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Existence of a positive solution for a singular fractional boundary value problem with fractional boundary conditions using convolution and lower order problems

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Abstract: Existence of a positive solution is shown for two singular two-point fractional boundary value problems with fractional boundary conditions using fixed point theory, lower order problems, and convolution of Green's functions. A nontrivial example is included.

Key words: Fractional differential equation, singular problem, fixed point, positive solution, convolution, change of variable

1. Introduction

Let $n \in \mathbb{N}$, $n \geq 5$. For $\alpha \in (n-1, n]$, consider the singular fractional differential equation

$$D_{0^+}^{\alpha} u = f(t, u), \quad t \in (0,1), \tag{1.1}$$

satisfying the boundary conditions

$$u^{(i)}(0) = 0,\ i = 0, \dots, n-4, \quad D_{0^+}^{\alpha-2} u(0) = 0, \quad D_{0^+}^{\alpha-2} u(1) = 0, \quad D_{0^+}^{\beta} u(1) = 0, \tag{1.2}$$

or

$$u^{(i)}(0) = 0,\ i = 0, \dots, n-4, \quad D_{0^+}^{\alpha-2} u(0) = 0, \quad D_{0^+}^{\alpha-1} u(1) = 0, \quad D_{0^+}^{\beta} u(1) = 0, \tag{1.3}$$

where $\beta \in [1, n-3]$, and $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are the Riemann–Liouville fractional derivatives of order α and β, respectively, and $f(t,u)$ is singular at $t = 0$, $t = 1$, and $u = 0$. Here, we assume

(H1) $f(t,u) : (0,1) \times (0,\infty) \to (0,\infty)$ is continuous;

(H2) $f(t,u)$ is decreasing in u for each t; and

(H3) $\lim_{u \to 0^+} f(t,u) = \infty$ and $\lim_{u \to \infty} f(t,u) = 0$, uniformly on compact subsets of $(0,1)$.

The main purpose of this paper is to show that by writing the associated Green's function of each these problems as a convolution of Green's functions of lower order problems, we attain properties that allow us to show existence of positive solutions of (1.1), (1.2) and (1.1), (1.3). Since the Green's function of the higher

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order problem inherits some properties of the Green's function of the lower order problem, similar arguments to ones showing the existence of solutions of the lower order problem can be used to show the existence of positive solutions of (1.1), (1.2) and (1.1), (1.3).

The study of singular fractional boundary value problems has been extensive and growing in recent years. Several authors have used a range of fixed point theorems to establish the existence of positive solutions for various types of problems [1, 10, 12, 16–19]. As a specific recent example, we point the reader to Henderson and Luca [7] for their study of the singular fractional problem

$$D^{\alpha}_{0+}u(t) + \lambda f(t, u(t)) = 0, \quad t \in (0, 1),$$

$$u^{(i)}(0) = 0,\ i = 0, 1, \ldots, n-2, \quad D^{p}_{0+}u(1) = \sum_{i=1}^{m} a_i D^{q}_{0+}u(\xi_i),$$

where λ is a positive parameter, $\alpha \in \mathbb{R}$, $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, $n \geq 3$, $\xi_i \in \mathbb{R}$ for all $i = 1, \ldots, m$, $m \in \mathbb{N}$, $0 < \xi_1 < \cdots < \xi_m < 1$, $p, q \in \mathbb{R}$, $p \in [1, n-2]$, $q \in [0, p]$. The function f may be singular and the nonlinearity of f may change sign at $t = 0$ or $t = 1$. Here the authors employed Krasnosel'skii's fixed point theorem [9].

Of primary interest to this work are the techniques employed by Cui [2] and then Neugebauer in [14] in seeking positive solutions for a singular fractional two-point boundary value problem using the Gatica, Oliker, Waltman fixed point theorem. We will follow the convolution procedure outlined by Lyons and Neugebauer in [11] to construct the associated Green's functions for (1.1), (1.2) and (1.1), (1.3).

Section two establishes background information and states the Gatica, Oliker, Waltman fixed point theorem. The following section introduces the lower order problem and associated Green's function found in [4]. Next, we present the main result over two sections. We conclude with an example.

2. Preliminary definitions and the fixed point theorem

Definition 2.1 *Let $\nu > 0$. The Riemann–Liouville fractional integral of a function u of order ν, denoted $I^{\nu}_{0+}u$, is defined as*

$$I^{\nu}_{0+}u(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} u(s)ds,$$

provided the right-hand side exists. Moreover, let n denote a positive integer and assume $n - 1 < \alpha \leq n$. The Riemann–Liouville fractional derivative of order α of the function $u : [0, 1] \to \mathbb{R}$, denoted $D^{\alpha}_{0+}u$, is defined as

$$D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)ds = D^n I^{n-\alpha}_{0+} u(t),$$

provided the right-hand side exists.

