

1-1-2021

## Korovkin type approximation via triangular $\$A$ - $\$s$ statistical convergence on an infinite interval

SELİN ÇINAR

SEVDA YILDIZ

KAMİL DEMİRCİ

Follow this and additional works at: <https://dctubitak.researchcommons.org/math>



Part of the [Mathematics Commons](#)

---

### Recommended Citation

ÇINAR, SELİN; YILDIZ, SEVDA; and DEMİRCİ, KAMİL (2021) "Korovkin type approximation via triangular  $\$A$ - $\$s$ statistical convergence on an infinite interval," *Turkish Journal of Mathematics*: Vol. 45: No. 2, Article 21. <https://doi.org/10.3906/mat-2012-57>

Available at: <https://dctubitak.researchcommons.org/math/vol45/iss2/21>

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals.

## Korovkin type approximation via triangular $A$ –statistical convergence on an infinite interval

Selin ÇINAR , Sevda YILDIZ\* , Kamil DEMİRCİ 

Department of Mathematics, Faculty of Sciences and Arts, Sinop University, Sinop, Turkey

Received: 14.12.2020

Accepted/Published Online: 16.02.2021

Final Version: 26.03.2021

**Abstract:** In the present paper, using the triangular  $A$ –statistical convergence for double sequences, which is an interesting convergence method, we prove a Korovkin-type approximation theorem for positive linear operators on the space of all real-valued continuous functions on  $[0, \infty) \times [0, \infty)$  with the property that have a finite limit at the infinity. Moreover, we present the rate of convergence via modulus of continuity. Finally, we give some further developments.

**Key words:** Triangular  $A$ –statistical convergence, positive linear operator, the Korovkin type theorem, Szasz-Mirakyan operator

### 1. Introduction and preliminaries

The Korovkin theory [10, 17] deals with an approximation to a function  $f$  by means of a sequence  $(L_j(f))$  of positive linear operators, which is mainly based on the classical limit  $\lim_j L_j(f) = f$ . In 1960, first of all, Korovkin [17] determined the sufficient conditions for the uniform convergence of  $L_j(f)$  to a function  $f$  by the test function  $1, u, u^2$ . Many authors have researched these conditions for various operators defined on different spaces (see i.e. [1, 14]). Obviously, using the statistical convergence in approximation theory, which is known as statistical approximation, brings along many advantages. We should note that the statistical approximation was initiated by Gadjiev and Orhan (see [13]) and improved by Anastassiou and Duman (see [2]). These types results have a crucial role in the approximation. Especially, the matrix summability methods of Cesàro type are strong enough to correct the absence of convergence of various sequences of linear operators such as Hermite-Fejér [7]. As it is well known, these types of operators do not converge at points of simple discontinuity. Therefore, such facts present why we need summability methods in the approximation theory (see i.e. [3, 4, 21, 22]). Furthermore, presently, with the help of the concept of statistical convergence, which is a regular (nonmatrix) summability transformation, various statistical approximation results have been proved [12, 16]. Then, it was shown that these new versions of Korovkin type theorems extended the classical one. Recently, Bardaro et al. [5] introduced an approximation theorem with the help of the notion of new statistical convergence called *triangular  $A$ –statistical convergence*. Let  $C_*(D)$  be a Banach space of all real-valued continuous functions on  $D := [0, \infty) \times [0, \infty)$  with the property that  $\lim_{(u,v) \rightarrow (\infty, \infty)} f(u, v)$  exists and is finite. Moreover, it is endowed with the supremum norm  $\|f\| = \sup_{(u,v) \in D} |f(u, v)|$  for  $f \in C_*(D)$ . In the present work,

\*Correspondence: sevdaorhan@sinop.edu.tr

2010 AMS Mathematics Subject Classification: 40G15, 41A36

using this convergence method and test functions  $1$ ,  $e^{-u}$ ,  $e^{-v}$ , and  $e^{-2u} + e^{-2v}$ , we provide a Korovkin-type approximation theorem for positive linear operators on the space  $C_*(D)$ . We construct two examples such that in the first one, our new approximation theorem works but its statistical case does not work and in the second one, statistical case works but our result does not work. We also present a rate of convergence by means of the modulus of continuity. Finally, we give some further developments.

We will now mention some basic definitions and notations used in the paper.

A double sequence  $u = (u_{i,j})$ ,  $i, j \in \mathbb{N}$ , is said to be convergent in Pringsheim's sense if, for every  $\varepsilon > 0$ , there exists  $J = J(\varepsilon) \in \mathbb{N}$  such that  $|u_{i,j} - L| < \varepsilon$  whenever  $i, j > J$ . Then,  $L$  is the Pringsheim limit of  $u$  (see [19]). In this case, we say that  $u = (u_{i,j})$  is “ $P$ -convergent to  $L$ ” and is denoted by  $P - \lim u = L$ . Moreover, if there exists a positive number  $M$  such that  $|u_{i,j}| \leq M$  for all  $(i, j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ , then  $u = (u_{i,j})$  is said to be bounded. As it is known, different from a convergent single sequence, a convergent double sequence need not be bounded.

