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On the spectral and scattering properties of eigenparameter dependent discrete impulsive Sturm–Liouville equations

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Abstract: This work develops scattering and spectral analysis of a discrete impulsive Sturm–Liouville equation with spectral parameter in boundary condition. Giving the Jost solution and scattering solutions of this problem, we find scattering function of the problem. Discussing the properties of scattering function, scattering solutions, and asymptotic behavior of the Jost solution, we find the Green function, resolvent operator, continuous and point spectrum of the problem. Finally, we give an example in which the main results are made explicit.

Key words: Impulsive condition, discrete Sturm–Liouville equation, spectral parameter, eigenvalue, resolvent operator, scattering solution, scattering function

1. Introduction

Discrete impulsive equations, that is, difference equations involving impulsive effect, appear as a natural description of observed evolution phenomena of several real world problems. It is well-known that the theory of impulsive difference equations takes form under favor of the theory of the differential equations with impulses. In that way, for the mathematical theory of such impulsive equations, we refer to the monographs $[2, 3, 7, 22, 27]$. Impulsive difference equations are a basic tool to study dynamics that are subjected to sudden changes in their states. The theory of these equations has been motivated by a number of applied problems arising, in particular, in control theory, mechanical systems with impact, biological systems such as heart beats, blood flows, population dynamics, theoretical physics, chemistry, pharmacokinetics, mathematical economy, electric technology, metallurgy, ecology, infectious diseases, medicine, industrial robotics, biotechnology processes, engineering, navigational control of ships, and aircraft (see $[4, 10, 11, 15, 17, 18, 20, 21, 23, 26]$).

The theory of difference equations with impulses is a new and important branch of difference equations. In spite of its importance, the studies on spectral and scattering analysis of impulsive problems have been inadequate. Although there are many books and papers devoted to scattering analysis of differential and difference equations $[1, 8, 9, 13, 14, 19]$, there are only a few study about impulsive cases of such equations in literature $[5, 6, 12, 25]$. Because of this reason, scattering analysis of such equations have became a popular topic for mathematicians. None of the mentioned studies given in the literature about scattering problems of impulsive equations do not consist a spectral parameter in boundary condition. In this study, we will be interested in spectral properties and scattering solutions of discrete impulsive Sturm–Liouville equation with spectral parameter in boundary condition. As a result of this, the problem becomes more applicable in many

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parts of physics, mathematics, and other disciplines. Also, it creates different perspective for the solution method.

Let us consider an impulsive discrete Sturm–Liouville boundary value problem (IBVP) generated by the following difference equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n,\ n \in \mathbb{N} \setminus \{m_0 - 1,\ m_0,\ m_0 + 1\} \tag{1.1}$$

with the boundary condition

$$(\mu_0 + \lambda\mu_1)\, y_1 + (\nu_0 + \lambda\nu_1)\, y_0 = 0 \tag{1.2}$$

and the impulsive conditions

$$\begin{aligned} y_{m_0+1} &= \gamma_1 y_{m_0-1} \\ y_{m_0+2} &= \gamma_2 y_{m_0-2},\ \gamma_1\gamma_2 \neq 0,\ \gamma_1,\ \gamma_2 \in \mathbb{R} \end{aligned} \tag{1.3}$$

where $\lambda = 2\cos z$ is a spectral parameter, $\mu_0\nu_1 - \mu_1\nu_0 \neq 0$ for $i = 0, 1,$ $\mu_i,\ \nu_i$ are real numbers, $\{a_n\}_{n\in\mathbb{N}\cup\{0\}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are real sequences satisfying the condition

$$\sum_{n\in\mathbb{N}} n\left(|1 - a_n| + |b_n|\right) < \infty. \tag{1.4}$$

Throught this work, we will assume that $a_n \neq 0$ for all $n \in \mathbb{N}\cup\{0\}$. The paper consists, besides this introductory section, of six sections. Section 2 is an auxiliary section that we give some basic notations and definitions. In Section 3, we find the Jost solution and scattering solutions of (1.1)–(1.3). By using the properties of these solutions, we get the scattering function of IBVP (1.1)–(1.3) and we investigate the properties of scattering function. In Section 4, we give the resolvent operator and continuous spectrum of (1.1)–(1.3). Also, we give an asymptotic equation to get the properties of eigenvalues in this section. In Section 5, we are interested in unperturbated impulsive boundary value problem of (1.1)–(1.3). Discussing the properties of Jost solution and scattering function of this unperturbed problem, we determine the region of eigenvalues and continuous spectrum of this problem. Finally, in Section 6, we made some conclusions.