For a detailed view of fractional calculus, see the books by Diethelm [3], Kilbas, Srivastava, and Trujillo [8], Miller and Ross [13], or Podlubny [15].

Definition 2.2 *Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided*

(i) $\alpha u + \beta v \in \mathcal{P}$ for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \geq 0$, and

(ii) $u \in \mathcal{P}$ *and* $-u \in \mathcal{P}$ *implies* $u = 0$.

Given a cone $\mathcal{P}$ over a Banach space $\mathcal{B}$, $u \le v$ for $u, v \in \mathcal{B}$ if $v - u \in \mathcal{P}$.

If $u, v \in \mathcal{B}$ with $u \le v$, let $\langle u, v\rangle$, the closed order interval between u and v, be defined by

$$\langle u, v\rangle = \{w \in \mathcal{B} : u \le w \le v\}.$$

Definition 2.3 *A cone* $\mathcal{P}$ *is normal in* $\mathcal{B}$ *if there exists a* $\delta > 0$ *such that* $\|e_1 + e_2\| \ge \delta$ *for all* $e_1, e_2 \in \mathcal{P}$ *with* $\|e_1\| = \|e_2\| = 1$.

If $\mathcal{P}$ being a normal cone, then closed order intervals are norm bounded.

The following theorem, due to Gatica, Oliker, and Waltman [6] is used to show the existence of a positive solution of (1.1), (1.2) and (1.1), (1.3).

Theorem 2.4 *Let* $\mathcal{B}$ *be a Banach space,* $\mathcal{P}$ *a normal cone,* J *a subset of* $\mathcal{P}$ *such that if* $u, v \in J$ *with* $u \le v$, *then* $\langle u, v\rangle \subset J$, *and let* $T : J \to \mathcal{P}$ *be a continuous decreasing mapping which is compact on any closed order interval contained in* J. *Suppose there exists a* $u_0 \in J$ *such that* T^2u_0 *is defined, and furthermore,* Tu_0 *and* T^2u_0 *are order comparable to* u_0. *Then* T *has a fixed point in* J *provided that, either*

1. $Tu_0 \le u_0$ *and* $T^2u_0 \le u_0$, *or* $Tu_0 \ge u_0$ *and* $T^2u_0 \ge u_0$; *or*
2. *the complete sequence of iterates* $\{T^n u_0\}_{n=0}^{\infty}$ *is defined, and there exists a* $v_0 \in J$ *such that* $v_0 \le T^n u_0$ *for every* n.

We define the Banach space $\mathcal{B}$ by $\mathcal{B} = C[0,1]$ with the norm

$$\|u\| = \max_{0 \le t \le 1} |u(t)|.$$

Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{u \in \mathcal{B} : u(t) \ge 0 \text{ for all } t \in [0,1]\}.$$

Notice $\mathcal{P}$ is normal.

3. The lower order problem

In the process of obtaining the existence of positive solutions of (1.1), (1.2) and (1.1), (1.3), we will require properties of the Green's function associated with the following lower order fractional boundary value problem. The details and results of this section are found in [14].

Let $m \in \mathbb{N}$, $m \ge 3$. For $\mu \in (m-1, m]$,

$$D_{0^+}^{\mu} u + f(t, u) = 0, \quad 0 < t < 1, \tag{3.1}$$

satisfying the boundary conditions

$$u^{(i)}(0) = 0,\ i = 0, \dots, m-2, \quad D_{0^+}^{\beta} u(1) = 0, \tag{3.2}$$

where $\beta \in [1, m-1]$, and $D_{0^+}^{\mu}$, $D_{0^+}^{\beta}$ are the Riemann–Liouville fractional derivatives of order μ and β, respectively.

