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Self-adjoint extensions for a class of singular operators

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Abstract: In this study, we consider the domains of the minimal and maximal operators generated of singular differential-expression-type Sturm–Liouville and obtain all self-adjoint extensions of the operator in terms of boundary conditions.

Key words: Sturm–Liouville equation, minimal and maximal operators, self-adjoint extensions

1. Introduction

One of the fundamental problems in the theory of symmetric operators is to construct all those extensions of a given symmetric operator \mathcal{A} which are themselves symmetric operators. A special case of this is the problem of setting up conditions under which an operator shall have a self-adjoint extension and to construct all the self-adjoint extensions when these conditions hold.

If \mathcal{B} is a symmetric extension of a symmetric operator \mathcal{A} , then $\mathcal{A} \subset \mathcal{B}$, so we $\mathcal{B}^* \subset \mathcal{A}^*$. However, \mathcal{B} is a symmetric operator, i.e. $\mathcal{B} \subset \mathcal{B}^*$, so we have $\mathcal{A} \subset \mathcal{B} \subset \mathcal{B}^* \subset \mathcal{A}^*$, i.e. every symmetric extension of an operator \mathcal{A} is a restriction of the operator \mathcal{A}^* . A symmetric operator \mathcal{A} is said to be maximal if it has no proper symmetric extension. Every self-adjoint operator \mathcal{A} is a maximal symmetric operator. For, in this case, $\mathcal{A}^* = \mathcal{A}$, and the $\mathcal{A} \subset \mathcal{B} \subset \mathcal{B}^* \subset \mathcal{A}^*$ implies that $\mathcal{B} = \mathcal{A}$, i.e. every symmetric extension \mathcal{B} of \mathcal{A} coincides with \mathcal{A} . There are maximal operators which are not self-adjoint.

Boundary value problems for the differential operators with integrable coefficient in the finite interval were well learned (see for exp., [1, 6]). In the last years, learning the differential operators with the nonintegrable or singular coefficient has become even more important according to physical science needs.

In the present paper, in interval $(0, 1)$, all self-adjoint extensions have been considered for Sturm–Liouville operator which is generated by differential expression

$$l[u] = -u''(x) + cx^{-\alpha}u(x) + l(l+1)x^{-2}u(x) + q(x)u(x) \quad (1.1)$$

where c , α , and l are real numbers, $\alpha \in [1, 2)$, $l \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $q(x)$ is a real-valued measurable bounded function and define $\sigma(x) = \int q(x)dx$.

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Note that a similar problem was completely studied in [1, 4, 6, 7] for the Sturm–Liouville equation with integrable potential and with Bessel potential ($c = 0$). Boundary value problems with another regularization were given in [2] for $\alpha < \frac{3}{2}$ ($l = 0$). Self-adjoint extensions of the operator generated by differential expression (1.1) in the case $l = 0$ with potential $\sigma'(\cdot)$, $\sigma(\cdot) \in L_2(0, 1)$ were given in [5]. Regularizations used in the paper coincide with those in [5] only for $\alpha < \frac{3}{2}$ in the case $l = 0$. More generally than studies [2] and [5], that is, in the case of $\alpha \in [1, 2)$ and $l = 0$, the operators self-adjoint extensions produced by (1.1) differential expression were obtained in [3].

2. Main results

On the set $D_0 = C_0^\infty(0, 1)$, we introduce an operator L'_0 by the formula $L'_0 u \equiv l[u]$. Obviously, L'_0 is a symmetric operator in $L_2(0, 1)$. Its closure L_0 is called the minimal operator generated by differential expression (1.1). The adjoint operator L_0^* is called the maximal operator. Since the operators L_0 and L'_0 generated by the expression (1.1) differ from the corresponding operators generated by the expression (1.1) with $q(x) \equiv 0$ by a bounded operator, in the following, we assume that

$$l[u] = -u''(x) + cx^{-\alpha}u(x) + l(l + 1)x^{-2}u(x)$$

For the determination of the domains of the minimal and maximal operators we need a fundamental (principal) solution system for the equation

$$-u''(x) + cx^{-\alpha}u(x) + l(l + 1)x^{-2}u(x) = 0 \tag{2.1}$$

First of all, let us give the following lemma:

Lemma 2.1 *If $M \neq n - 1$, $\forall n \in \mathbb{N}$, then the differential equation*

$$u''(t) - \frac{M}{t}u'(t) = \frac{Q}{t}u(t), \quad Q \neq 0 \tag{2.2}$$

has only one solution as the following:

$$u(t) = \sum_{k=0}^{+\infty} c_k t^k$$

If $M = n - 1$ for any $n \in \mathbb{N}$, then the differential equation (2.2) has only one solution as the following:

$$u(t) = \sum_{k=0}^{n-1} c_k t^k + t^n \frac{Q}{n} c_{n-1} \ln t + \sum_{k=n+1}^{+\infty} (\tilde{A}_k + \tilde{B}_k \ln t) t^k$$

where $c_k = Q^k (k!)^{-1} \prod_{p=0}^{k-1} (p - M)^{-1}$, $c_0 = 1$, $\tilde{B}_{k+1}(k + 1)(k - n + 1) = \tilde{B}_k$, $k \geq n$, $\tilde{B}_n = \frac{Q}{n} c_{n-1}$,

$$\tilde{A}_{n+1}(n + 1) + \tilde{B}_{n+1}(n + 2) = 0,$$

$$Q\tilde{A}_k = (k + 1)(k - n + 1)\tilde{A}_{k+1} + (2k + 2 - n)\tilde{B}_{k+1}, \quad k > n.$$

It is clearly seen that the function $u(t)$ given in the expression of the lemma satisfies equation (2.2) in both $M \neq n - 1$ and $M = n - 1$ cases.

Let us investigate the fundamental system of solution of differential equation (2.1) as

$$u_1(x) = x^{l+1}\psi(x), \quad u_2(x) = x^{-l}\varphi(x).$$

Then, we obtain the differential equations

$$\psi''(x) + \frac{2(l+1)}{x}\psi'(x) = \frac{c}{x^\alpha}\psi(x), \quad \varphi''(x) - \frac{2l}{x}\varphi'(x) = \frac{c}{x^\alpha}\varphi(x)$$

for functions $\psi(x)$ and $\varphi(x)$.

If the transformation $t = x^{2-\alpha}$ in the obtained differential equations for the functions $\psi(x)$ and $\varphi(x)$ and representation $\tilde{\psi}(t) = \psi\left(t^{\frac{1}{2-\alpha}}\right)$, $\tilde{\varphi}(t) = \varphi\left(t^{\frac{1}{2-\alpha}}\right)$ are used, we get the following differential equations:

$$\begin{aligned} \tilde{\psi}''(t) + \frac{2(l+1) + 1 - \alpha}{2 - \alpha} \cdot \frac{1}{t} \tilde{\psi}'(t) &= \frac{c}{(2 - \alpha)^2} \cdot \frac{1}{t} \tilde{\psi}(t) \\ \tilde{\varphi}''(t) - \frac{2l - 1 + \alpha}{2 - \alpha} \cdot \frac{1}{t} \tilde{\varphi}'(t) &= \frac{c}{(2 - \alpha)^2} \cdot \frac{1}{t} \tilde{\varphi}(t) \end{aligned}$$

Therefore, it was shown that functions $\tilde{\psi}(t)$ and $\tilde{\varphi}(t)$ satisfy the differential equation with the type (2.2). By using the solutions of differential equation (2.2), we obtained the representations in Lemma 2.1 for the functions $\tilde{\psi}(t)$ and $\tilde{\varphi}(t)$, so also functions $u_1(x)$ and $u_2(x)$ are fundamental solution of equation (2.1).

Lemma 2.2 Equation (2.1) has a solution $u_1(x)$ with the initial conditions

$$u_1(x) = x^{l+1} [1 + o(1)], \quad u_1'(x) = (l + 1)x^l + o(x^l) \quad x \rightarrow +0$$

and a solution $u_2(x)$ that can be represented in the form

$$u_2(x) = x^l \begin{cases} \sum_{k=0}^{+\infty} a_k x^{(2-\alpha)k} & \text{if } \alpha \neq \frac{2n-1-2l}{n}, \quad (n \in \mathbb{N}) \\ \sum_{k=0}^{n-1} a_k x^{(2-\alpha)k} + \frac{c}{2l+1} a_{n-1} x^{2l+1} \ln x & \\ + \sum_{k=n+1}^{+\infty} (A_k + B_k \ln x) x^{(2-\alpha)k} & \text{if } \alpha = \frac{2n-1-2l}{n}, \quad (n \in \mathbb{N}) \end{cases} \quad (2.3)$$

where $a_0 = 1$, $a_k = \frac{1}{k!} \frac{c^k}{(2-\alpha)^k} \prod_{p=1}^k (2p-1-\alpha p-2l)^{-1}$.