We denote the set of all  $P$ -convergent double sequences by  $c^2$ .

Now let  $A = (a_{k,l,i,j})$ ,  $k, l, i, j \in \mathbb{N}$ , be a four-dimensional summability matrix. For a given double sequence  $u = (u_{i,j})$ , the  $A$ -transform of  $u$ , denoted by  $Au := ((Au)_{k,l})$ , is given by

$$(Au)_{k,l} = \sum_{(i,j) \in \mathbb{N}^2} a_{k,l,i,j} u_{i,j},$$

provided the double series converges in Pringsheim's sense for every  $(k, l) \in \mathbb{N}^2$ . As it is known,  $A = (a_{k,l,i,j})$  is said to be  $RH$ -regular if it maps every bounded  $P$ -convergent sequence into a  $P$ -convergent sequence with the same  $P$ -limit. The well-known characterization of regularity for four-dimensional matrices is known as Robison-Hamilton conditions, or briefly,  $RH$ -regularity (see, [15, 20]):

- (i)  $P - \lim_{k,l} a_{k,l,i,j} = 0$  for each  $i$  and  $j$ ,
- (ii)  $P - \lim_{k,l} \sum_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j} = 1$ ,
- (iii)  $P - \lim_{k,l} \sum_{i=1}^{\infty} |a_{k,l,i,j}| = 0$  for each  $j \in \mathbb{N}$ ,
- (iv)  $P - \lim_{k,l} \sum_{j=1}^{\infty} |a_{k,l,i,j}| = 0$  for each  $i \in \mathbb{N}$ ,
- (v)  $\sum_{i,j=1,1}^{\infty,\infty} |a_{k,l,i,j}|$  is  $P$ -convergent for every  $(k, l) \in \mathbb{N}^2$ ,
- (vi) There exists finite positive integers  $M$  and  $N$  such that  $\sum_{i,j>N} |a_{k,l,i,j}| < M$  holds for every  $(k, l) \in \mathbb{N}^2$ .

Now, let  $A = (a_{k,l,i,j})$  be a nonnegative  $RH$ -regular summability matrix, and let  $K \subset \mathbb{N}^2$ . Then, a real double sequence  $u = (u_{i,j})$  is said to be  $A$ -statistically convergent to a number  $L$  if, for every  $\varepsilon > 0$ ,

$$P - \lim_{k,l} \sum_{(i,j) \in K(\varepsilon)} a_{k,l,i,j} = 0,$$

where

$$K(\varepsilon) := \{(i, j) \in \mathbb{N}^2 : |u_{i,j} - L| \geq \varepsilon\}.$$

In this case we write  $st_A^2 - \lim_{i,j} u_{i,j} = L$  ([18]). Observe that, a  $P$ -convergent double sequence is  $A$ -statistically convergent to the same value but the converse is not always true.

Let  $C(1, 1) = (c_{k,l,i,j})$  be a double Cesàro matrix defined by  $c_{k,l,i,j} = \frac{1}{kl}$  if  $1 \leq i \leq k, 1 \leq j \leq l$ , and  $c_{k,l,i,j} = 0$  otherwise. Note that if we take  $A = C(1, 1)$ , then  $C(1, 1)$ -statistical convergence coincides with the concept of statistical convergence for double sequence. Moreover, if the matrix  $A$  is replaced by the identity matrix for four-dimensional matrices, then  $A$ -statistical convergence reduces to the Pringsheim convergence.

We denote the set of all  $A$ -statistically convergent double sequences by  $st_A^2$ .

Consider the double sequence  $u = (u_{i,j})$  and suppose that it is neither  $A$ -statistical convergent nor convergent in the Pringsheim's sense. For answering the question if there is any kind of statistical convergence which is different from  $A$ -statistical convergence and Pringsheim convergence, Bardaro et al. ([5]) used two-dimensional regular matrices for double sequences. We now recall this interesting convergence method:

Let  $A = (a_{i,j})$  be a two-dimensional matrix transformation. For a double sequence  $u = (u_{i,j})$  of real numbers, put

$$(Au)_i := \sum_{j=1}^{\infty} a_{i,j} u_{i,j},$$

if the series is convergent.

