Turkish Journal of Mathematics

Volume 45 | Number 1 Article 11

1-1-2021

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 167 – 184
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doi:10.3906/mat-1911-79

# Widths and entropy of sets of smooth functions on compact homogeneous manifolds

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**Received:** 22.11.2019 • **Accepted/Published Online:** 11.11.2020 • **Final Version:** 21.01.2021

**Abstract:** We develop a general method to calculate entropy and $n$-widths of sets of smooth functions on an arbitrary compact homogeneous Riemannian manifold $\mathbb{M}^d$. Our method is essentially based on a detailed study of geometric characteristics of norms induced by subspaces of harmonics on $\mathbb{M}^d$. This approach has been developed in the cycle of works [1, 2, 10–19]. The method's possibilities are not confined to the statements proved but can be applied in studying more general problems. As an application, we establish sharp orders of entropy and $n$-widths of Sobolev's classes $W_p^\gamma\left(\mathbb{M}^d\right)$ and their generalisations in $L_q\left(\mathbb{M}^d\right)$ for any $1<p,q<\infty$. In the case $p,q=1,\infty$ sharp in the power scale estimates are presented.

**Key words:** $n$-widths, compact homogeneous manifold, Lévy mean, volume

## 1. Introduction

Let $(\Omega,\upsilon)$ be a measure space and $\{\xi_k\}_{k\in\mathbb{N}}$ be a sequence of orthonormal, functions on $\Omega$. Let $X$ be a Banach space with the norm $\|\cdot\|_X$ and $\{\xi_k\}_{k\in\mathbb{N}}\subset X$. Clearly, $\Xi_n(X):=\mathrm{lin}\{\xi_1,\cdots,\xi_n\}\subset X$, $\forall n\in\mathbb{N}$ is a sequence of closed subspaces of $X$ with the norm induced by $X$. Consider the coordinate isomorphism $\mathrm{J}$ defined as

$$\begin{array}{rcl}\mathrm{J}:\ \mathbb{R}^n & \longrightarrow & \Xi_n(X)\\ \alpha=(\alpha_1,\cdots,\alpha_n) & \longmapsto & \sum_{k=1}^n\alpha_k\xi_k.\end{array}$$

Hence, the definition

$$\|\alpha\|_{\mathrm{J}^{-1}\Xi_n(X)}=\|\mathrm{J}\alpha\|_X$$

induces the norm on $\mathbb{R}^n$ which appears to be useful in various applications. Of course, not much can be said regarding such kind of norms even in lower dimensions. To be able to apply methods of geometry of Banach spaces to various open problems in different spaces of functions on $\Omega$ we will need to calculate an expectation $\mathbf{E}\left[\rho_n(\alpha)\right]$ of the function $\rho_n(\alpha):=\|\alpha\|_{\mathrm{J}^{-1}\Xi_n(X)}$ on the unit sphere $\mathbb{S}^{n-1}\subset\mathbb{R}^n$ with respect to Haar measure $d\mu_n$, i.e. to find the Lévy $\mathrm{M}(\|\cdot\|_{\mathrm{J}^{-1}\Xi_n(X)})$ mean

$$\mathrm{M}(\|\cdot\|_{\mathrm{J}^{-1}\Xi_n(X)})=\mathbf{E}\left[\rho_n(\alpha)\right]=\int_{\mathbb{S}^{n-1}}\|\alpha\|_{\mathrm{J}^{-1}\Xi_n(X)}\,d\mu_n(\alpha).$$

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2010 *AMS Mathematics Subject Classification:* 41A46, 42B15

Observe that the sequence of Lévy means $\mathrm{M}(\|\cdot\|_{\mathrm{J}^{-1}\Xi_n(X)})$ contain more information then the sequence of volumes $\mathrm{Vol}_n\left(B_{J^{-1}\Xi_n(X)}\right)$, $n\in\mathbb{N}$, where $B_{\mathrm{J}^{-1}\Xi_n(X)} := \{\alpha|\, \alpha\in\mathbb{R}^n,\ \|\alpha\|_{\mathrm{J}^{-1}\Xi_n(X)}\le 1\}$ is the unit ball induced by the norm $\|\cdot\|_{\mathrm{J}^{-1}\Xi_n(X)}$ and therefore is more useful in various applications.

As a motivating example consider the case $\Omega=\mathbb{M}^d$, where $\mathbb{M}^d$ is a compact homogeneous Riemannian manifold, $\upsilon$ its normalized volume element, $\{\xi_k\}_{k\in\mathbb{N}}$ is a sequence of orthonormal harmonics on $\mathbb{M}^d$ and $X=L_p=L_p(\mathbb{M}^d,\upsilon)$, $p\ge 2$. In general, the sequence $\{\xi_k\}_{k\in\mathbb{N}}$ is not uniformly bounded on $\mathbb{M}^d$. Hence, the method of estimating of Lévy means developed in [10–13] cannot give sharp order result. Various modifications of this method presented in [15–17] give an extra $(\log n)^{1/2}$ factor even if $p<\infty$. Our general result is presented in Lemma 3 which gives sharp order estimates for the Lévy means which correspond to the norm induced on $\mathbb{R}^n$ by the subspace $\oplus_{s=1}^m \mathrm{H}_{k_s}\cap L_p$, $\dim \oplus_{s=1}^m \mathrm{H}_{k_s} := n$ with an arbitrary index set $(k_1,\cdots,k_m)$, where $\mathrm{H}_{k_s}$ are the eigenspaces of the Laplace–Beltrami operator for $\mathbb{M}^d$ defined by (2.2). To show the boundness of the respective Lévy means as $n\to\infty$ we impose a technical condition (2.1) which holds in particular for any compact homogeneous Riemannian manifold because of the addition formula (2.4) and employ the equality

$$\int_{\mathbb{R}^n} h(\alpha)\, d\gamma(\alpha) = \lim_{m\to\infty}\int_0^1 h\left(\frac{\delta_1^m(\theta)}{(2\pi)^{1/2}},\cdots,\frac{\delta_n^m(\theta)}{(2\pi)^{1/2}}\right)d\theta,$$

where $h:\mathbb{R}^n\to\mathbb{R}$ is a continuous function, $h(\alpha_1,\cdots,\alpha_n)\exp\left(-\sum_{k=1}^n|\alpha_k|\right)\to 0$ uniformly when $\sum_{k=1}^n|\alpha_k|\to\infty$, $d\gamma(\alpha)=\exp\left(-\pi\sum_{k=1}^n\alpha_k^2\right)d\alpha$ is the Gaussian measure on $\mathbb{R}^n$, $\delta_k^m(\theta)=m^{-1/2}(r_{(k-1)m}(\theta)+\cdots+r_{km})$, $1\le k\le n$ and $r_s(\theta)=\operatorname{sign}\sin(2^s\pi\theta)$, $s\in\mathbb{N}\cup\{0\}$, $\theta\in[0,1]$ is the sequence of Rademacher functions [21], [19]. To extend our estimates to the case $p=\infty$ we apply Lemma 3.1 which gives a useful inequality between norms of polynomials on $\mathbb{M}^d$ with an arbitrary spectrum. It seems that the factor $(\log n)^{1/2}$ obtained in Lemma 3.2 is essential because of the lower bound for the Lévy means found in [9] in the case of trigonometric system. This fact explains a logarithmic slot in our estimates of entropy numbers presented in Theorem 3.12. Section 3 deals with estimates of entropy numbers and $n$-widths. Theorem 3.3 establishes general lower bounds for entropy numbers in terms of Lévy means and is of independent interest. We derive lower bounds for the entropy numbers of Sobolev's classes (3.8) using Theorem 3.3 and estimates of Lévy means given by Lemma 3.2. At this point we apply Lemma 2.2 to get the dependence between eigenvalues and dimensions of eigenspaces of the Laplace–Beltrami operator. The proof of Lemma 2.2 is based on Weyl's formula (see [23])