The Green's function for $-D^{\mu}_{0+}u = 0$ satisfying the boundary conditions (3.2) is given by (see [4])

$$G(t,s) = \begin{cases} \frac{t^{\mu-1}(1-s)^{\mu-1-\beta}}{\Gamma(\mu)} - \frac{(t-s)^{\mu-1}}{\Gamma(\mu)}, & 0 \le s < t \le 1, \\ \frac{t^{\mu-1}(1-s)^{\mu-1-\beta}}{\Gamma(\mu)}, & 0 \le t \le s < 1. \end{cases} \tag{3.3}$$

Therefore, u is a solution of (3.1), (3.2) if and only if

$$u(t) = \int_0^1 G(t,s)f(s,u(s))ds, \quad 0 \le t \le 1.$$

Define $g : [0,1] \to [0,1]$ by

$$g(t) = t^{\alpha-1}.$$

For $\theta > 0$, define

$$g_\theta(t) = \theta g(t).$$

Now $g(t) > 0$ on $(0,1]$ and $\max_{t\in[0,1]} g(t) = 1$. This implies $\max_{t\in[0,1]} g_\theta(t) = \theta$.

We note that the properties in the following two lemmas are carried over in a similar fashion to the higher order problem, and hence, we present the results here.

Lemma 3.1 *Let G be defined as in (3.3).*

1. *$G(t,s) \in C([0,1]\times[0,1))$ with $G(t,s) > 0$.*
2. *$g(t)G(1,s) \le G(t,s) \le G(1,s)$ for $(t,s) \in [0,1]\times[0,1)$.*

Lemma 3.2 *Suppose $D^{\alpha}_{0+}u \in C[0,1]$. If $-D^{\alpha}_{0+}u(t) \ge 0$ for all $t \in [0,1]$ and u satisfies (1.2), then*

$$g(t)u(1) \le u(t) \le u(1), \quad \textit{for } t \in [0,1].$$

Finally, we make the following additional assumption.

(H4) $\int_0^1 G(1,s)f(s,g_\theta(s))ds < \infty$ for all $\theta > 0$.

Theorem 3.3 *If f satisfies (H1)-(H4), then (3.1), (3.2) has at least one positive solution $u \in D$.*

4. Existence result for (1.1), (1.2)

Now, consider the fractional boundary value problem (1.1), (1.2). We use methods similar to the ones used in [5] and [11] to construct the associated Green's function. For completeness, the construction is given here.

Recall that we have $n \in \mathbb{N}$ with $n \ge 5$ and $\alpha \in (n-1,n]$. To apply the results in the previous section, we set $m = n-2$ so $m \in \mathbb{N}$ with $m \ge 3$. Thus, $\alpha - 2 \in (m-1,m]$.

Let h be a continuous function on $[0,1]$ and consider the linear fractional differential equation

$$D^{\alpha}_{0+}u = h(t), \quad t \in (0,1), \tag{4.1}$$

satisfying (1.2). Make the change of variable $v(t) = D_{0^+}^{\alpha-2}u(t)$. By the definition of the fractional integral,

$$D^2v(t) = D^2D_{0^+}^{\alpha-2}u(t) = D^2D^mI_{0^+}^{m-(\alpha-2)}u(t) = D^{m+2}I_{0^+}^{n-\alpha}u(t) = D^nI_{0^+}^{n-\alpha}u(t) = D_{0+}^{\alpha}u(t).$$

Using the boundary conditions $D_{0^+}^{\alpha-2}u(0) = 0$, $D_{0^+}^{\alpha-2}u(1) = 0$, we see that v satisfies the differential equation

$$v'' = h(t), \quad t \in (0,1),$$

and the Dirichlet boundary conditions

$$v(0) = 0, \quad v(1) = 0.$$

Thus,

$$v(t) = \int_0^1 G_{conj}(t,s)(-h(s))ds,$$

where $G_{conj}(t,s)$ has the form

$$G_{conj}(t,s) = \begin{cases} t(1-s), & 0 \le t < s \le 1, \\ s(1-t), & 0 \le s < t \le 1. \end{cases}$$

Now u satisfies the boundary value problem

$$D_{0^+}^{\alpha-2}u = v(t), \quad t \in (0,1),$$

$$u^{(i)}(0) = 0, \ i = 0, \dots, m-2, \quad D_{0^+}^{\beta}u(1) = 0.$$

The Green's function is given by (3.3) (with $\mu = \alpha - 2$), which has the form

$$G(\alpha-2;t,s) = \begin{cases} \frac{t^{\alpha-3}(1-s)^{\alpha-3-\beta}}{\Gamma(\alpha-2)} - \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)}, & 0 \le s < t \le 1, \\ \\ \frac{t^{\alpha-3}(1-s)^{\alpha-3-\beta}}{\Gamma(\alpha-2)}, & 0 \le t \le s < 1. \end{cases} \tag{4.2}$$