2. Preliminaries

Let us define two semi-strips

$$D_0 := \left\{ z \in \mathbb{C} : \operatorname{Im} z > 0,\ \ -\frac{\pi}{2} \leq \operatorname{Re} z \leq \frac{3\pi}{2} \right\}$$

and $D := D_0 \cup \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Assume that $P(z) = \{P_n(z)\}$ and $Q(z) = \{Q_n(z)\}$ are the fundamental solutions of (1.1) for $z \in D$ and $n = 0, 1, \ldots, m_0 - 1$ satisfying the initial conditions

$$P_0(z) = 0,\ \ P_1(z) = 1$$

and

$$Q_0(z) = \frac{1}{a_0},\ \ Q_1(z) = 0,$$

respectively. For each $n \geq 0$, $P_n(z)$ is polynomial of degree $(n-1)$ and is called a polynomial of the first kind and $Q_n(z)$ is a polynomial of degree $(n-2)$ and is known as a polynomial of the second kind.

It is well-known that the wronskian of two solutions $y = \{y_n(z)\}$ and $u = \{u_n(z)\}$ of the equation (1.1) defined by the following equation is independently of n.

$$W[y,u] := a_n [y_n(z) u_{n+1}(z) - y_{n+1}(z) u_n(z)].$$

It is clear from that the solutions $P(z)$ and $Q(z)$ are linear independent solutions of (1.1) because the wronskian of these solutions is equal to -1 for all $z \in \mathbb{C}$. Moreover, $P_n(z)$ and $Q_n(z)$ are entire functions with respect to z. Note that, we can write the other solution of (1.1) as a linear combination of fundamental solutions and by using the boundary condition, we can express this solution as

$$\psi_n(z) = -(\nu_0 + \lambda\nu_1) P_n(z) + a_0(\mu_0 + \lambda\mu_1) Q_n(z), \; n = 0, 1, \ldots, m_0 - 1. \tag{2.1}$$

On the other hand, we will show by $e(z) = \{e_n(z)\}$, $n = m_0+1, m_0+2, \ldots$ the bounded solution of (1.1) satisfying the condition $\lim\limits_{n\to\infty} e^{-inz} e_n(z) = 1$, $z \in D$. The solution $e(z)$ is represented by

$$e_n(z) = \rho_n e^{inz}\left(1 + \sum_{m=1}^{\infty} A_{nm} e^{imz}\right), \; n = m_0+1, m_0+2, \ldots$$

in literature [22], where ρ_n and A_{nm} are given in terms of the sequences $\{a_n\}$ and $\{b_n\}$ as

$$\begin{aligned} \rho_n &:= \prod_{k=n}^{\infty} a_k^{-1}, \\ A_{n1} &:= -\sum_{k=n+1}^{\infty} b_k, \\ A_{n2} &:= \sum_{k=n+1}^{\infty}\left\{1 - a_k^2 + b_k \sum_{p=k+1}^{\infty} b_p\right\}, \\ A_{n,m+2} &:= A_{n+1,m} + \sum_{k=n+1}^{\infty}\left\{\left(1-a_k^2\right) A_{k+1,m} - b_k A_{k,m+1}\right\} \end{aligned}$$

for $m \geq 1$. The function $e_n(z)$ is analytic according to z in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, continuous in $\overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$ and 2π periodic. The equation (1.1) also has an unbounded solution. We will show the unbounded solution by $\widehat{e_n}(z)$ for $n = m_0+1, m_0+2, \ldots$ providing $\lim\limits_{n\to\infty} e^{inz}\widehat{e_n}(z) = 1$, $z \in \overline{\mathbb{C}}_+$. It is evident from the definitions of $e_n(z)$, $\widehat{e_n}(z)$ and wronskian that

$$W[e_n(z), \widehat{e_n}(z)] = -2i \sin z \tag{2.2}$$

for $n = m_0+1, m_0+2, \ldots$ and $z \in D \setminus \{0, \pi\}$.