$$\lim_{a\to\infty} a^{-d/2}n(a) = (2\pi^{1/2})^{-d}\Gamma\left(1+\frac{d}{2}\right)\mathrm{V}(\mathbb{M}^d), \tag{1.1}$$

where $\mathrm{V}(\mathbb{M}^d)$ is the volume of $\mathbb{M}^d$ and $n(a)$ is the number of eigenvalues (each counted with its multiplicity) smaller than $a$. To get upper bounds for entropy numbers contained in Theorem 3.12 we apply estimates of Lévy means established in Lemma 3.2 and make use of the Pajor–Tomczak-Jaegermann inequality [24] which states in our notations that for any $\lambda\in(0,1)$ there exists a subspace $X_s\subset \mathrm{J}^{-1}\Xi_n(X)$, $\dim X_s=s>\lambda n$ and a universal constant $C>0$ such that

$$\|\alpha\|_2^* \le C\frac{\mathrm{M}(\|\cdot\|^o_{\mathrm{J}^{-1}\Xi_n(X)})}{(1-\lambda)^{1/2}}\|\alpha\|_{\mathrm{J}^{-1}\Xi_n(X)},\ \ \forall\alpha\in X_s, \tag{1.2}$$

where $\|\cdot\|_2^* = \langle\cdot,\cdot\rangle^{1/2}$ is the Euclidean norm on $\mathbb{R}^n$ induced by the scalar product $\langle\mathbf{x},\mathbf{y}\rangle=\sum_{k=1}^n x_k y_k$ and $\|\cdot\|^o_{\mathrm{J}^{-1}\Xi_n(X)}$ is the dual norm with respect to $\|\cdot\|_{\mathrm{J}^{-1}\Xi_n(X)}$. Remark that (1.2) is essentially based on a technical result due to Gluskin [8]. However, for our applications is sufficient to apply a less sharp result established by Bourgain and Milman [3] which is based on averaging arguments and isoperimetric inequality.

The paper ends with estimates of different $n$-widths and their applications in calculation of entropy which extend previous results [1, 2, 17].

In this article there are several universal constants which enter into the estimates. These positive constants are mostly denoted by $C, C_1, \cdots$. We will only distinguish between the different constants where confusion is likely to arise, but we have not attempted to obtain good estimates for them. For ease of notation we will write $a_n \ll b_n$ for two sequences, if $a_n \leq Cb_n$, $\forall n \in \mathbb{N}$ and $a_n \asymp b_n$, if $C_1 b_n \leq a_n \leq C_2 b_n$, $\forall n \in \mathbb{N}$ and some constants $C$, $C_1$ and $C_2$.

Though the main purpose of this paper is to present new results, we have tried to make the text self contained by presenting well-known definitions and elementary properties of entropy numbers and $n$-widths.

Let $X$ and $Y$ be Banach spaces with the closed unit balls $B_X$ and $B_Y$ respectively. Let $v : X \to Y$ be a compact operator. Then the $n^{\text{th}}$ entropy number $e_n(v) = e_n(v : X \to Y)$ is the infimum over all positive $\epsilon$ such that there exist $y_1, \cdots, y_{2^{n-1}}$ in $Y$ such that

$$v(B_X) \subset \bigcup_{k=1}^{2^{n-1}} (y_k + \epsilon B_Y).$$

Similarly, for a compact set $A \subset Y$ we define the entropy number $e_n(A, Y)$ as the infimum of all positive $\epsilon$ such that there exist $\{y_k\}_{k=1}^{2^{n-1}} \subset Y$ such that $A \subset \bigcup_{k=1}^{2^{n-1}} (y_k + \epsilon B_Y)$. Suppose that $A$ is a convex, compact, centrally symmetric subset of a Banach space $X$ with unit ball $B_X$. The Kolmogorov $n$-width of $A$ in $X$ is defined by

$$d_n(A, X) := d_n(A, B_X) := \inf_{X_n} \sup_{f \in A} \inf_{g \in X_n} \|f - g\|_X,$$

where $X_n$ runs over all subspaces of $X$ of dimension $n$. The Gelfand $n$-width of $A$ in $X$ is defined by

$$d^n(A, X) := d^n(A, B_X) := \inf_{L^n} \sup_{x \in L^n \cap A} \|x\|_X,$$

where $L^n$ runs over all subspaces of $X$ of codimension $n$. The Bernstein $n$-width of $A$ in $X$ is defined by

$$b_n(A, X) := b_n(A, B_X) := \sup_{X_{n+1}} \sup\{\epsilon > 0 : \epsilon B_X \cap X_{n+1} \subset A\},$$

where $X_{n+1}$ is any $(n+1)$-dimensional subspace of $X$. For a compact operator $v : X \to Y$ we define Kolmogorov's numbers

$$d_n(v) = d_n(v : X \to Y) = \inf_{L \subset Y, \dim L \leq n} \sup_{x \in B_X} \inf_{y \in L} \|vx - y\|_Y$$

and Gelfand numbers

$$d^n(v) = d^n(v : X \to Y) = \inf\{\|v|L\| \,|L \subset X, \operatorname{codim} L \leq n\}.$$

**Proposition 1.1** *This proposition records some simple properties of $n$-widths and entropy numbers which we will need.*

1. *If* $X \subset Y$, *then* $d_n(A,Y) \le d_n(A,X)$.

2. *Let* $n = i + j$ *and* $A = A_1 + A_2$. *Then* $d_n(A,X) \le d_i(A_1,X) + d_j(A_2,X)$.

3. *Kolmogorov and Gelfand* $n$*-widths are dual. Let* $X$ *and* $Y$ *be Banach spaces,* $v \in \mathcal{L}(X,Y)$. *If* $X$ *is reflexive and* $v(X)$ *is dense in* $Y$, *then* $d_n(v) = d^n(v^*)$ *(see e.g., [22], p.408).*

4. *Later we will wish to restrict estimation of entropy numbers over infinite-dimensional sets to finite-dimensional sets. In order to do this let* $i$ *be any linear isometry,* $i : Y \to \tilde{Y}$ *(here we will think of* $Y$ *as finite dimensional and* $i$ *as the imbedding into the infinite dimensional space). Then ([26, Proposition 5.1])* $2^{-1}e_n(v) \le e_n(i \circ v) \le e_n(v), \ \forall n \in \mathbb{N}$.

## 2. Harmonic analysis

**Definition 2.1** *Let* $(\Omega, \upsilon)$ *be a measure space for some compact set* $\Omega \in \mathbb{R}^s$, $s \in \mathbb{N}$. *Let* $\Xi = \{\xi_k\}_{k\in\mathbb{N}}$ *be a set of orthonormal functions* $\xi_k = \xi_k(x)$ *in* $L_2(\Omega, \nu)$. *Suppose that there exists a sequence* $\kappa = \{k_j\}_{j\in\mathbb{N}}$, $k_1 = 1$, *such that for any* $j \in \mathbb{N}$ *and some* $C > 0$

$$\sum_{k=k_j}^{k_{j+1}-1} |\xi_k(x)|^2 \le C d_j, \, \forall x \in \Omega \tag{2.1}$$

*a.e. on* $\Omega$, *where* $d_j := k_{j+1} - k_j$. *Then we say that* $(\Omega, \upsilon, \Xi, \kappa) \in \mathcal{K}$.

