L}_{\hbar_1,\hbar_2}$. □

7. ⋆-Structures on the algebras $\mathcal{O}(\mathbb{A}_h^{1|2})$ and $\mathcal{O}(\mathbb{A}_{h'}^{2|1})$

It is possible to define the star operation (or involution) on the Grassmann generators. However, there are two possibilities to do so*. If α and β are two Grassmann generators and λ is a complex number and $\bar{\lambda}$ its complex conjugate, the star operation, denoted by $\star$, is defined by

$$(\lambda\alpha)^\star = \bar{\lambda}\alpha^\star, \quad (\alpha\beta)^\star = \beta^\star\alpha^\star, \quad (\alpha^\star)^\star = \alpha$$

and the superstar operation, denoted by $\#$, is defined by

$$(\lambda\alpha)^\# = \bar{\lambda}\alpha^\#, \quad (\alpha\beta)^\# = \alpha^\#\beta^\#, \quad (\alpha^\#)^\# = -\alpha.$$

It is easily shown that there exists a star operation on the algebra $\mathcal{O}(\mathbb{A}_q^{1|2})$ if q is a complex number of modulus one:

Proposition 7.1 *(i) If $\bar{q} = q^{-1}$ then the algebra $\mathcal{O}(\mathbb{A}_q^{1|2})$ equipped with the involution determined by*

$$X^\star = X, \quad \Theta_i^\star = \Theta_i \qquad (i = 1, 2) \tag{7.1}$$

becomes a $\star$-algebra.

(ii) If $\bar{p} = p^{-1}$ and $\bar{q} = q^{-1}$ then the algebra $\mathcal{O}(\mathbb{A}_{p,q}^{2|1})$ equipped with the involution determined by

$$\Phi^\star = \Phi, \quad Y_i^\star = -Y_i \qquad (i = 1, 2) \tag{7.2}$$

becomes a $\star$-algebra.

*arXiv.org e-Print archive (1996). Dictionary on Lie Superalgebras [online]. Website https://arxiv.org/abs/hep-th/9607161 [18 July 1996].

7.1. $\star$-Structures on the algebra $\mathcal{O}(\mathbb{A}_h^{1|2})$

As noted in Section 3, the relations in (3.3) do not include the parameter h'. Thus, we can rearrange the change of basis in the coordinates (see, equation (3.1)) as

$$\begin{pmatrix} X \\ \Theta_1 \\ \Theta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{h}{q-1} & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \theta_1 \\ \theta_2 \end{pmatrix}. \tag{7.3}$$

This case can help us to define a star operation on the algebra $\mathcal{O}(\mathbb{A}_h^{1|2})$ by a coordinate transformation using the generators of the algebra $\mathcal{O}(\mathbb{A}_q^{1|2})$ and to prove the following lemma.

Lemma 7.2 *For a certain special choice of h, there exists an involution on the algebra $\mathcal{O}(\mathbb{A}_h^{1|2})$.*

Proof Using the equation (7.3), we introduce the coordinates x, θ_1, and θ_2 with the change of basis in the coordinates of the superspace $\mathbb{A}_q^{1|2}$ as follows:

$$x = X, \quad \theta_1 = \Theta_1, \quad \theta_2 = \Theta_2 - \frac{h}{q-1} X.$$

Then, with $|q| = 1$ and (7.1)

$$\theta_2^\star = \Theta_2^\star - \frac{q\bar{h}}{1-q} X^\star = \theta_2 + \frac{h + q\bar{h}}{q-1} x$$

so that, if we demand that $\bar{h} = -h$, we obtain $\theta_2^\star = \theta_2 - hx$. Note that

$$(x^\star)^\star = x, \quad (\theta_1^\star)^\star = \theta_1, \quad (\theta_2^\star)^\star = \theta_2,$$

for all h. □

Proposition 7.3 *If $\bar{h} = -h$, then the algebra $\mathcal{O}(\mathbb{A}_h^{1|2})$ supplied with the involution determined by*

$$x^\star = x, \quad \theta_1^\star = \theta_1, \quad \theta_2^\star = \theta_2 - hx \tag{7.4}$$

becomes a $\star$-algebra.

Proof Since $\bar{h} = -h$, we have

$$\begin{aligned}
(x\theta_1 - \theta_1 x)^\star &= \theta_1 x - x\theta_1, \\
(x\theta_2 - \theta_2 x - hx^2)^\star &= (\theta_2 - hx)x - x(\theta_2 - hx) + hx^2 = (\theta_2 x - x\theta_2 + hx^2), \\
(\theta_1\theta_2 + \theta_2\theta_1)^\star &= (\theta_2 - hx)\theta_1 + \theta_1(\theta_2 - hx) = \theta_2\theta_1 + \theta_1\theta_2, \\
(\theta_2^2 + h\theta_2 x)^\star &= (\theta_2 - hx)(\theta_2 - hx) + x(\theta_2 - hx)(-h) = \theta_2^2 + h\theta_2 x.
\end{aligned}$$

Hence, the ideal $(x\theta_1 - \theta_1 x,\, x\theta_2 - \theta_2 x - hx^2,\, \theta_1\theta_2 + \theta_2\theta_1,\, \theta_1^2,\, \theta_2^2 + h\theta_2 x)$ is $\star$-invariant and the quotient algebra

$$\mathbb{K}\langle x, \theta_1, \theta_2\rangle / (x\theta_1 - \theta_1 x,\, x\theta_2 - \theta_2 x - hx^2,\, \theta_1\theta_2 + \theta_2\theta_1,\, \theta_1^2,\, \theta_2^2 + h\theta_2 x)$$

becomes a $\star$-algebra. □

Remark 7.4 *Of course, we can consider the change of basis in the coordinates of the superspace* $\mathbb{A}_q^{1|2}$ *in (3.1). In this case, since*

$$\begin{aligned} x^\star &= (1 + \tilde{h}\bar{\tilde{h}}' - \overline{\tilde{h}\tilde{h}'})x + (\tilde{h}' - \bar{\tilde{h}}')\theta_2, \\ \theta_1^\star &= \theta_1, \\ \theta_2^\star &= (1 - \bar{\tilde{h}}(\tilde{h}' - \bar{\tilde{h}}'))\theta_2 + (\tilde{h} - \bar{\tilde{h}})x, \end{aligned}$$

we have again (7.4) with the choices $\bar{h} = -h$ *and* $\bar{h}' = h'$.

7.2. $\star$-Structure on the algebra $\mathcal{O}(\mathbb{A}_{h'}^{2|1})$

Since the relations in (3.6) do not include the parameter h, we can rearrange the change of basis in the coordinates (see, equation (3.1)) as

$$\begin{pmatrix} \Phi \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{h'}{pq-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi \\ y_1 \\ y_2 \end{pmatrix}. \tag{7.5}$$

There exists a special case, where the algebra $\mathcal{O}(\mathbb{A}_{h'}^{2|1})$ admits an involution. The proofs of the following lemma and proposition can be done in a similar way to Lemma 7.2 and Proposition 7.3.

Lemma 7.5 *If* $\bar{h}' = h'$, *there exists an involution on the algebra* $\mathcal{O}(\mathbb{A}_{h'}^{2|1})$.

Proposition 7.6 *If* $\bar{h}' = h'$, *then the algebra* $\mathcal{O}(\mathbb{A}_{h}^{2|1})$ *supplied with the involution determined by*

$$\varphi^\star = \varphi - h'y_2, \quad y_i^\star = -y_i, \quad (i = 1, 2) \tag{7.6}$$

becomes a $\star$*-algebra.*

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