Thus,

$$\begin{aligned} u(t) &= \int_0^1 G(\alpha-2;t,s)(-v(s))ds \\ &= \int_0^1 G(\alpha-2;t,s)\left(-\int_0^1 G_{conj}(s,r)(-h(r))dr\right)ds \\ &= \int_0^1\int_0^1 (G(\alpha-2;t,s)G_{conj}(s,r)ds)\,h(r)dr. \end{aligned}$$

Therefore,

$$u(t) = \int_0^1 \mathcal{G}(t,s)h(s)ds, \quad t \in [0,1],$$

where

$$\mathcal{G}(t,s) = \int_0^1 G(\alpha-2;t,r)G_{conj}(r,s)dr, \quad (t,s) \in [0,1]\times[0,1]. \tag{4.3}$$

So the Green's function for $D^{\alpha}_{0+}u = 0$ satisfying (1.2) is given by (4.3). Thus, u is a solution of (1.1), (1.2) if and only if

$$u(t) = \int_0^1 \mathcal{G}(t,s)f(s,u(s))ds.$$

Define $a : [0,1] \to [0,1]$ by

$$a(t) = t^{\alpha-3}.$$

For $\theta > 0$, define

$$a_\theta(t) = \theta a(t).$$

Now, $a(t) > 0$ and $\max_{t\in[0,1]} a(t) = 1$. This implies $\max_{t\in[0,1]} a_\theta(t) = \theta$. The next two lemmas are proven in [11].

Lemma 4.1 *Let $\mathcal{G}(t,s)$ be defined as in (4.3). Then*

1. *$\mathcal{G}(t,s) \in C([0,1]\times[0,1))$ with $\mathcal{G}(t,s) > 0$ for $(t,s) \in (0,1]\times(0,1)$.*
2. *$a(t)\mathcal{G}(1,s) \le \mathcal{G}(t,s) \le \mathcal{G}(1,s)$ for $(t,s) \in [0,1]\times[0,1]$.*

Lemma 4.2 *Suppose $D^{\alpha}_{0+}u \in C[0,1]$. If $D^{\alpha}_{0+}u(t) \ge 0$ for all $t \in [0,1]$ and u satisfies (1.2), then*

$$a(t)u(1) \le u(t) \le u(1) \text{ for } t \in [0,1].$$

We make the following additional assumption.

($\mathcal{G}$4) $\int_0^1 \mathcal{G}(1,s)f(s,a_\theta(s))ds < \infty$ for all $\theta > 0$.

Remark 4.3 *We note here that while $\mathcal{G}(t,s)$ would be difficult to calculate, it is not difficult to calculate $\mathcal{G}(1,s)$. In fact, notice*

$$\begin{aligned}
\mathcal{G}(1,s) &= \int_0^1 G(\alpha-2;1,r)G_{conj}(r,s)dr \\
&= \int_0^s \frac{(1-r)^{\alpha-3-\beta}-(1-r)^{\alpha-3}}{\Gamma(\alpha-2)} r(1-s)dr \\
&\quad + \int_s^1 \frac{(1-r)^{\alpha-3-\beta}-(1-r)^{\alpha-3}}{\Gamma(\alpha-2)} s(1-r)dr \\
&= \frac{1}{(\alpha-1-\beta)(\alpha-2-\beta)\Gamma(\alpha)}\big((1-s)^{\alpha-1}(\alpha-1+\beta)(\alpha-2+\beta) \\
&\quad -(1-s)^{\alpha-1-\beta}(\alpha-2)(\alpha-1)+(1-s)\beta(2\alpha-3-\beta)\big)
\end{aligned}$$

Therefore, checking condition ($\mathcal{G}$4) is not difficult.

Define a subset $D \subset \mathcal{P}$ by

$$D = \{u \in \mathcal{P} : \text{there exists a } \theta(u) > 0 \text{ so } a_\theta(t) \le u(t) \text{ for all } t \in [0,1]\}.$$

Notice if u is a positive solution of (1.1), (1.2), then, by Lemma 4.2,

$$a_{u(1)}(t) \leq u(t), \text{ for all } t \in [0,1],$$

implying $u \in D$.

Define an integral operator $T : D \to \mathcal{P}$ by

$$(Tu)(t) = \int_0^1 \mathcal{G}(t,s) f(s,u(s)) ds, \quad u \in D.$$

The singularity in f at $u = 0$ means we cannot define T on all of $\mathcal{P}$. Thus, we define T on D instead.