Consider the set of $p$-integrable functions on $(\Omega, \upsilon)$, $L_p = L_p(\Omega, \upsilon)$. It follows from (2.1) that the functions $\xi_k$ are a.e. bounded for every $n \in \mathbb{N}$. Hence, for an arbitrary $\varphi \in L_p$, $1 \le p \le \infty$ it is possible to construct the Fourier coefficients

$$c_k(\varphi) = \int_\Omega \varphi \bar{\xi}_k d\upsilon, \quad k \in \mathbb{N},$$

and consider the formal Fourier series

$$\varphi \sim \sum_{l\in\mathbb{N}} \sum_{k=k_l}^{k_{l+1}-1} c_k(\varphi)\xi_k.$$

Let $U_p := \{\varphi | \, \|\varphi\|_p \le 1\}$ be the unit ball in $L_p$, and $\Lambda = \{\lambda_l\}_{l\in\mathbb{N}}$ be a fixed sequence of complex numbers. We shall say that the multiplier operator $\Lambda$ is of type $(\kappa, p, q)$ with the norm $\|\Lambda\|_{p,q}^{\kappa} := \sup_{\varphi\in U_p} \|\Lambda\varphi\|_q$, , if for any $\varphi \in L_p$ there is such $f \in L_q$ that

$$f \sim \sum_{l\in\mathbb{N}} \lambda_l \sum_{k=k_l}^{k_{l+1}-1} c_k(\varphi)\xi_k.$$

Let us present here several important examples of measure spaces
$(\Omega, \upsilon, \Xi, \kappa) \in \mathcal{K}$. Consider a compact, connected, $d$-dimensional $C^\infty$ Riemannian manifold $\mathbb{M}^d$ with $C^\infty$ metric. Let $g$ its metric tensor, $\upsilon$ its normalized volume element and $\Delta$ its Laplace–Beltrami operator. In local coordinates $x_l$, $1 \le l \le d$,

$$\Delta = -(\overline{g})^{-1/2} \sum_k \frac{\partial}{\partial x_k} \left( \sum_j g^{jk} (\overline{g})^{1/2} \frac{\partial}{\partial x_j} \right), \tag{2.2}$$

where $g_{jk} := g(\partial/x_j, \partial/x_k)$, $\overline{g} := |\det(g_{jk})|$, and $(g^{jk}) := (g_{jk})^{-1}$. It is well-known that $\Delta$ is an elliptic, self adjoint, invariant under isometry, second order operator. The eigenvalues $\theta_k$, $k \geq 0$, of $\Delta$ are discrete, nonnegative and form an increasing sequence $0 \leq \theta_0 \leq \theta_1 \leq \cdots \leq \theta_n \leq \cdots$ with $+\infty$ the only accumulation point. The corresponding eigenspaces $\mathrm{H}_k$, $k \geq 0$ are finite dimensional, $d_k := \dim(\mathrm{H}_k)$, orthogonal and $L_2 = L_2(\mathbb{M}^d, \upsilon) = \oplus_{k=0}^{\infty}\mathrm{H}_k$. Let us fix an orthonormal basis $\{\mathrm{Y}_m^k\}_{m=1}^{d_k}$ of $\mathrm{H}_k$. Using multiplier operators we can introduce a wide range of sets of smooth functions on $\mathbb{M}^d$. Let $\varphi$ be an arbitrary function, $\varphi \in L_p$, $1 \leq p \leq \infty$ with the formal Fourier series

$$\varphi \sim c_0 + \sum_{k\in\mathbb{N}} \sum_{m=1}^{d_k} c_{k,m}(\varphi)\mathrm{Y}_m^k, \;\; c_{k,m}(\varphi) = \int_{\mathbb{M}^d} \varphi \overline{\mathrm{Y}}_m^k d\upsilon$$

and $\lambda(\cdot) : (0,\infty) \mapsto \mathbb{R}$ be a continuous function. If for any $\varphi \in L_p$ there is a function $f := \Lambda\varphi \in L_q$ such that

$$\varphi \sim c_0 + \sum_{k\in\mathbb{N}} \lambda(\theta_k) \sum_{m=1}^{d_k} c_{k,m}(\varphi)\mathrm{Y}_m^k$$

then we shall say that the multiplier operator $\Lambda$ is of $(p,q)$-type.

Consider the sets $\Lambda U_p$ generated by multiplier sequences $\{\lambda(\theta_k)\}$. In particular, let $\lambda(t) = t^{-\gamma/2}$ then the $\gamma$-th fractional integral $\mathrm{I}_\gamma\varphi := \varphi_\gamma$, $\gamma > 0$, is defined as

$$\varphi_\gamma \sim c + \sum_{k\in\mathbb{N}} \theta_k^{-\gamma/2} \sum_{m=1}^{d_k} c_{k,m}(\varphi)\mathrm{Y}_m^k, \;\; c \in \mathbb{R}. \tag{2.3}$$

The function $D_\gamma\varphi := \varphi^{(\gamma)} \in L_p$, $1 \leq p \leq \infty$ is called the $\gamma$-th fractional derivative of $\varphi$ if

$$\varphi^{(\gamma)} \sim \sum_{k\in\mathbb{N}} \theta_k^{\gamma/2} \sum_{m=1}^{d_k} c_{k,m}(\varphi)\mathrm{Y}_m^k.$$

The Sobolev classes $W_p^\gamma$ are defined as sets of functions with formal Fourier expansions (2.3) where $\|\varphi\|_p \leq 1$ and $\int_{\mathbb{M}^d} \varphi d\upsilon = 0$. In this article we assume $c_0, c = 0$ to guarantee compactness of the set $W_p^\gamma$ in $L_q$.

We recall that a Riemannian manifold $\mathbb{M}^d$ is called homogeneous if its group of isometries $\mathcal{G}$ acts transitively on it, i.e. for every $x, y \in \mathbb{M}^d$, there is a $\mathbf{g} \in \mathcal{G}$ such that $\mathbf{g}x = y$. For a compact homogeneous Riemannian manifold $\mathbb{M}^d$ the following addition formula is known [7]

$$\sum_{k=1}^{d_k} |\mathrm{Y}_m^k(x)|^2 = d_k, \;\; \forall x \in \mathbb{M}^d, \tag{2.4}$$

where $\{\mathrm{Y}_m^k\}_{m=1}^{d_k}$ is an arbitrary orthonormal basis of $\mathrm{H}_k$, $k \geq 0$. Hence, any such manifold possesses the property $\mathcal{K}$, and these include real and complex Grassmannians, the $n$-torus, the Stiefel manifold, two point homogeneous spaces (the spheres, the real, complex and quaternionic projective spaces and the Cayley elliptic plane), and the complex sphere.

**Lemma 2.2** *Let $\mathbb{M}^d$ be a compact, connected, homogeneous Riemannian manifold, $\{\theta_k\}_{k\in\mathbb{N}\cup\{0\}}$ be the sequence of eigenvalues and $\{\mathrm{H}_k\}_{k\in\mathbb{N}\cup\{0\}}$ be the corresponding sequence of eigenspaces of the Laplace–Beltrami operator $\Delta$ on $\mathbb{M}^d$. Put $\mathcal{T}_N = \oplus_{k=0}^N \mathrm{H}_k$ and $\tau_N = \dim \mathcal{T}_N$. Then*

$$\lim_{N\to\infty} \frac{\theta_{N+1}}{\theta_N} = 1 \tag{2.5}$$

*and*

$$\lim_{N\to\infty} \frac{\tau_{N+1}}{\tau_N} = 1. \tag{2.6}$$

**Proof** Applying Weyl's formula (1.1) for $a = \theta_N$ we get

$$\lim_{N\to\infty} \theta_N^{-d/2} n(\theta_N) = (2\pi^{1/2})^{-d}\Gamma\left(1+\frac{d}{2}\right)\mathrm{V}(\mathbb{M}^d), \tag{2.7}$$

and it follows that

$$\lim_{N\to\infty} \frac{\theta_{N+1}^{-d/2} n(\theta_{N+1}) - \theta_N^{-d/2} n(\theta_N)}{\theta_N^{-d/2} n(\theta_N)} \to 0, \quad N\to\infty.$$

Now, $n(\theta_N) = \tau_N$, so that

$$\frac{\theta_{N+1}^{-d/2}\tau_{N+1} - \theta_N^{-d/2}\tau_N}{\theta_N^{-d/2}\tau_N}$$

$$= \frac{\theta_{N+1}^{-d/2}(\tau_N + \dim H_{N+1}) - \theta_N^{-d/2}\tau_N}{\theta_N^{-d/2}\tau_N}$$

$$= \frac{\theta_{N+1}^{-d/2} - \theta_N^{-d/2}}{\theta_N^{-d/2}} + \frac{\theta_{N+1}^{-d/2}}{\theta_N^{-d/2}}\frac{\dim H_{N+1}}{\tau_N} \to 0, \quad N\to\infty. \tag{2.8}$$