The existence of the solution $u_1(x)$ is well known (see [8]). The assertion concerning the solution $u_2(x)$ can be obtained by a straightforward substitution of the expression (2.3) into equation (2.1).

Lemma 2.3 The domain $D(L_0^*)$ of the maximal operator L_0^* generated by the expression (1.1) is the set of functions $u(x)$ that can be represented in the form

$$u(x) = u_1(x)h_1 + u_2(x)h_2 + \int_0^x [u_2(x)u_1(t) - u_1(x)u_2(t)] f(t)dt$$

where $h_1, h_2 \in \mathbb{C}$ and $f \in L_2(0, 1)$ (\mathbb{C} is the set of complex numbers).

It follows from this lemma that the function $u \in L_2(0, 1)$ belongs to $D(L_0^*)$ if and only if $u'(x)$ is absolutely continuous on $(0, 1)$ and $l[u] \in L_2(0, 1)$. Next, on $D(L_0^*)$, the operator L_0^* acts by the formula $L_0^*u = l[u]$.

It follows from Lemma 2.2 and Lemma 2.3 that the function $u'(x)$, where $u \in D(L_0^*)$, does not necessarily have a boundary value at the point $x = 0$. However, it has a regularized boundary value. We set $b_1 = a_1$ and $b_k = ka_k - b_1a_{k-1} - \dots - b_{k-1}a_1$, $k > 1$ and introduce the functions

$$\sigma_{\alpha,l}(x) = \begin{cases} 0 & \alpha < 1 - 2l \\ c \ln x & \alpha = 1 - 2l \\ (2 - \alpha) \sum_{k=1}^{n-1} b_k x^{2k-1-\alpha k-2l} + ca_{n-1} \ln x & \alpha = \frac{2n-1-2l}{n}, n = 2, 3, \dots \\ (2 - \alpha) \sum_{k=1}^n b_k x^{2k-1-\alpha k-2l} & \alpha \in \left(\frac{2n-1-2l}{n}, \frac{2n+1-2l}{n+1} \right), \\ & n = 1, 2, \dots \end{cases}$$

$$(\Gamma_1 y)(x) = x^l y(x), (\Gamma_2 y)(x) = x^{-l-1} [xy'(x) + ly(x)] - \sigma_{\alpha,l}(x)x^l y(x)$$

Lemma 2.4 Let $u \in D(L_0^*)$. Then the functions $(\Gamma_1 u)(x)$ and $(\Gamma_2 u)(x)$ has limits

$$\lim_{x \rightarrow 0^+} (\Gamma_1 u)(x) = (\Gamma_1 u)(0), \quad \lim_{x \rightarrow 0^+} (\Gamma_2 u)(x) = (\Gamma_2 u)(0).$$

Now one can readily show (see [1, 6]) that the domain $D(L_0)$ of the minimal operator L_0 generated by expression (1.1) is the set of functions $u \in D(L_0^*)$ such that

$$(\Gamma_1 u)(0) = (\Gamma_2 u)(0) = u(1) = u'(1) = 0.$$

For each function $u \in D(L_0^*)$, we define an ordered pair $\{U, U'\}$ of vectors of the two-dimensional euclidean space \mathbb{C}^2 as follows:

$$U = \{-(\Gamma_1 u)(0), u(1)\}, \quad U' = \{(\Gamma_2 u)(0), u'(1)\} \tag{2.4}$$

The Lagrange formula for functions $u, v \in D(L_0^*)$ can be written out in the form

$$(L_0^*u, v) - (u, L_0^*v) = \langle U, V' \rangle - \langle U', V \rangle$$

where the symbols (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$ stand for the inner products in the spaces $L_2(0, 1)$ and \mathbb{C}^2 , respectively. Now, by using theorem on the representation of Hermitian relations [1, 7], one can prove the following theorem:

Theorem 2.5 For each self-adjoint extensions of operator L_0^*

$$(Cos A).U' - (Sin A).U = 0 \tag{2.5}$$

$$(\dot{U} - I)U' + (\dot{U} + I)U = 0 \tag{2.6}$$

any of boundary conditions have been determined, where A , \dot{U} , and I are self-adjoint, uniter, and identical operators respectively. Conversely, helping any of boundary conditions (2.5) and (2.6), the self-adjoint extensions can be generated.

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