Lemma 4.4 *The operator T is well-defined, decreasing, and $T : D \to D$.*

Proof

Let $u, v \in D$ with $v(t) \leq u(t)$ for all $t \in [0,1]$. Then there exists a $\theta(v) > 0$ such that $a_\theta(t) \leq v(t)$. Assumptions (H2) and ($\mathcal{G}$4) and the positivity of $\mathcal{G}$ give that

$$\begin{aligned} 0 &\leq \int_0^1 \mathcal{G}(t,s) f(s,u(s)) ds \\ &\leq \int_0^1 \mathcal{G}(t,s) f(s,v(s)) ds \\ &\leq \int_0^1 \mathcal{G}(t,s) f(s,a_\theta(s)) ds \\ &< \infty. \end{aligned}$$

Thus T is well-defined on D, and T is a decreasing operator.

Now, for $u \in D$, $D_{0^+}^{\alpha}(Tu)(t) = f(t,u(t)) > 0$ for all $t \in (0,1)$, and Tu satisfies the boundary conditions (1.2). So by Lemma 4.2, $Tu \in D$. □

In the next two lemmas, we show the existence of *a priori* bounds of solutions of (1.1), (1.2).

Lemma 4.5 *If f satisfies (H1)-(H3) and ($\mathcal{G}$4), then there exists an $S > 0$ such that if $u \in D$ is a solution of (1.1), (1.2), then $\|u\| \leq S$.*

Proof Assume by way contradiction that the conclusion is false. So there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of solutions of (1.1), (1.2) in D such that $u_n(t) > 0$ for all $0 < t \leq 1$, and

$$\|u_n\| \leq \|u_{n+1}\| \text{ and } \lim_{n\to\infty} \|u_n\| = \infty.$$

Now, for $n \in \mathbb{N}$, $\|u_n\| = u_n(1)$,

$$u_n(t) \geq a_{\|u_n\|}(t) \geq a_{\|u_1\|}(t) = \text{ for all } t \in [0,1].$$

So, for $n \in \mathbb{N}$ and $t \in [0,1]$, by assumptions (H2), ($\mathcal{G}4$) and by Lemma 4.1,

$$\begin{aligned} u_n(t) &= Tu_n(t) \\ &= \int_0^1 \mathcal{G}(t,s) f(s, u_n(s)) ds \\ &\leq \int_0^1 \mathcal{G}(1,s) f(s, a_{\|u_1\|}(s)) ds := N < \infty. \end{aligned}$$

Thus, $\|u_n\| \leq N$ for all $n \geq 1$, which is a contradiction. □

Lemma 4.6 *If f satisfies (H1)-(H3) and ($\mathcal{G}4$), then there exists an $R > 0$ such that if $u \in D$ is any solution of* (1.1), (1.2), *then $\|u\| \geq R$.*

Proof Assume by way of contradiction that the conclusion is false. Then, there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of solutions of (1.1), (1.2) in D such that $u_n(t) > 0$ for all $0 < t \leq 1$, and

$$\|u_{n+1}\| \leq \|u_n\| \text{ and } \lim_{n\to\infty} \|u_n\| = 0 \text{ on } [0,1].$$

Since $\mathcal{G}(t,s) > 0$ for $(t,s) \in (0,1) \times (0,1)$,

$$M := \inf\{\mathcal{G}(t,s) : (t,s) \in [1/4, 3/4] \times [1/4, 3/4]\} > 0.$$

By assumption (H3), $\lim_{u\to 0^+} f(t,u) = \infty$ uniformly on compact subsets of $(0,1)$. So there exists a $\delta > 0$ such that for $t \in [1/4, 3/4]$ and $u \in (0, \delta)$, $f(t,u) \geq \frac{2}{M}$. However, since $\lim_{n\to\infty} \|u_n\| = 0$ on $[0,1]$, there exists an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$,

$$0 < u_n(t) < \frac{\delta}{2} \text{ for all } t \in [1/4, 3/4].$$

So, for $t \in [1/4, 3/4]$ and $n \geq n_0$,

$$\begin{aligned} u_n(t) &= (Tu_n)(t) \\ &= \int_0^1 \mathcal{G}(t,s) f(s, u_n(s)) ds \\ &\geq \int_{1/4}^{3/4} \mathcal{G}(t,s) f(s, u_n(s)) ds \\ &\geq M \int_{1/4}^{3/4} f\left(s, \frac{\delta}{2}\right) ds \\ &\geq M \int_{1/4}^{3/4} \frac{2}{M} ds \\ &= 1, \end{aligned}$$

which is a contradiction. □

Theorem 4.7 *If f satisfies (H1)–(H3) and ($\mathcal{G}$4), then* (1.1), (1.2) *has at least one positive solution $u \in D$.*