Since both quotients in the last equation are positive we have

$$\frac{\theta_{N+1}^{-d/2} - \theta_N^{-d/2}}{\theta_N^{-d/2}} \to 0, \quad N\to\infty,$$

which gives us (2.5). Equation (2.6) follows since

$$\lim_{N\to\infty} \frac{\tau_{N+1}}{\tau_N} = \lim_{N\to\infty} \frac{\tau_N + \dim H_{N+1}}{\tau_N} = 1,$$

using the second quotient in (2.8), and (2.5). □

## 3. Estimates of entropy and $n$-widths

In this section we give several estimates of entropy and $n$-widths which are order sharp in many important cases. Fix a measure space $(\Omega, \upsilon)$, an orthonormal system $\Xi$ and a sequence $\{k_j\}_{j\in\mathbb{N}}$ such that $(\Omega, \upsilon, \Xi, \kappa) \in \mathcal{K}$. Let

$$\Xi^j := \mathrm{span}\,\{\xi_k\}_{k=k_j}^{k_{j+1}-1}, \ \ \Omega_m := \{j_1, \cdots, j_m\}, \ \ \Xi(\Omega_m) := \mathrm{span}\,\{\Xi^{j_s}\}_{s=1}^m.$$

Put $l_0 := 0$, $l_k := \sum_{s=1}^{k} d_{j_s}$, $k = 1, \cdots, m$, and $n := l_m = \dim(\Xi(\Omega_m))$.

Unfortunately, we need to introduce a reenumeration of the functions $\xi_k$, since we are selecting separated blocks of them. Let us write

$$(\Xi(\Omega_m)) = \text{span}\,\{\eta_i : i = 1, \cdots, n\},$$

with the $\eta_i$ organized so that $\Xi^{j_s} = \text{span}\,\{\eta_i : l_{s-1} + 1 \le i \le l_s\}$. Consider the coordinate isomorphism

$$\mathrm{J} : \mathbb{R}^n \to \Xi(\Omega_m)$$

that assigns to $\alpha = (\alpha_1 \cdots, \alpha_n) \in \mathbb{R}^n$ the function $\mathrm{J}\alpha = \xi^\alpha = \sum_{l=1}^{n} \alpha_l \eta_l \in \Xi(\Omega_m)$. Let $X$ and $Y$ be given Banach space such that $\Xi(\Omega_m) \subset X \cap Y$ for any $\Omega_m \subset \mathbb{N}$. Put $X_n = \Xi(\Omega_m) \cap X$ and $Y_n = \Xi(\Omega_m) \cap Y$. Let $\lambda_i \in \mathbb{R}$, $i = 1, \cdots, m$, and

$$\Lambda_n = \text{diag}\{\lambda_1 \mathrm{I}_{d_{j_1}}, \cdots, \lambda_m \mathrm{I}_{d_{j_m}}\},$$

where $\mathrm{I}_s$ is the identity matrix of dimension $s$. Now, if $\Lambda_n$ is invertible then $\mathrm{J}\Lambda_n\mathrm{J}^{-1} : \Xi(\Omega_m) \to \Xi(\Omega_m)$ is an invertible operator which essentially multiplies each block $\Xi^{j_s}$ by $\lambda_s$, $s = 1, \cdots, m$. Since it should not cause any confusion we will refer to this operator also as $\Lambda_n$.

In what follows, a $*$ will be used to denote norms and balls in Euclidean space, and lack of a $*$ will indicate the same quantities in function spaces. Let us define the norms

$$\|\alpha\|^*_{X_n} = \|\xi^\alpha\|_{X_n} = \|\xi^\alpha\|_X.$$

Put $B^*_{X_n} := \{\alpha \in \mathbb{R}^n,\ \|\alpha\|^*_{X_n} \le 1\}$, and $B_{X_n} := \mathrm{J}B^*_{X_n}$.

**Lemma 3.1** *For any $\Omega_m$ and any $\xi \in \Xi(\Omega_m) \subset L_\infty$, $m \in \mathbb{N}$ we have*

$$\|\xi\|_\infty \le Cn^{1/2}\|\xi\|_2,$$

*where $n := \dim \Xi(\Omega_m)$.*

**Proof** Let

$$K_n(x, y) := \sum_{i=1}^{n} \eta_i(x)\overline{\eta_i(y)}.$$

be the reproducing kernel for $\Xi(\Omega_m)$. Clearly,

$$K_n(x, y) = \int_{\mathbb{M}^d} K_n(x, z)K_n(z, y)d\upsilon(z),$$

and $K_n(x, y) = \overline{K_n(y, x)}$. Since $(\Omega, \upsilon, \Xi, \kappa) \in \mathcal{K}$, from (2.1), we have $\|K_n(x, \cdot)\|_2 \le Cn^{1/2}$, $\forall x \in \mathbb{M}^d$. Then applying Hölder inequality we get

$$\|\xi\|_\infty = \max_{x \in \mathbb{M}^d} \left| \int_{\mathbb{M}^d} K_n(x, z)\xi(z)d\upsilon(z) \right|$$

$$\le \max_{x \in \mathbb{M}^d} \|K_n(x, \cdot)\|_2 \|\xi\|_2$$

$$\leq Cn^{1/2}\|\xi\|_2.$$

□

Let us fix a norm $\|\cdot\|^*$ on $\mathbb{R}^n$ and let $E = (\mathbb{R}^n, \|\cdot\|^*)$ be a Banach space with the unit ball $B_E^*$. The dual space $E^{*o} = (\mathbb{R}^n, \|\cdot\|^{*o})$ is endowed with the norm $\|\xi\|^{*o} = \sup_{\sigma\in B_E^*} |\langle\xi,\sigma\rangle|$ and has the unit ball $B_{E^{*o}}$. In these notations the Lévy mean $M_{B_{E^*}}$ is

$$\mathrm{M}_{B_{E^*}} = \int_{\mathbb{S}^{n-1}} \|\xi\|^* d\mu,$$

where $d\mu$ denotes the normalised invariant measure on $\mathbb{S}^{n-1}$, the unit sphere in $\mathbb{R}^n$. We are interested in the case where $\|\cdot\| = \|\alpha\|^*_{\mathrm{I}_n, L_p}$. In the case $\Omega_m = \{1, \cdots, m\}$ the estimates of the associated Lévy means were obtained in [19]. For an arbitrary index set the respective result was established in [20]. This we state as

**Lemma 3.2** *Let $X_n = L_p \cap \Xi(\Omega_m)$ with the unit ball $B^{*n}_{L_p}$, and $n := \dim \Xi(\Omega_m)$. Then*

$$\mathrm{M}_{B^{*n}_{L_p}} \leq C \left\{ \begin{array}{ll} p^{1/2}, & p < \infty, \\ (\log n)^{1/2}, & p = \infty. \end{array} \right.$$

We can now give lower bounds for entropy in terms of Lévy means. In the following the reader should be identifying the spaces $X$ and $Y$ with $L_p$ and $L_q$ respectively for some $1 \leq p, q \leq \infty$. However, we wished to state the result in greater generality, and then apply the previous result to extract particular results in these cases.