3. Jost solution, Jost function, scattering solutions and scattering function of discrete ımpulsive equation

In this section, we present some new definitions and results about scattering solutions and scattering function of the IBVP (1.1)–(1.3). Using $P(z)$, $Q(z)$ and $e(z)$, we define the following solution of (1.1)–(1.3)

$$E_n(z) = \begin{cases} \alpha(z) P_n(z) + \beta(z) Q_n(z) & ; \quad n = 1, 2, \dots, m_0 - 1 \\ e_n(z) & ; \quad n = m_0 + 1, m_0 + 2, \dots \end{cases} \tag{3.1}$$

for z in D, where α and β are z-dependent coefficients. The impulsive conditions (1.3) imply

$$E_{m_0-1}(z) = \frac{1}{\gamma_1} E_{m_0+1}(z)$$

$$E_{m_0-2}(z) = \frac{1}{\gamma_2} E_{m_0+2}(z)$$

and

$$\begin{aligned} \frac{1}{\gamma_1} e_{m_0+1}(z) &= \alpha(z) P_{m_0-1}(z) + \beta(z) Q_{m_0-1}(z) \\ \frac{1}{\gamma_2} e_{m_0+2}(z) &= \alpha(z) P_{m_0-2}(z) + \beta(z) Q_{m_0-2}(z). \end{aligned} \tag{3.2}$$

By using (3.2) and the definition of wronskian, we obtain the coefficients α and β uniquely

$$\alpha(z) = -\frac{a_{m_0-2}}{\gamma_1\gamma_2} \left[\gamma_1 e_{m_0+2}(z) Q_{m_0-1}(z) - \gamma_2 e_{m_0+1}(z) Q_{m_0-2}(z)\right] \tag{3.3}$$

$$\beta(z) = \frac{a_{m_0-2}}{\gamma_1\gamma_2} \left[\gamma_1 e_{m_0+2}(z) P_{m_0-1}(z) - \gamma_2 e_{m_0+1}(z) P_{m_0-2}(z)\right] \tag{3.4}$$

for $z \in D$. The function $E(z) = \{E_n(z)\}$ is the Jost solution of IBVP (1.1)–(1.3), where $\alpha(z)$ and $\beta(z)$ defined as in (3.3) and (3.4), respectively. Since $P_n(z) = P_n(-z)$ and $Q_n(z) = Q_n(-z)$ for $z \in D$, we obtain that $\alpha(-z) = \overline{\alpha(z)}$ and $\beta(-z) = \overline{\beta(z)}$.

It is clear from the definition of wronskian and $e_n(z)$ that for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$, we write

$$W\left[e_n(z), e_n(-z)\right] = -2i \sin z. \tag{3.5}$$

Next, we consider another solution $F(z) = \{F_n(z)\}$ of (1.1)-(1.3) by

$$F_n(z) := \begin{cases} \psi_n(z) & , \quad n = 1, 2, \dots, m_0 - 1 \\ \delta(z) e_n(z) + d(z) e_n(-z) & , \quad n = m_0 + 1, m_0 + 2, \dots \end{cases} \tag{3.6}$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$. By (1.3) and (3.5), it is easy to get

$$\delta(z) = -\frac{a_{m_0+1}}{2i \sin z} \left[\gamma_1 e_{m_0+2}(-z) \psi_{m_0-1}(z) - \gamma_2 e_{m_0+1}(-z) \psi_{m_0-2}(z)\right] \tag{3.7}$$

and

$$d(z) = \frac{a_{m_0+1}}{2i \sin z} \left[\gamma_1 e_{m_0+2}(z) \psi_{m_0-1}(z) - \gamma_2 e_{m_0+1}(z) \psi_{m_0-2}(z)\right] \tag{3.8}$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$.

Corollary 3.1 *Since* $\psi_n(z) = \psi_n(-z)$, *for all* $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$ *the coefficients* $d(z)$ *and* $\delta(z)$ *have the following relationship*

$$d(z) = \delta(-z) = \overline{\delta(z)}. \tag{3.9}$$

Lemma 3.2 *The wronskian of the solutions* $E(z)$ *and* $F(z)$ *is given by*

$$W[E_n(z), F_n(z)] := \begin{cases} \frac{a_{m_0-2}}{a_{m_0+1}} \frac{2i\sin z}{\gamma_1\gamma_2} d(z) & , \quad n = 1, 2, \ldots, m_0 - 1 \\ -2i\sin z d(z) & , \quad n = m_0 + 1, m_0 + 2, \ldots \end{cases}$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$.