Proof For each $n \geq 1$, define $u_n : [0,1] \to [0,\infty)$ by

$$u_n(t) = \int_0^1 \mathcal{G}(t,s) f(s,n) ds.$$

By condition (H2),

$$0 < u_{n+1}(t) < u_n(t) \text{ for all } t \in (0,1).$$

By (H3),

$$\lim_{n\to\infty} u_n(t) = 0 \text{ uniformly on } [0,1].$$

Next, define $f_n(t,u) : (0,1) \times [0,\infty) \to [0,\infty)$ by

$$f_n(t,u) = f(t, \max\{u, u_n(t)\}).$$

Then, f_n is continuous, and f_n does not have a singularity at $u = 0$. Also, for $(t,u) \in (0,1) \times (0,\infty)$,

$$f_n(t,u) \leq f(t,u),$$

and

$$f_n(t,u) \leq f(t, u_n(t)).$$

Now, define a sequence of operators $T_n : \mathcal{P} \to \mathcal{P}$, $n \geq 1$, by

$$T_n u(t) = \int_0^1 \mathcal{G}(t,s) f_n(s, u(s)) ds, \quad t \in [0,1].$$

Since f_n is continuous and does not have a singularity at $u = 0$, each T_n is well-defined. A standard application of the Arzelà–Ascoli theorem gives that each T_n is compact.

Next, notice that, for $u_0 = 0 \in \mathcal{P}$,

$$\begin{aligned} T_n(0)(t) &= \int_0^1 \mathcal{G}(t,s) f_n(s,0) ds \\ &= \int_0^1 \mathcal{G}(t,s) f(s, \max\{0, u_n(s)\}) ds \\ &= \int_0^1 \mathcal{G}(t,s) f(s, u_n(s)) ds \\ &\geq 0. \end{aligned}$$

This implies $T_n(0) \in \mathcal{P}$ (or $0 \leq T_n(0)$), which implies $T_n^2(0)(t) = T_n(T_n(0))(t) \geq 0$. Thus, $0 \leq T_n^2(0)$. By Theorem 2.4, with $u_0 = 0$, for each $n \in \mathbb{N}$, T_n has a fixed point $\varphi_n \in \mathcal{P}$. This implies φ_n satisfies the boundary conditions (1.2) and, for $t \in [0,1]$,

$$\begin{aligned} T_n \varphi_n(t) &= \int_0^1 \mathcal{G}(t,s) f_n(s, \varphi_n(s)) ds \\ &\leq \int_0^1 \mathcal{G}(t,s) f(s, u_n(s)) ds \\ &= T u_n(t). \end{aligned}$$

Arguments similar to those in the proofs of Lemmas 4.5 and 4.6 give the existence of $R > 0$, $S > 0$ such that

$$R \leq \|\varphi_n\| \leq S$$

for each n. Since for each $n \in \mathbb{N}$, $D^{\alpha}_{0^+}\varphi_n(t) = f_n(t, \varphi(t)) \geq 0$ and φ_n satisfies (1.2), the conclusions of Lemma 4.2 hold for φ_n. Thus for $t \in [0, 1]$,

$$\varphi_n(t) \geq a(t)\varphi_n(1) \geq Ra(t) = a_R(t) \text{ for all } n \in \mathbb{N}.$$

Therefore, the sequence $\{\varphi_n\}$ is contained in the closed order interval $\langle a_R, S\rangle$. So $\{\varphi_n\} \subset D$. Since $T : D \to D$ is a compact mapping, there is a subsequence of $\{T\varphi_n\}$ which converges to some $\varphi^* \in D$. We assume without loss of generality that $\lim_{n\to\infty} T\varphi_n = \varphi^*$.

Finally, we show φ^* is a solution of (1.1), (1.2). By assumption ($\mathcal{G}4$), there exists a $\delta \in (0, 1)$ such that

$$\int_0^{\delta} \mathcal{G}(1, s) f(s, a_R(s))ds < \frac{\epsilon}{2}.$$

Since $\lim_{n\to\infty} u_n(t) = 0$ uniformly on $[0, 1]$, there exists an n_0 such that for all $n \geq n_0$,

$$u_n(t) \leq a_R(t) \text{ for all } t \in [\delta, 1].$$

This implies

$$u_n(t) \leq a_R(t) \leq \varphi_n(t) \text{ for all } t \in [\delta, 1].$$

For $t \in [\delta, 1]$ and $n \geq n_0$,

$$f_n(t, \varphi_n(t)) = f(t, \max\{\varphi_n(t), u_n(t)\}) = f(t, \varphi_n(t)).$$