**Theorem 3.3** *Viewing $\Xi(\Omega_m)$ as a subspace $X_n \subset X$ and $Y_n \subset Y$, we have*

$$e_k(\Lambda U_X, Y) \geq 2^{-1-k/n} \frac{|\det \Lambda_n|^{1/n}}{\mathrm{M}_{B^*_{X_n}} \mathrm{M}_{(B^*_{Y_n})^o}},$$

*where $k, n \in \mathbb{N}$ are arbitrary.*

**Proof** First, we use Proposition 1.1 (4) to obtain the estimate

$$e_k(\Lambda U_X, Y) \geq 2^{-1} e_k(\Lambda U_X \cap \Xi_n, Y \cap \Xi_n) = 2^{-1} e_k(\Lambda_n(B^*_{X_n}), B^*_{Y_n}), \tag{3.1}$$

using the appropriate norms in $X_n$ and $Y_n$. Let $\vartheta_1, \cdots, \vartheta_{N(\epsilon)}$ be a minimal $\epsilon$-net for $\Lambda_n(B^*_{X_n})$ in $\left(B^*_{Y_n}, \mathbb{R}^n\right)$. Then,

$$\Lambda_n B^*_{X_n} \subset \bigcup_{k=1}^{N(\epsilon)} \left(\epsilon B^*_{Y_n} + \vartheta_k\right).$$

By comparing volumes we get

$$\begin{aligned} \mathrm{Vol}_n \left(\Lambda_n B^*_{X_n}\right) &= |\det \Lambda_n| \mathrm{Vol}_n \left(B^*_{X_n}\right) \\ &\leq \epsilon^n N(\epsilon) \mathrm{Vol}_n \left(B^*_{Y_n}\right). \end{aligned}$$

If we put $N(\epsilon) = 2^{k-1}$, then from the last inequality and the definition of entropy numbers we obtain

$$\epsilon = e_k \left(\Lambda_n B^*_{X_n}, B^*_{Y_n}\right) \geq 2^{-k/n} |\det \Lambda_n|^{1/n} \left( \frac{\mathrm{Vol}_n \left(B^*_{X_n}\right)}{\mathrm{Vol}_n \left(B^*_{Y_n}\right)} \right)^{1/n}. \tag{3.2}$$

Let $B_2^*$ be the unit Euclidian ball in $\mathbb{R}^n$, $V \subset \mathbb{R}^n$ be a convex symmetric body and $V^o$ its dual. From Uryson's inequality ([26], p.6),

$$\left(\frac{\mathrm{Vol}_n V}{\mathrm{Vol}_n B_2^*}\right)^{1/n} \leq \mathrm{M}_{V^o}$$

it follows that

$$\left(\mathrm{Vol}_n\left(B_{Y_n}^*\right)\right)^{1/n} \leq \mathrm{M}_{(B_{Y_n}^*)^o}\left(\mathrm{Vol}_n\left(B_2^*\right)\right)^{1/n},$$

so that

$$\left(\frac{\mathrm{Vol}_n\left(B_{X_n}^*\right)}{\mathrm{Vol}_n\left(B_{Y_n}^*\right)}\right)^{1/n} \geq \frac{\left(\mathrm{Vol}_n\left(B_{X_n}^*\right)\right)^{1/n}}{\mathrm{M}_{(B_{Y_n}^*)^o}\left(\mathrm{Vol}_n\left(B_2^*\right)\right)^{1/n}}. \tag{3.3}$$

Also, a direct calculation shows that

$$\left(\frac{\mathrm{Vol}_n\left(B_{X_n}^*\right)}{\mathrm{Vol}_n\left(B_2^*\right)}\right)^{1/n} = \left(\int_{\mathbb{S}^{n-1}} \|\alpha\|^{-n} d\mu(\alpha)\right)^{1/n} \geq \mathrm{M}_{B_{X_n}^*}^{-1}. \tag{3.4}$$

Combining (3.1), (3.3) and (3.4) we have

$$\left(\frac{\mathrm{Vol}_n\left(B_{X_n}^*\right)}{\mathrm{Vol}_n\left(B_{Y_n}^*\right)}\right)^{1/n} \geq \frac{1}{\mathrm{M}_{B_{X_n}^*}\mathrm{M}_{(B_{Y_n}^*)^o}}.$$

and substitution into (3.2) completes the proof. □

**Remark 3.4** *Assume that* $|\lambda_1| \geq \cdots \geq |\lambda_n|$. *Then* $|\det \Lambda_n|^{1/n} \geq |\lambda_n|$, *and for* $k = n$ *we have*

$$e_n(\Lambda U_X, Y) \geq \frac{|\lambda_n|}{4M_{B_{X_n}^*}M_{(B_{Y_n}^*)^o}}. \tag{3.5}$$

**Remark 3.5** *Let* $(\Omega, \upsilon, \Xi, \kappa) \in \mathcal{K}$, $X = L_p$ *and* $Y = L_q$, $1 \leq q \leq 2 \leq p \leq \infty$. *Then using Hölder's inequality we get*

$$\mathrm{M}_{(B_{L_q}^{*n})^o} = \int_{\mathbb{S}^{n-1}} \|\xi\|_{L_q}^{*o} d\mu \leq \int_{\mathbb{S}^{n-1}} \|\xi\|_{L_{q'}}^{*} d\mu = \mathrm{M}_{B_{L_{q'}}^{*n}}, \tag{3.6}$$

*where* $1/q + 1/q' = 1$. *Comparing Lemma 3.2 with (3.5) and (3.6) we find*

$$e_n(\Lambda U_{L_p}, L_q) \gg |\lambda_n| \begin{cases} (pq')^{-1/2}, & p < \infty, q > 1, \\ (p\log n)^{-1/2}, & p < \infty, q = 1, \\ (q'\log n)^{-1/2}, & p = \infty, q > 1, \\ (\log n)^{-1}, & p = \infty, q = 1. \end{cases} \tag{3.7}$$

To proceed with calculation of entropy numbers we need to assume a technical condition on the multiplyer $\Lambda$.

**Definition 3.6** *We say that* $\Lambda \in \mathcal{A}$ *if* $\lambda(\cdot) : (0, \infty) \longmapsto (0, \infty)$ *is a decreasing continuous function such that* $\lambda(ct) \gg \lambda(t)$, $t \to \infty$ *for any* $c \geq 1$.

**Remark 3.7** *Remind that* $\tau_N = \dim \mathcal{T}_N$. *Put* $n = \tau_N$ *and* $\tilde{n} = \tau_{N+1}$ *in (3.7). By (2.7) we have* $\tau_N = C\theta_N^{d/2}(1+\epsilon_N)$, *where* $\epsilon_N \to 0$ *as* $N \to \infty$. *Let*

$$\varrho_N := \begin{cases} (p/(q-1))^{-1/2}, & p<\infty, q>1, \\ (p\log N)^{-1/2}, & p<\infty, q=1, \\ (\log N/(q-1))^{-1/2}, & p=\infty, q>1, \\ (\log N)^{-1}, & p=\infty, q=1. \end{cases}$$

*and* $\Lambda \in \mathcal{A}$. *From (3.7) it follows that*

$$e_{\tilde{n}}(\Lambda U_p, L_q) \gg \lambda\left(\theta_{N+1}\right)\varrho_N = \lambda\left(\frac{\theta_{N+1}}{\theta_N}\theta_N\right)\varrho_N$$

$$\gg \lambda\left(c\theta_N\right)\varrho_N \gg \lambda\left(\theta_N\right)\varrho_N,$$

*where the last step follows from Lemma 2.2. Since the sequence of entropy numbers is not increasing, then*

$$e_m \geq C\lambda\left(\theta_N\right)\varrho_N \gg \lambda\left(\frac{\theta_N}{n^{2/d}}n^{2/d}\right) \gg \lambda\left(n^{2/d}\right)\varrho_N, \ \ \forall m \in [n,\tilde{n}],$$

*where we have used (2.7) in the third inequality. From the last inequality we get*

$$e_n(\Lambda U_p, L_q) \gg \lambda\left(n^{2/d}\right)\varrho_n. \tag{3.8}$$

*In particular, if* $\lambda(t) = t^{-\gamma/2}, \gamma > 0$, *then* $\Lambda \in \mathcal{A}$ *and* $\Lambda U_p = W_p^\gamma$. *In this case,*

$$e_n(W_p^\gamma, L_q) \gg n^{-\gamma/d}\varrho_n.$$

**Remark 3.8** *From [19, Theorem 4], [20] it follows that*

$$d^{[C\theta_N^{d/2}]}(\Lambda U_p, L_q) \gg \lambda\left(\theta_N^{2/d}\right), \ \ 1 < p,q < \infty.$$