Proof Using the definition of wronskian for $n = 1, 2, \ldots, m_0 - 1$, we find

$$\begin{aligned} W[E_n(z), F_n(z)] &= a_0[E_0(z) F_1(z) - E_1(z) F_0(z)] \\ &= a_0[\alpha(z) P_0(z) + \beta(z) Q_0(z)] \psi_1(z) \\ &- a_0[\alpha(z) P_1(z) + \beta(z) Q_1(z)] \psi_0(z). \end{aligned}$$

Since $P_0(z) = 0$, $P_1(z) = 1$, $Q_0(z) = \frac{1}{a_0}$ and $Q_1(z) = 0$, it follows from that

$$\begin{aligned} W[E_n(z), F_n(z)] &= \beta(z) \psi_1(z) - a_0\alpha(z) \psi_0(z) \\ &= -(\nu_0 + \lambda\nu_1) \beta(z) - a_0\alpha(z)(\mu_0 + \lambda\mu_1) \end{aligned}$$

for $n = 0, 1, \ldots, m_0 - 1$. If we apply the definitions of $\alpha(z)$, $\beta(z)$ and $d(z)$ given in (3.3), (3.4), (3.6) and (3.8), respectively, we find

$$W[E_n(z), F_n(z)] = \frac{a_{m_0-2}}{a_{m_0+1}} \frac{2i\sin z}{\gamma_1\gamma_2} d(z) \text{ for } n = 0, 1, \ldots, m_0 - 1.$$

Similarly, we obtain

$$W[E_n(z), F_n(z)] = -2i\sin z d(z) \text{ for } n = m_0 + 1, m_0 + 2, \ldots$$

by using (3.5) and (3.8). This completes the proof. □

Note that, using the boundary condition (1.2) and (3.1), we define the function J:

$$\begin{aligned} J(z) &= (\mu_0 + \lambda\mu_1) E_1 + (\nu_0 + \lambda\nu_1) E_0 \\ &= \alpha(z)(\mu_0 + \lambda\mu_1) + \frac{\beta(z)}{a_0}(\nu_0 + \lambda\nu_1). \end{aligned}$$

Analogously to the Sturm–Liouville equation, the function J is called the Jost function of (1.1)–(1.3). The function J is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}}_+$. It is evident that

$$J(-z) = \alpha(-z)(\mu_0 + \lambda\mu_1) + \frac{\beta(-z)}{a_0}(\nu_0 + \lambda\nu_1) \tag{3.10}$$

Lemma 3.3 *For all* $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$, *there is a following relation between the function* J *and* $d(z)$

$$J(z) = -\frac{1}{a_0}\frac{a_{m_0-2}}{a_{m_0+1}}\frac{2i\sin z}{\gamma_1\gamma_2}d(z). \tag{3.11}$$

Proof It follows from (2.1), (3.9) and (3.10) that

$$\begin{aligned} d(z) &= \frac{a_{m_0+1}}{2i\sin z}\left[\gamma_1 e_{m_0+2}(z)\psi_{m_0-1}(z) - \gamma_2 e_{m_0+1}(z)\psi_{m_0-2}(z)\right] \\ &= \frac{a_{m_0+1}}{2i\sin z}\left\{-(\nu_0+\lambda\nu_1)\frac{\beta(z)\gamma_1\gamma_2}{a_{m_0-2}} + a_0(\mu_0+\lambda\mu_1)\alpha(z)\frac{\gamma_1\gamma_2}{a_{m_0-2}}\right\} \\ &= \frac{a_{m_0+1}}{2i\sin z}\frac{\gamma_1\gamma_2}{a_{m_0-2}}\left\{-(\nu_0+\lambda\nu_1)\beta(z) + a_0(\mu_0+\lambda\mu_1)\alpha(z)\right\} \\ &= -a_0\frac{a_{m_0+1}}{a_{m_0-2}}\frac{\gamma_1\gamma_2}{2i\sin z}\left[\frac{1}{a_0}(\nu_0+\lambda\nu_1)\beta(z) + (\mu_0+\lambda\mu_1)\alpha(z)\right] \\ &= -a_0\frac{a_{m_0+1}}{a_{m_0-2}}\frac{\gamma_1\gamma_2}{2i\sin z}J(z). \end{aligned}$$

It completes the proof of Lemma 3.3. □

Theorem 3.4 *For all* $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$, $d(z) \neq 0$.