Thus, for $t \in [0, 1]$,

$$\begin{aligned} |T\varphi_n(t) - \varphi_n(t)| &\leq \int_0^{\delta} \mathcal{G}(t, s) f(s, \varphi_n(s))ds + \int_0^{\delta} \mathcal{G}(t, s) f(s, u_n(s))ds \\ &\leq \int_0^{\delta} \mathcal{G}(t, s) f(s, \varphi_n(s))ds + \int_0^{\delta} \mathcal{G}(t, s) f(s, \varphi_n(s))ds \\ &\leq 2\int_0^{\delta} \mathcal{G}(1, s) f(s, a_R(s))ds \\ &< \epsilon. \end{aligned}$$

Thus, $\|T\varphi_n - \varphi_n\| \leq \epsilon$ for all $n \geq n_0$. Hence, $\lim_{n\to\infty}(T\varphi_n(t) - \varphi_n(t)) = 0$ uniformly on $[0, 1]$. So for $t \in [0, 1]$,

$$T\varphi^*(t) = T\left(\lim_{n\to\infty} T\varphi_n(t)\right) = T\left(\lim_{n\to\infty} \varphi_n(t)\right) = \lim_{n\to\infty} T\varphi_n(t) = \varphi^*(t).$$

Therefore, $\varphi^* \in D$ is a solution of (1.1), (1.2). □

5. Existence result for (1.1), (1.3)

Now, consider the fractional boundary value problem (1.1), (1.3). Again, we use methods similar to the ones used in [5] and [11] to construct the associated Green's function. For completeness, the construction is given here.

Recall that we have $n \in \mathbb{N}$ with $n \geq 5$ and $\alpha \in (n-1, n]$. Similar to the previous section, we set $m = n-2$ so $m \in \mathbb{N}$ with $m \geq 3$. Thus, $\alpha - 2 \in (m-1, m]$.

Let h be a continuous function on $[0,1]$ and consider the linear fractional differential equation

$$D_{0+}^{\alpha} u = h(t), \quad t \in (0,1), \tag{5.1}$$

satisfying (1.3). Make a change of variable $v(t) = D_{0+}^{\alpha-2} u(t)$. By the definition of the fractional integral,

$$D^2 v(t) = D^2 D_{0+}^{\alpha-2} u(t) = D^2 D^m I_{0+}^{m-(\alpha-2)} u(t) = D^{m+2} I_{0+}^{n-\alpha} u(t) = D^n I_{0+}^{n-\alpha} u(t) = D_{0+}^{\alpha} u(t).$$

Using only the boundary conditions $D_{0+}^{\alpha-2} u(0) = 0$, $D_{0+}^{\alpha-1} u(1) = 0$, v satisfies the differential equation

$$v'' = h(t), \quad t \in (0,1),$$

and the right focal boundary conditions

$$v(0) = 0, \quad v'(1) = 0.$$

Thus,

$$v(t) = \int_0^1 G_{foc}(t,s)(-h(s))ds,$$

where $G_{foc}(t,s)$ is well-known and has the form

$$G_{foc}(t,s) = \begin{cases} t, & 0 \leq t < s \leq 1, \\ s, & 0 \leq s < t \leq 1. \end{cases}$$

The function u satisfies the boundary value problem

$$D_{0+}^{\alpha-2} u = v(t), \quad t \in (0,1),$$

$$u^{(i)}(0) = 0, \ i = 0, \ldots, m-2, \quad D_{0+}^{\beta} u(1) = 0.$$

The Green's function is given by (3.3). Thus,

$$\begin{aligned} u(t) &= \int_0^1 G(\alpha-2; t,s)(-v(s))ds \\ &= \int_0^1 G(\alpha-2; t,s) \left(-\int_0^1 G_{foc}(s,r)(-h(r))dr \right) ds \\ &= \int_0^1 \int_0^1 \left(G(\alpha-2; t,s) G_{foc}(s,r) ds \right) h(r) dr. \end{aligned}$$

Therefore,

$$u(t) = \int_0^1 \mathcal{H}(t,s) h(s) ds, \quad t \in [0,1],$$

where

$$\mathcal{H}(t,s) = \int_0^1 G(\alpha-2;t,r)G_{foc}(r,s)dr, \quad (t,s) \in [0,1]\times[0,1]. \tag{5.2}$$

So the Green's function for $D_{0^+}^{\alpha}u = 0$ satisfying (1.3) is given by (5.2). Thus, u is a solution of (1.1), (1.3) if and only if

$$u(t) = \int_0^1 \mathcal{H}(t,s)f(s,u(s))ds.$$

Recall that $a:[0,1]\to[0,1]$ where $a(t) = t^{\alpha-3}$, and for $\theta > 0$, that $a_\theta(t) = \theta a(t)$. Thus, $a(t) > 0$ and $\max_{t\in[0,1]} a(t) = 1$ which implies $\max_{t\in[0,1]} a_\theta(t) = \theta$. The next two lemmas are proven in [11].