*Let* $\varphi \in L_p$, $2 \leq p \leq \infty$ *and* $1 \leq q \leq 2$. *Then* $\|\Lambda\varphi\|_q \leq \|\Lambda\varphi\|_2 \leq C\|\varphi\|_2 \leq C\|\varphi\|_p$, *i.e.,* $\Lambda \in \mathcal{L}(L_p, L_q)$. *It is easy to check that* $\Lambda L_p$ *is dense in* $L_q$ *since* $L_2 = \overline{\oplus_{k=0}^\infty \mathrm{H}_k}^{L_2}$ *and* $L_2$ *is dense in* $L_q$. *Also,* $L_p$ *is reflexive if* $2 \leq p < \infty$. *Hence, for any* $N \in \mathbb{N}$ *and* $1 < p,q < \infty$, *from the duality of Kolmogorov and Gel'fand* $n$*-widths given by Proposition 1.1,*

$$d_{[C\theta_{N+1}^{d/2}]}(\Lambda U_p, L_q) \gg \lambda\left(\frac{\theta_{N+1}}{\theta_N}\theta_N\right).$$

*By Lemma 2.2* $\lim_{N\to\infty}\theta_{N+1}/\theta_N = 1$. *Thus,* $\theta_{N+1}/\theta_N \leq 2$ *for some* $N_0 \in \mathbb{N}$ *and any* $N \geq N_0$. *Assume that* $\Lambda \in \mathcal{A}$ *then the last estimate can be rewritten as*

$$d_{[C\theta_{N+1}^{d/2}]}(\Lambda U_p, L_q) \gg \lambda\left(\theta_N\right).$$

*Since the sequence of Kolmogorov's* $n$*-widths* $d_n$ *is nonincreasing, then*

$$d_{[Cn^{d/2}]}(\Lambda U_p, L_q) \gg \lambda\left(n\right)$$

*for any* $n$, $\theta_N \leq n \leq \theta_{N+1}$. *Therefore, for any* $n \in \mathbb{N}$,

$$d_n(\Lambda U_p, L_q) \gg \lambda\left(n^{2/d}\right), \ \ 1 < p, q < \infty.$$

*In the case of Sobolev's classes we get*

$$d_n(W_p^\gamma, L_q) \gg n^{-\gamma/d}, \ 1 < p, q < \infty, \ \gamma > 0.$$

**Remark 3.9** *Lower bounds for Bernstein's* $n$*-widths may also be obtained. Let* $\Lambda \in \mathcal{A}$, $\Lambda = \{\lambda^{-1}(\theta_k)\}$ *and* $\Lambda^{-1} := \{\lambda^{-1}(\theta_k)\}$. *Then,* $\forall z = \sum_{k=0}^{M}\sum_{m=1}^{d_k} c_{k,m}Y_{k,m} \in \mathcal{T}_M$ *we have*

$$\left\|\Lambda^{-1}z\right\|_2^2$$

$$= \left\|\Lambda^{-1}\left(\sum_{k=1}^{M}\sum_{m=1}^{d_k} c_{k,m}(z)Y_m^k\right)\right\|_2^2 = \left\|\sum_{k=1}^{M} \lambda^{-1}(\theta_k) \sum_{m=1}^{d_k} c_{k,m}(z)Y_m^k\right\|_2^2$$

$$= \sum_{k=1}^{M} \lambda^{-2}(\theta_k) \sum_{m=1}^{d_k} |c_{k,m}(z)|^2 \ \leq \ (\max_{1\leq k\leq M} \lambda^{-2}(\theta_k)) \sum_{k=1}^{M}\sum_{m=1}^{d_k} |c_{k,m}(z)|^2$$

$$= \lambda^{-2}(\theta_M)\,\|z\|_2^2,$$

*so that* $\left\|\Lambda^{-1}z\right\|_2 \leq \lambda^{-1}(\theta_M)\,\|z\|_2$. *Therefore,* $\lambda(\theta_M)\,U_2 \cap \mathcal{T}_M \subset \Lambda U_2$ *and*

$$b_n(\Lambda U_2, L_q) \geq b_n(\lambda(\theta_M)\,U_2 \cap \mathcal{T}_M, L_q) = \lambda(\theta_M)\,b_n(U_2 \cap \mathcal{T}_M, L_q).$$

*Set* $m = \dim \mathcal{T}_M$. *By [24, Theorem 1] there exists a subspace* $X_s \subset \{\mathbb{R}^m, \|\cdot\|_{q'}^*\}$, $1 \leq q \leq 2$, $1/q + 1/q^{'} = 1$, $\dim X_s = s > \lambda l$, $0 < \lambda < 1$, *such that*

$$\|\alpha\|_2^* \leq C M_{B_{L_{q'}}^{*m}} (1-\lambda)^{-1/2}(\|\alpha\|_{q'}^*)^o, \ \ \forall \alpha \in X_s. \tag{3.9}$$

*Let* $\lambda = 1/2$. *Then* $\|\alpha\|_2^* \leq C_1 \mathrm{M}_{B_{L_{q'}}^{*m}}(\|\alpha\|_{q'}^*)^o$ *and by Hölder's inequality* $(\|\alpha\|_{q'}^*)^o \ \leq \|\alpha\|_q^*$. *Hence,*

$$\|\alpha\|_2^* \leq C_1 \mathrm{M}_{B_{L_{q'}}^{*m}} \|\alpha\|_q^*, \quad 1 \leq q \leq 2.$$

*Since, by Lemma 3.2,* $\mathrm{M}_{B_{L_{q'}}^{*m}} \leq C_2$, $2 < q^{'} < \infty$, *we have*

$$\|\alpha\|_2^* \leq C_3\,\|\alpha\|_q^* \ \ \forall \alpha \in X_s.$$

*Therefore* $X_s \cap B_q^* \subset C_4 X_s \cap B_2^*$ *and since the spaces* $\mathbb{R}^m$ *and* $\mathrm{J}\mathbb{R}^m = \mathcal{T}_M$ *are isometrically isomorphic we get* $\|z\|_2 \leq C_3\,\|z\|_q$, $\forall z \in \mathrm{J}X_s \subset \mathcal{T}_M, s \geq [m/2]$. *Hence, denoting an arbitrary* $s$*-dimensional subspace of* $\mathcal{T}_M$ *by* $Y_s$,

$$b_{s-1}(U_2 \cap \mathcal{T}_M, L_q) = \sup_{Y_s \subset L_q}\sup_{\varepsilon>0}\{\varepsilon U_q \cap Y_s \subset U_2\}$$

$$\geq \sup_{\varepsilon>0}\{\varepsilon U_q \cap \mathrm{J}X_s \subset U_2 \cap \mathcal{T}_M\}$$

$$\geq \sup_{\varepsilon>0}\{\varepsilon C_3^{-1} U_2 \cap \mathrm{J}X_s \subset U_2 \cap \mathcal{T}_M\} \geq C_3^{-1}.$$

*Consequently,*

$$b_{s-1}(\Lambda U_2, L_q) \gg \lambda\left(\theta_M\right), \ \ s \geq [m/2].$$

*Finally, applying the same line of arguments as in Remark 3 we get*

$$b_n(\Lambda U_2, L_q) \gg \lambda\left(n^{2/d}\right), \ \ q > 1.$$

We now turn to estimates for the upper bounds of entropy and $n$-widths. To avoid long technical notations we shall present here just results in the case of Sobolev's classes, i.e. if $\lambda\left(t\right) = t^{-\gamma/2}$.