A two-dimensional matrix transformation is regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization of regularity for two dimensional matrix  $A = (a_{i,j})$  is known as Silverman Toeplitz ([8]) conditions:

- (i)  $\|A\| = \sup_{i \rightarrow \infty} \sum_{j=1}^{\infty} |a_{i,j}| < \infty$ ,
- (ii)  $\lim_{i \rightarrow \infty} a_{i,j} = 0$  for each  $j \in \mathbb{N}$ ,
- (iii)  $\lim_{i \rightarrow \infty} \sum_{j=1}^{\infty} a_{i,j} = 1$ .

**Definition 1.1** [5] Let  $A = (a_{i,j})$  be a nonnegative regular summability matrix,  $K \subset \mathbb{N}^2$  be a nonempty set, and for every  $i \in \mathbb{N}$ , consider  $K_i = \{j \in \mathbb{N} : (i, j) \in K, j \leq i\}$ . Let  $|K_i|$  be the cardinality of  $K_i$ . We say that the triangular  $A$ -density of  $K$  is defined by

$$\delta_A^T(K) := \lim_i \sum_{j \in K_i} a_{i,j}$$

provided that the limit on the right-hand side exists.

**Definition 1.2** [5] Let  $A = (a_{i,j})$  be a nonnegative regular summability matrix. The double sequence  $u = (u_{i,j})$  is triangular  $A$ -statistically convergent to  $L$  iff for every  $\varepsilon > 0$

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where  $K_i(\varepsilon) = \{ j \in \mathbb{N} : j \leq i, |u_{i,j} - L| \geq \varepsilon \}$  and this is denoted by  $st_A^T - \lim_i u_{i,j} = L$ .

Note that if  $A = C_1$ , that is Cesàro matrix, then triangular  $C_1$ -statistical convergence coincides with the notion of triangular statistical convergence.

The set of all triangular  $A$ -statistically convergent sequences denoted by  $st_A^T$ .

Now, in the view of above definition, we compare triangular statistical convergence to statistical convergence. First, we present below an example to show that a sequence is triangular  $A$ -statistically convergent whenever it is not statistically convergent.

**Example 1.3** Let  $A = C_1$  and the double sequence  $u = (u_{i,j})$  is given by

$$u_{i,j} = \begin{cases} 1, & i = j = n^2, \\ \frac{n}{4(n+1)}, & i = 2n, j = 2(n-1), \\ \frac{n}{5(n+1)}, & i = 2n+1, j = 2n+3, \\ 0, & \text{otherwise,} \end{cases} \quad n \in \mathbb{N}.$$

For every  $\varepsilon \in (0, \frac{1}{10}]$ ,

$$\frac{1}{i} |\{ j \in \mathbb{N} : j \leq i, |u_{i,j}| \geq \varepsilon \}| = \begin{cases} 1, & i = 1, \\ \frac{2}{(2n)^2}, & i = (2n)^2, \\ \frac{1}{(2n+1)^2}, & i = (2n+1)^2, \\ \frac{1}{2n}, & i = 2n \text{ and } i \text{ is not square,} \\ 0, & \text{otherwise,} \end{cases} \quad n \in \mathbb{N},$$

then clearly,

$$\lim_i \frac{1}{i} |\{ j \in \mathbb{N} : j \leq i, |u_{i,j}| \geq \varepsilon \}| = 0.$$

Hence, we obtain  $st_{C_1}^T - \lim_i u_{i,j} = 0$ . Nevertheless,  $u = (u_{i,j})$  is not Pringsheim's and  $C(1, 1)$ -statistically convergent.

From Example 1.3, we can say that the concept of triangular  $A$ -statistical convergence does not characterize as given below:

A double sequence  $u = (u_{i,j})$  is triangular  $A$ -statistically convergent to  $L$  if and only if there exists a set  $K \subset \mathbb{N}^2$  such that the triangular  $A$ -density of  $K$  is 1 and

$$P - \lim_{i,j \rightarrow \infty \text{ and } (i,j) \in K} u_{i,j} = L. \tag{1.1}$$

By (1.1) we mean that for every  $\varepsilon > 0$  there exists an integer  $N$  such that

$$|u_{i,j} - L| \leq \varepsilon \text{ if } i, j \geq N \text{ and } (i, j) \in K.$$

The following example shows that a sequence is  $A$ -statistically convergent whenever it is not triangular  $A$ -statistically convergent.

**Example 1.4** Take  $A = C(1, 1)$  and the double sequence  $u = (u_{i,j})$  is given by

$$u_{i,j} = \begin{cases} \sqrt{ij}, & \text{if } i \text{ and } j \text{ are squares,} \\ \frac{1}{ij}, & \text{otherwise.} \end{cases}$$

It is easy to see that  $st_{C(1,1)}^2 - \lim_{i,j} u_{i,j} = 0$ . Nevertheless,  $u$  is not Pringsheim's and triangular statistically convergent.