Proof Assume that, there exists a z_0 in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$ such that $d(z_0) = 0$. As a result of Corollary 3.1, it gives that $\delta(z_0) = d(z_0) = 0$. It follows from that $F_n(z_0) = 0$ for all $n \in \mathbb{N} \cup \{0\}$, but this is a trivial solution of IBVP (1.1)–(1.3). Since this is contradictory, $d(z)$ is not equal to zero for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$. □

In the following, we can give a definition for scattering function due to the existence of Theorem 3.4.

Definition 3.5 *The function*

$$S(z) := \frac{\overline{J(z)}}{J(z)}, \; z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$$

is called the scattering function of IBVP (1.1)–(1.3).

It easy to see from Lemma 3.3 and Definition 3.1 that the scattering function can be rewritten in terms of $d(z)$ as

$$S(z) = \frac{\overline{J(z)}}{J(z)} = \frac{J(-z)}{J(z)} = -\frac{d(-z)}{d(z)} \tag{3.12}$$

for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$.

Theorem 3.6 *The function* $S(z)$ *satisfies*

$$S(-z) = S^{-1}(z) = \overline{S(z)}$$

for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$ *and* $|S(z)| = 1$.

Proof Using the definition of function J and (3.12), we get

$$\begin{aligned} S(-z) &= \frac{J(z)}{J(-z)} \\ &= \frac{\alpha(z)(\mu_0 + \lambda\mu_1) + \frac{\beta(z)}{a_0}(\nu_0 + \lambda\nu_1)}{\alpha(-z)(\mu_0 + \lambda\mu_1) + \frac{\beta(-z)}{a_0}(\nu_0 + \lambda\nu_1)} \\ &= \frac{J(z)}{\overline{J(z)}} \\ &= S^{-1}(z) \\ &= \overline{S(z)} \end{aligned}$$

for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$. Finally, it is obvious that; since $|S(z)|^2 = \overline{S(z)} S(z)$, (3.12) gives us

$$|S(z)| = \frac{J(z)}{J(-z)} \frac{J(-z)}{J(z)} = 1.$$

It completes the proof of theorem. □

4. Resolvent operator, eigenvalues and continuous spectrum of IBVP

Let us define a solution $G(z) = \{G_n(z)\}$ of (1.1)-(1.3)

$$G_n(z) := \begin{cases} \psi_n(z) & , \quad n = 1, 2, \ldots, m_0 - 1 \\ q(z) e_n(z) + k(z) \widehat{e_n}(z) & , \quad n = m_0 + 1, m_0 + 2, \ldots \end{cases} \tag{4.1}$$

for all $z \in D$, where $\widehat{e_n}(z)$ denotes the unbounded solution of (1.1) given in Section 2. Similar to previous solutions, it is possible to find the coefficients $q(z)$ and $k(z)$ uniquely. By using impulsive condition (1.3), we get

$$q(z) = -\frac{a_{m_0+1}}{2i \sin z} \left[\gamma_1(z) \psi_{m_0-1}(z) \widehat{e_{m_0+2}}(z) - \gamma_2 \psi_{m_0-2}(z) \widehat{e_{m_0+2}}(z)\right] \tag{4.2}$$

and

$$k(z) = \frac{a_{m_0+1}}{2i \sin z} \left[\gamma_1 \psi_{m_0-1}(z) e_{m_0+2}(z) - \gamma_2 \psi_{m_0-2}(z) e_{m_0+1}(z)\right] \tag{4.3}$$

for $z \in D$. Note that for all $z \in D$, $k(z) = d(z)$. It follows from (3.1) and (4.1) that for all $z \in D$,

$$W[E_n(z), G_n(z)] = \begin{cases} \frac{a_{m_0-2}}{a_{m_0+1}} \frac{2i \sin z}{\gamma_1 \gamma_2} d(z) & , \quad n = 1, 2, \ldots, m_0 - 1 \\ -2i \sin z d(z) & , \quad n = m_0 + 1, m_0 + 2, \ldots \end{cases}. \tag{4.4}$$

It is evident that for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$ above wronskian is same with the $W\left[E_n(z), F_n(z)\right]$ given in Lemma 3.2.

Theorem 4.1 *For all $z \in D \setminus \{0, \pi\}$ and $d(z) \neq 0$, the resolvent operator of IBVP (1.1)–(1.3) is defined by*

$$R_\lambda g_n := \sum_{k=1}^{\infty} R_{nk}(z)\, g_k, \ \ \{g_k\} \in l_2(\mathbb{N}),$$

where

$$R_{nk}(z) = \begin{cases} -\frac{G_k E_n}{W[E_k, G_k]} & ; \quad k \leq n \\ -\frac{G_n E_k}{W[E_k, G_k]} & ; \quad k > n \end{cases}$$

is the Green function of the IBVP (1.1)–(1.3) for $k, n \neq m_0$.