**Theorem 3.10** *Let* $2 \leq p, q \leq \infty$ *and* $\gamma > d/2$. *Then*

$$d_n(\Lambda U_p, L_q) \ll n^{-\gamma/d} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\log n)^{1/2}, & q = \infty. \end{cases}$$

**Proof** It is sufficient to consider the case $p = 2$, since the case $p \geq 2$ follows by imbedding. For a given $N \in \mathbb{N}$ let $\theta_N$ be the corresponding eigenvalue of the Laplace–Beltrami operator for $\mathbb{M}^d$. Put $N_{-1} := 1$, $N_0 := N$ and for any $k \geq 0$ let $N_{k+1}$ be such that $\theta_{N_{k+1}-1} \leq 2^{2/\gamma}\theta_{N_k} \leq \theta_{N_{k+1}}$. This is always possible to do since the sequence of eigenvalues forms an increasing sequence with $+\infty$ as the only accumulation point. By Lemma 2.2, $\lim_{k\to\infty} \theta_{N_{k+1}}/\theta_{N_{k+1}-1} = 1$. Then, a simple argument shows that,

$$\lim_{k\to\infty} \theta_{N_{k+1}}/\theta_{N_k} = 2^{2/\gamma}.$$

From here we conclude that there is a $\delta(k)$, with $\delta \to 0$ as $k \to \infty$, and constants $C_1, C_2 > 0$ such that

$$C_1(1+\delta)^{-k} 2^{2k/\gamma}\theta_N \leq \theta_{N_k} \leq C_2(1+\delta)^k 2^{2k/\gamma}\theta_N. \tag{3.10}$$

Let $\mathcal{T}_{N_k,N_{k+1}} := \oplus_{l=N_k}^{N_{k+1}} \Xi_l$, and $\dim \mathcal{T}_{N_k,N_{k+1}} = l_k$. Using (2.7) we get

$$l_k < \dim \mathcal{T}_{N_{k+1}} \leq C\theta_{N_{k+1}}^{d/2}. \tag{3.11}$$

It is easy to check that

$$\mathrm{I}_\gamma(U_2 \cap \mathcal{T}_{N_k,N_{k+1}}) \subset \theta_{N_k}^{-\gamma/2}(U_2 \cap \mathcal{T}_{N_k,N_{k+1}}). \tag{3.12}$$

Thus, by Lemma 3.1 and (3.11),

$$U_2 \cap \mathcal{T}_{N_k,N_{k+1}} \subset C l_k^{1/2}(U_\infty \cap \mathcal{T}_{N_k,N_{k+1}})$$

$$\subset C\theta_{N_{k+1}}^{d/4}(U_\infty \cap \mathcal{T}_{N_k,N_{k+1}}). \tag{3.13}$$

Clearly, $\|\mathrm{P}\|_{L_2\to L_2\cap \mathcal{T}_{N_k,N_{k+1}}} = 1$, where P is the orthogonal projection. Hence, by (3.12),

$$W_2^\gamma = \mathrm{I}_\gamma U_2 \subset \bigoplus_{k=-1}^{\infty} \mathrm{I}_\gamma (U_2 \cap \mathcal{T}_{N_k,N_{k+1}})$$

$$\subset \bigoplus_{k=-1}^{\infty} \theta_{N_k}^{-\gamma/2} (U_2 \cap \mathcal{T}_{N_k,N_{k+1}}). \tag{3.14}$$

Let $\epsilon > 0$ be a fixed parameter which will be specified later,

$$M := [\epsilon^{-1}\log(\tau_N)],\ m_0 := \tau_N,\ m_k := [2^{-\epsilon k}\tau_N] + 1$$

if $1 \le k \le M$ and $m_k := 0$ if $k > M$. Let

$$\mu := \sum_{k=0}^{M} m_k \le \tau_N + \sum_{k=1}^{M} 2^{-\epsilon k}\tau_N + M \le C\tau_N \le C\theta_N^{d/2}.$$

Using Proposition 1.1 (a) and (b), (3.13) and (3.14) we find

$$d_\mu(W_2^\gamma, L_q) \le C \sum_{k=0}^{M} \theta_{N_k}^{-\gamma/2} d_{m_k}(U_2 \cap \mathcal{T}_{N_k,N_{k+1}}, L_q \cap \mathcal{T}_{N_k,N_{k+1}})$$

$$+C \sum_{k=M+1}^{\infty} \theta_{N_k}^{-\gamma/2}\theta_{N_{k+1}}^{d/4} d_0(U_\infty \cap \mathcal{T}_{N_k,N_{k+1}}, L_\infty \cap \mathcal{T}_{N_k,N_{k+1}})$$

$$:= \sigma_1 + \sigma_2, \tag{3.15}$$

where using the fact that $d_0\left(U_\infty \cap \mathcal{T}_{N_k,N_{k+1}}, L_\infty \cap \mathcal{T}_{N_k,N_{k+1}}\right) = 1$,

$$\sigma_2 \le C \sum_{k=M+1}^{\infty} \theta_{N_k}^{-\gamma/2}\theta_{N_{k+1}}^{d/4}.$$

Using (3.10),

$$\sigma_2 \le C\theta_N^{-\gamma/2+d/4} \sum_{k\ge C\epsilon^{-1}\log\theta_N} 2^{-k(2/\gamma)(\gamma/2-d/4)}(1+\delta)^k.$$

Since $\delta > 0$, and for sufficiently large $N$ we can choose $\delta$ as small as we please, then the last series converges if $\gamma/d > 1/2 - 1/q$. In this case

$$\sigma_2 \le C\theta_N^{-\gamma/2+d/4} 2^{C(\log\theta_N)(-\gamma/2+d/4)/\epsilon\gamma}(1+\delta)^{C(\log\theta_N)/\epsilon}$$

$$\le C\theta_N^{-\gamma/2+d/4}\theta_N^{C(-\gamma/2+d/4)/\epsilon\gamma}\theta_N^{C\eta/\epsilon},$$

where $\eta := \log(1+\delta)$. Hence, if

$$0 < \epsilon < C(\gamma - d/2), \tag{3.16}$$

then

$$\sigma_2 \leq C\theta_N^{-\gamma/2}. \tag{3.17}$$

To complete the proof we need to get upper bounds for $\sigma_1$. From (3.9), there exists a subspace $L_s^l \subset \{\mathbb{R}^l, \|\cdot\|_q^*\}$, $\dim L_s^l = s > \lambda l$, $0 < \lambda < 1$, such that

$$\|\alpha\|_2^* \leq C\mathrm{M}_{B_{L_q}^{*l}}(1-\lambda)^{-1/2}(\|\alpha\|_q^*)^o$$

for any $\alpha \in L_s^l$. Put $m := l - s$, then

$$\|z\|_2 \leq C(l/m)^{1/2}\mathrm{M}_{B_{L_q}^{*l}}\|z\|_q^o$$

for any $z \in \mathrm{J}L_s^l$. By duality of Kolmogorov's and Gel'fand's $n$-widths, recalling the definition of $m_k$, and letting $X_{m_k}^{l_k} \subset \mathcal{T}_{N_k,N_{k+1}}$ be an arbitrary subspace of codimension $m_k$, we get

$$d_{m_k}(B_2 \cap \mathcal{T}_{N_k,N_{k+1}}, L_q \cap \mathcal{T}_{N_k,N_{k+1}})$$

$$= d^{m_k}((B_q \cap \mathcal{T}_{N_k,N_{k+1}})^o, L_2 \cap \mathcal{T}_{N_k,N_{k+1}})$$

$$= \inf_{X_{m_k}^{l_k} \subset \mathcal{T}_{N_k,N_{k+1}}} \sup_{z \in X_{m_k}^{l_k} \cap (B_q \cap \mathcal{T}_{N_k,N_{k+1}})^o} \|z\|_2$$

$$\leq \sup_{z \in \mathrm{J}L_{s_k}^{l_k} \cap (B_q \cap \mathcal{T}_{N_k,N_{k+1}})^o} \|z\|_2,$$

where $s_k = l_k - m_k$, since $\mathrm{J}L_{s_k}^{l_k}$ is a specific subspace of codimension $m_k$. Thus, using Lemma 2.2 and (3.11),

$$d_{m_k}(B_2 \cap \mathcal{T}_{N_k,N_{k+1}}, L_q \cap \mathcal{T}_{N_k,N_{k+1}})$$