Lemma 5.1 *Let $\mathcal{H}(t,s)$ be defined as in (5.2). Then*

1. *$\mathcal{H}(t,s) \in C([0,1]\times[0,1))$ with $\mathcal{H}(t,s) > 0$ for $(t,s)\in(0,1]\times(0,1)$.*
2. *$a(t)\mathcal{H}(1,s) \le \mathcal{H}(t,s) \le \mathcal{H}(1,s)$ for $(t,s)\in[0,1]\times[0,1]$.*

Lemma 5.2 *Suppose $D_{0^+}^{\alpha}u \in C[0,1]$. If $D_{0^+}^{\alpha}u(t) \ge 0$ for all $t\in[0,1]$ and u satisfies (1.3), then*

$$a(t)u(1) \le u(t) \le u(1) \text{ for } t\in[0,1].$$

We make the following additional assumption.

($\mathcal{H}4$) $\int_0^1 \mathcal{H}(1,s)f(s,a_\theta(s))ds < \infty$ for all $\theta > 0$.

Remark 5.3 *Similar to $\mathcal{G}(t,s)$, $\mathcal{H}(t,s)$ would be difficult to calculate. However, one can easily obtain a result for $\mathcal{H}(1,s)$ similar to that of $\mathcal{G}(1,s)$ from the previous section.*

Define a subset $D \subset \mathcal{P}$ as before by

$$D = \{u \in \mathcal{P} : \text{there exists a } \theta(u) > 0 \text{ so } a_\theta(t) \le u(t) \text{ for all } t\in[0,1]\}.$$

Notice if u is a positive solution of (1.1), (1.3), then, by Lemma 5.2,

$$a_{u(1)}(t) \le u(t), \text{ for all } t\in[0,1],$$

implying $u \in D$.

Define an integral operator $T: D \to \mathcal{P}$ by

$$(Tu)(t) = \int_0^1 \mathcal{H}(t,s)f(s,u(s))ds, \quad u \in D.$$

Because of the singularity in f at $u = 0$, we cannot define T on all of $\mathcal{P}$. Hence, we define T on D instead. A similar argument to the previous section provides that T is well-defined, decreasing, and $T: D \to D$.

The next two lemmas provide the existence of a priori bounds of solutions of (1.1), (1.3). The proofs are similar to the analogues of the previous section, and therefore, are omitted.

Lemma 5.4 *If f satisfies (H1)–(H3) and ($\mathcal{H}4$), then there exists an $S > 0$ such that if $u \in D$ is a solution of* (1.1), (1.3)*, then $\|u\| \leq S$.*

Lemma 5.5 *If f satisfies (H1)–(H3) and ($\mathcal{H}4$), then there exists an $R > 0$ such that if $u \in D$ is any solution of* (1.1), (1.3)*, then $\|u\| \geq R$.*

Finally, we present an existence result for (1.1), (1.3). Again, the proof is similar to that of the previous section and is omitted.

Theorem 5.6 *If f satisfies (H1)–(H3) and ($\mathcal{H}4$), then* (1.1), (1.3) *has at least one positive solution $u \in D$.*

6. An example

Let $\alpha = \frac{11}{2}$, $\beta = \frac{3}{2}$. Define $f(t,u) : (0,1) \times (0,\infty) \to (0,\infty)$ by

$$f(t,u) = \frac{1}{((1-t)u)^{1/11}(1-(1-t)^{3/2})}.$$

Then (H1)–(H3) are satisfied. Also for $\theta > 0$,

$$\int_0^1 \mathcal{G}(1,s) f(s, \theta s^{5/2}) ds \approx 0.02090001\theta^{11},$$

so ($\mathcal{G}4$) is satisfied. Notice that for $\theta > 0$

$$\int_0^1 \mathcal{H}(1,s) f(s, \theta s^{5/2}) ds \approx 0.05489925\theta^{11}.$$

Thus, ($\mathcal{G}4$) and ($\mathcal{H}4$) are satisfied. So, under the conditions above, (1.1), (1.2), and (1.1), (1.3) each have at least one positive solution.

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