Proof It is necessary to solve the equation

$$\nabla(a_n \triangle y_n) + h_n y_n - \lambda y_n = g_n \tag{4.5}$$

to get the Green function of IBVP (1.1)-(1.3), where $h_n = a_{n-1} + a_n + b_n$, $\triangle$ is the forward difference operator, ∇ is the backward difference operator defined by $\triangle y_n = y_{n+1} - y_n$ and $\nabla y_n = y_n - y_{n-1}$, respectively. Since $E_n(z)$ and $G_n(z)$ are the fundamental solutions of IBVP (1.1)-(1.3), we can write the general solution of $y = \{y_n(z)\}$ of (4.5)

$$y_n(z) = c_n E_n(z) + t_n G_n(z), \tag{4.6}$$

where c_n, t_n are coefficients and are different from zero. Using the method of variation of parameters, we get c_n and t_n by

$$c_n = -\sum_{k=1}^{n} \frac{G_k g_k}{W[E_k, G_k]}, \ \ k \neq m_0 \tag{4.7}$$

$$t_n = -\sum_{k=n+1}^{\infty} \frac{E_k g_k}{W[E_k, G_k]}, \ k \neq m_0. \tag{4.8}$$

It follows from (4.6), (4.7) and (4.8) that the Green function of (1.1)–(1.3) is $R_{nk}(z)$ given in Theorem 4.1 and it is easy to write the resolvent operator of IBVP (1.1)–(1.3) by using Green function. □

Theorem 4.1 is a main tool to define the set of eigenvalues of IBVP (1.1)–(1.3). By using this theorem and the definition of eigenvalues, we write the set of eigenvalues of (1.1)–(1.3) generated by σ_d as

$$\sigma_d = \{\lambda = 2\cos z : z \in D_0, \ d(z) = 0\}.$$

Theorem 4.2 *Assume (1.4). Then $d(z)$ satisfies the following asymptotic equation for all $z \in D_0$*

$$d(z) = e^{4iz}(B + o(1)), \ \ |z| \to \infty, \ B \neq 0.$$

Proof Since the polynomial function $P_n(z)$ is of $(n-1)$. degree and the polynomial function $Q_n(z)$ is of $(n-2)$. degree according to λ, we can immediately obtain that

$$\lim_{|z|\to\infty}\left\{\psi_n(z)e^{inz}\right\} = -\frac{\upsilon_1}{a_1a_2\ldots a_{n-1}},\ n=1,2,\ldots,m_0-1 \tag{4.9}$$

and

$$\lim_{|z|\to\infty}\left\{e_n(z)e^{-inz}\right\} = \rho_n,\ n=1,2,\ldots,m_0-1 \tag{4.10}$$

by using (1.2), (2.1) and (3.8), where $\rho_n := \left(\prod_{k=n}^{\infty} a_k\right)^{-1}$.

It follows from (3.8), (4.9) and (4.10) that

$$d(z) = \frac{a_{m_0+1}}{2i\sin z}\left\{\begin{array}{c}\gamma_1\psi_{m_0-1}(z)e^{i(m_0-1)z}e_{m_0+2}(z)e^{-i(m_0+2)z}e^{3iz}- \\ \gamma_2\psi_{m_0-2}(z)e^{i(m_0-2)z}e_{m_0+1}(z)e^{-i(m_0+1)z}e^{3iz}\end{array}\right\}$$

and

$$d(z)e^{-4iz} = \frac{a_{m_0+1}}{2i\sin z}\left\{-\gamma_1\frac{\upsilon_1}{a_1a_2\ldots a_{m_0-2}}\rho_{m_0+2} + \frac{\gamma_2\upsilon_1}{a_1a_2\ldots a_{m_0-3}}\rho_{m_0+1}\right\}.$$

Last equation gives

$$d(z)e^{-4iz} = \frac{B}{(e^{2iz}-1)}, \tag{4.11}$$

where

$$B = -\frac{a_{m_0+1}\upsilon_1\gamma_1\rho_{m_0+1}}{(a_1a_2\ldots a_{m_0-3})}\left(\frac{a_{m_0+1}}{a_{m_0-2}} - \frac{\gamma_2}{\gamma_1}\right).$$

Using (4.11), we get $\lim_{|z|\to\infty} d(z)e^{-4iz} = B$ for all $z\in D_0$ and it completes the proof of theorem. □

Theorem 4.2 shows that the set of eigenvalues of IBVP (1.1)-(1.3) is bounded under the condition (1.4). If we denote the continuous spectrum of (1.1)-(1.3) by σ_c, we can give the following Theorem.