$$\leq C\left(\frac{l_k}{m_k}\right)^{1/2}\mathrm{M}_{B_{L_q}^{*l_k}} \sup_{z \in \mathrm{J}L_{s_k}^{l_k} \cap (B_q \cap \mathcal{T}_{N_k,N_{k+1}})^o} \|z\|_{(B_q \cap \mathcal{T}_{N_k,N_{k+1}})^o}$$

$$\leq C\left(\frac{\tau_{N_{k+1}}}{m_k}\right)^{1/2}\mathrm{M}_{B_{L_q}^{*l_k}}$$

$$\leq C\left(\frac{\theta_{N_{k+1}}^{d/2}}{m_k}\right)^{1/2}\mathrm{M}_{B_{L_q}^{*l_k}}$$

$$\leq C\left(\frac{((1+\delta)^k 2^{2k/\gamma}\theta_N)^{d/2}}{2^{-\epsilon k}\tau_N + 1}\right)^{1/2}\mathrm{M}_{B_{L_q}^{*l_k}},$$

from (3.10). Simplifying this last expression, it follows from Lemma 3.2 that

$$d_{m_k}(B_2 \cap \mathcal{T}_{N_k,N_{k+1}}, L_q \cap \mathcal{T}_{N_k,N_{k+1}})$$
$$\leq \quad C2^{k(d/\gamma+\epsilon)/2}(1+\delta)^{kd/4}\begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\log l_k)^{1/2}, & q = \infty. \end{cases}$$

Let

$$\eta_N := \begin{cases} q^{1/2}, & 2 \le q < \infty, \\ (\log \theta_N)^{1/2}, & q = \infty. \end{cases}$$

Then, using estimate (3.15) and (3.10) again, we get

$$\sigma_1 \le C \sum_{k=0}^{M} \theta_{N_k}^{-\gamma/2} d_{m_k}(B_2 \cap \mathcal{T}_{N_k,N_{k+1}}, L_q \cap \mathcal{T}_{N_k,N_{k+1}})$$

$$\le C\eta_N \sum_{k=0}^{M} \theta_{N_k}^{-\gamma/2} 2^{k(d/\gamma+\epsilon)/2}(1+\delta)^{kd/4}$$

$$\le C\eta_N \sum_{k=0}^{\infty} (\theta_N 2^{2k/\gamma}(1+\delta)^k)^{-\gamma/2} 2^{k(d/\gamma+\epsilon)/2}(1+\delta)^{kd/4}$$

$$\le C\eta_N \theta_N^{-\gamma/2} \sum_{k=0}^{\infty} 2^{-k(1-d/(2\gamma)-\epsilon/2)}(1+\delta)^{-k(\gamma/2-d/4)}.$$

The last sum is bounded for some $\delta > 0$ if $\gamma > d/2$, and $0 < \epsilon < 2 - d/\gamma$. Thus we must choose $\epsilon$ less than the aforementioned and the bound given in (3.16). In this case,

$$\sigma_1 \le C\theta_N^{-\gamma/2}\eta_N. \tag{3.18}$$

Finally, comparing (3.17) and (3.18) we get

$$d_{C\theta_N}(W_p^\gamma, L_q) \le C\theta_N^{-\gamma/d}\eta_N,$$

or

$$d_n(W_p^\gamma, L_q) \le Cn^{-\gamma/d} \begin{cases} q^{1/2}, & 2 \le q < \infty, \\ (\log n)^{1/2}, & q = \infty. \end{cases}$$

□

**Remark 3.11** *Comparing the above theorem with Remark 3.8, and applying an embedding arguments we get*

$$d_n(W_p^\gamma, L_q) \asymp n^{-\gamma/d}, \ \ \gamma > d/2, \ 2 \le p < \infty, \ 1 < q < \infty.$$

We are prepared now to prove the main result of this article.

**Theorem 3.12** *Let $\gamma > d$. Then for any $n \in \mathbb{N}$ and $1 \le p, q \le \infty$,*

$$e_n(W_p^\gamma, L_q) \ge C_1 n^{-\gamma/d} \begin{cases} (p/(q-1))^{-1/2}, & p < \infty, q > 1, \\ (p\log n)^{-1/2}, & p < \infty, q = 1, \\ (\log n/(q-1))^{-1/2}, & p = \infty, q > 1, \\ (\log n)^{-1}, & p = \infty, q = 1, \end{cases}$$

*and*

$$e_n(W^\gamma_p, L_q) \le C_2 n^{-\gamma/d} \begin{cases} (q/(p-1))^{1/2}, & 2 \le q < \infty, 1 < p \le 2, \\ (q \log n)^{1/2}, & 2 \le q < \infty, p = 1, \\ (\log n/(p-1))^{1/2}, & q = \infty, 1 < p \le 2, \\ \log n, & q = \infty, p = 1, \end{cases}$$

*where* $C_1, C_2 > 0$. *In particular, if* $1 < p, q < \infty$, *then*

$$e_n(W^\gamma_p, L_q) \asymp n^{-\gamma/d}.$$

**Proof** From Theorem 3.10, and the duality of Kolmogorov and Gel'fand $n$-widths, we have

$$d^n(W^\gamma_{q'}, L_2) = d_n(W^\gamma_2, L_q) \ll n^{-\gamma/d} \begin{cases} q^{1/2}, & 2 \le q < \infty, \\ (\log n)^{1/2}, & q = \infty, \end{cases}$$

where $1/q + 1/q^{'} = 1$. Let $\{s_n\}$ denotes either of the sequences $\{d_n\}$ or $\{d^n\}$. Assume that $f(l)$, $f : \mathbb{N} \to \mathbb{R}$ is a positive and increasing (for large $l \in \mathbb{N}$) function such that $f(2^j) \le Cf(2^{j-1})$ for some fixed $C$ and any $j \in \mathbb{N}$. Then, there is a constant $C > 0$ such that for all $n \in \mathbb{N}$ we have

$$\sup_{1 \le l \le n} f(l) e_l(A, X) \le C \sup_{1 \le l \le n} f(l) s_l(A, X), \quad n \in \mathbb{N}$$

(see e.g., [4–6]). In particular, let $A = W^\gamma_2$, $X = L_q$,

$$f^*(l) := l^{\gamma/d} \begin{cases} q^{-1/2}, & 2 \le q < \infty, \\ (\log l)^{-1/2}, & q = \infty, \end{cases}$$

then $f^*(2^j) \le Cf^*(2^{j-1})$ for some $C > 0$ and

$$f^*(n) e_n(W^\gamma_2, L_q) \le \sup_{1 \le l \le n} f^*(l) d_l(W^\gamma_2, L_q) \le C.$$

or

$$e_n(W^\gamma_2, L_q) \le C n^{-\gamma/d} \begin{cases} q^{1/2}, & 2 \le q < \infty, \\ (\log n)^{1/2}, & q = \infty, \end{cases} \tag{3.19}$$

where $\gamma > d/2$ by the Theorem 3.10. Similarly, if $\gamma > d/2$, then

$$e_n(W^\gamma_p, L_2) \le C n^{-\gamma/d} \begin{cases} (p-1)^{-1/2}, & 1 < p \le 2, \\ (\log n)^{1/2}, & p = 1. \end{cases} \tag{3.20}$$

Applying the multiplicative property of entropy numbers (see e.g., [25]), (3.19) and (3.20) we get,

$$e_n(W^\gamma_p, L_q) = e_n(I_\gamma : L_p \to L_q)$$

$$= e_n(I_{\gamma/2} : L_p \to L_2) \cdot e_n(I_{\gamma/2} : L_2 \to L_q)$$

$$\le C n^{-\gamma/d} \begin{cases} (q/(p-1))^{1/2}, & 2 \le q < \infty, 1 < p \le 2, \\ (q \log n)^{1/2}, & 2 \le q < \infty, p = 1, \\ (\log n/(p-1))^{1/2}, & q = \infty, 1 < p \le 2, \\ \log n, & q = \infty, p = 1, \end{cases} \tag{3.21}$$

where $\gamma/2 > d/2$ or $\gamma > d$. Finally, comparing (3.8) and (3.21) we get the proof. □

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