Theorem 4.3 *Under the condition (1.4), $\sigma_c(L) = [-2,2]$, where L denotes the operator generated by the IBVP (1.1)-(1.3).*

Proof Let us introduce the operators L_1 and L_2 generated by the following difference expressions in $l_2(\mathbb{N})$ together with (1.2) and (1.3)

$$(L_1y)_n = y_{n-1} + y_{n+1},\ \mathbb{N}\backslash\{m_0-1,\ m_0+1\}$$

$$(L_2y)_n = (a_{n-1}-1)y_{n-1} + b_ny_n + (a_n-1)y_{n+1},\ \mathbb{N}\backslash\{m_0-1,\ m_0,\ m_0+1\},$$

respectively. It is clear from that $L = L_1 + L_2$ and L_2 is a compact operator in $l_2(\mathbb{N})$ under the assumption (1.4) (see [24]). We also can write $L_1 = L_3 + L_4$, where L_3 is a selfadjoint operator with $\sigma_c(L_3) = [-2,2]$ and L_4 is a finite dimensional operator in $l_2(\mathbb{N})$. Since L_4 is a finite dimensional operator in $l_2(\mathbb{N})$, it is a compact operator, and so, the sum of two compact operators $L_2 + L_4$ is also a compact operator. It follows from that $L = L_3 + L_4 + L_2$ and by using Weyl theorem [16] of a compact perturbation, we obtain that $\sigma_c(L_3) = \sigma_c(L) = [-2,2]$. □

5. Unperturbed ımpulsive equation

In this part, we will define an unperturbed discrete impulsive Sturm–Liouville equation and we will discuss our main results given in previous sections for this unperturbed problem. In that way, it gives an opportunity to readers applying the main results on a simple example. Let us consider the following unperturbed discrete impulsive problem

$$y_{n-1} + y_{n+1} = 2\cos z y_n,\ n \in \mathbb{N}\backslash \{2,3,4\}$$

$$(\mu_0 + \lambda\mu_1)\, y_1 + (\nu_0 + \lambda\nu_1)\, y_0 = 0 \tag{5.1}$$

$$y_4 = \gamma_1 y_2$$

$$y_5 = \gamma_2 y_1,$$

where $\mu_0,\ \mu_1,\ \nu_0,\ \nu_1,\ \gamma_1,\ \gamma_2 \in \mathbb{R}$ and $\gamma_1\gamma_2 \neq 0$. It is evident that in the problem (1.1)-(1.3), we suppose $a_n \equiv 1,\ b_n \equiv 0$ for all $n \in \mathbb{N},\ m_0 = 3$ for the problem (5.1). Then, the solution $e_n(z)$ turns into e^{inz}, and the fundamental solutions $P_n(z)$ and $Q_n(z)$ of (1.3) have the following values for $n = 0, 1, 2$.

$$P_0(z) = 0, \quad P_1(z) = 1, \quad P_2(z) = \lambda$$

$$Q_0(z) = \frac{1}{a_0}, \quad Q_1(z) = 0, \quad Q_2(z) = -\frac{1}{a_0}.$$

Thus, by using (3.1) and (3.8), we find $d(z)$ and Jost solution of this problem

$$d(z) = \frac{a_4}{2i\sin z}\left[\gamma_1 e_5(z)\,\psi_2(z) - \gamma_2 e_4(z)\,\psi_1(z)\right] \tag{5.2}$$

$$E_n(z) = \begin{cases} \alpha(z)\,P_n(z) + \beta(z)\,Q_n(z) & ; \quad n = 1,2 \\ e_n(z) & ; \quad n = 4,5,6,\ldots \end{cases}.$$

From the equation (5.2), we obtain the scattering function of (5.1)

$$S(z) = -e^{-8iz}\left[\frac{\gamma_1\psi_2(z)\,e^{-iz} - \gamma_2\psi_1(z)}{\gamma_1\psi_2(z)\,e^{iz} - \gamma_2\psi_1(z)}\right].$$

Moreover, continuous spectrum of the problem (5.1) is $[-2,2]$ from Theorem 4.3. To get the eigenvalues of the problem (5.1), it is necessary to find the zeros of $d(z)$ for $z \in D_0$. Because from the definition of eigenvalues, we write

$$\sigma_d = \{\lambda = 2\cos z : z \in D_0,\ d(z) = 0\} \tag{5.3}$$

for this problem, where $d(z)$ is defined by (5.2). By using the values of $P_i(z),\ Q_i(z);\ i = 1,2$ we obtain

$$\psi_1(z) = -(\nu_0 + \lambda\nu_1),$$

$$\psi_2(z) = -\lambda(\nu_0 + \lambda\nu_1) - (\mu_0 + \lambda\mu_1).$$

It follows from last equations and (5.2) that

$$d(z) = \frac{a_4}{2i\sin z}\left\{\gamma_1[-\lambda(\nu_0 + \lambda\nu_1) - (\mu_0 + \lambda\mu_1)]e^{5iz} + \gamma_2(\nu_0 + \lambda\nu_1)e^{4iz}\right\}. \tag{5.4}$$

Equation (5.4) implies that $d(z) = 0$ if and only if

$$\frac{\gamma_2}{\gamma_1} = \lambda e^{iz} + \frac{\mu_0 + \lambda\mu_1}{\nu_0 + \lambda\nu_1} e^{iz}. \tag{5.5}$$

For the simplicity on calculations, if we choose $\mu_1 = \nu_0 = 1$ and $\nu_1 = \mu_0 = 0$ in (5.5), we find

$$e^{2iz} = \frac{\gamma_2}{2\gamma_1} - 1.$$

Let $\gamma_2 = 2a\gamma_1,\ a \in \mathbb{R}$. By using last equation, we get $e^{2iz} = a - 1$. It gives us

$$2iz_k = \ln|a-1| + iArg\,(a-1) + 2ik\pi,\ k \in \mathbb{Z}$$

i.e.,

$$z_k = -\frac{i}{2}\ln|a-1| + \frac{1}{2}Arg\,(a-1) + k\pi,\ k \in \mathbb{Z}. \tag{5.6}$$

It is clear from (5.3) and (5.6) that the boundary value problem (5.1) has eigenvalues if and only if $\ln|a-1| < 0$. It implies that $-1 < a - 1 < 1$. Consequently, the necessary condition for the IBVP (5.1) to have an eigenvalue is that $0 < a < 2$. These eigenvalues are real and lie on $(-\infty, -2) \cup (2, \infty)$. Note that, since the impulsive conditions do not work when $a = 0$, the problem turns in to classical selfadjoint Sturm–Liouville problem, so $a \neq 0$. On the other hand, $a \neq 2$. Because, when a is equal to 2, we get $z_k = k\pi,\ k \in \mathbb{Z}$ in (5.6). But only for $k = 0, 1;\ z_0$ and $z_1 \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$. For $k = 0$, we obtain $\lambda_0 = 2$; for $k = 1$, we obtain $\lambda_1 = -2$. Since $\lambda = \pm 2$ are in continuous spectrum, they are not eigenvalues of (5.1).

6. Conclusions

This work is the first that informs readers about the scattering solutions of an discrete impulsive Sturm–Liouville equation with a boundary condition dependent on spectral parameter. These solutions help to find the scattering function of the problem by using the properties of scattering function. After finding the scattering function of this problem, we also find the resolvent operator, continuous spectrum, and discrete spectrum of the problem. As a result, discussing the main results, we are interested in an unperturbed equation as an example. This study will be a reference for researchers who study on scattering theory. On the other hand, for $\mu_0 = \mu_1 = \nu_1 = 0$, $\nu_0 \neq 0$ and $m_0 = 3$, the impulsive boundary value problem (1.1)–(1.3) can be written in the following form of boundary value problem

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n,\ n \geq 5$$
$$\gamma_2 a_1 y_4 + \gamma_1 (b_1 - \lambda) y_5 = 0$$

which is not an impulsive problem. But if m_0 can be chosen large enough, it can not be written in the form that is not an impulsive problem. Moreover, since the wronskian of the impulsive problem (1.1)-(1.3) is a constant on the left-side of m_0 and it is a different constant on the right-side of m_0, the proof of the finiteness of the eigenvalues of the problem (1.1)-(1.3) is difficult, even if it has not been solved yet.

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