**Example 1.5** Let  $A = C_1$  and the double sequence  $u = (u_{i,j})$  is given by

$$u_{i,j} = \begin{cases} 1, & i = j = n^2, \quad n \in \mathbb{N}. \\ 0, & \text{otherwise.} \end{cases}$$

Similarly,  $st_{C_1}^T - \lim_i u_{i,j} = 0$  and  $st_{C(1,1)}^2 - \lim_{i,j} u_{i,j} = 0$ .

**Example 1.6** Take  $A = C_1$  and the double sequence  $u = (u_{i,j})$  is given by

$$u_{i,j} = \begin{cases} 1, & i = j = n^2, \\ \frac{n}{2n+1}, & i = 2n + 1, \quad j = 2n - 1, \\ \frac{n}{4n+2}, & i = 2n, \quad j = 2(n + 1), \\ n, & i = n^2, \quad j = n^2 + 1, \\ 0, & \text{otherwise.} \end{cases} \quad n \in \mathbb{N}.$$

For every  $\varepsilon \in (0, \frac{1}{6}]$ ,

$$\frac{1}{i} |\{ j \in \mathbb{N} : j \leq i, |u_{i,j}| \geq \varepsilon \}| = \begin{cases} 1, & i = 1, \\ \frac{2}{(2n+1)^2}, & i = (2n + 1)^2, \\ \frac{1}{(2n)^2}, & i = (2n)^2, \\ \frac{1}{2n+1}, & i = 2n + 1 \text{ and } i \text{ is not square,} \\ 0, & \text{otherwise.} \end{cases} \quad n \in \mathbb{N},$$

Then, it is seen to be

$$\lim_i \frac{1}{i} |\{ j \in \mathbb{N} : j \leq i, |u_{i,j}| \geq \varepsilon \}| = 0.$$

Hence, we obtain  $st_{C_1}^T - \lim_i u_{i,j} = 0$ . Neither  $u = (u_{i,j})$  is Pringsheim's and  $C(1, 1)$  – statistically convergent nor bounded.

**Remark 1.7** [5] i) Statistical convergence and triangular statistical convergence are incompatible; i.e.  $st_A^2 \not\subseteq st_A^T$  and  $st_A^T \not\subseteq st_A^2$ .

ii)  $P$ –convergent double sequence is  $A$ –statistically convergent and triangular  $A$ –statistically convergent to the same value but the inverse implications are not true, i.e.  $st_A^2 \not\subseteq c^2$  and  $st_A^T \not\subseteq c^2$ .

## 2. A Korovkin-type theorem

Boyanov and Veselinov [9] have results for single sequences, where they proved the Korovkin theorem on  $C_*[0, \infty)$ , which is the Banach space of all real-valued continuous functions on  $[0, \infty)$  with the property that  $\lim_{u \rightarrow \infty} f(u)$  exists and is finite, endowed with the supremum norm  $\|f\| = \sup_{u \in [0, \infty)} |f(u)|$  for  $f \in C_*[0, \infty)$ , by

using the test function 1,  $e^{-u}$ ,  $e^{-2u}$ . In this section, using the concept of triangular  $A$ -statistical convergence for double sequences and test functions 1,  $e^{-u}$ ,  $e^{-v}$ , and  $e^{-2u} + e^{-2v}$ , we provide a Korovkin-type approximation for positive linear operators on the space  $C_*(D)$ .

Now, we remind Korovkin's theorem for double sequences of positive linear operators given by Demirci and Karakuş [11] and then, we give the Korovkin theorem in the triangular statistical sense.

Let  $L$  be a linear operator from  $C_*(D)$  into itself. Then, as usual, we say that  $L$  is positive linear operator provided that  $f \geq 0$  implies  $L(f) \geq 0$ . Moreover, we denote the value of  $L(f)$  at a point  $(u, v) \in D$  by  $L(f; u, v)$ .

We note that a function which is continuous on  $D$  and has a finite limit to infinity is uniformly continuous. Hence, Theorem 2.2. in [11] can be given as follow:

**Theorem 2.1** [11] *Let  $A = (a_{k,l,i,j})$  be a nonnegative RH-regular summability matrix. Let  $(L_{i,j})$  be a double sequence of positive linear operators acting from  $C_*(D)$  into itself. Then, for all  $f \in C_*(D)$*

$$st_A^2 - \lim_{i,j} \|L_{i,j}(f) - f\| = 0$$

if and only if the following statements hold:

- a)  $st_A^2 - \lim_{i,j} \|L_{i,j}(1) - 1\| = 0,$
- b)  $st_A^2 - \lim_{i,j} \|L_{i,j}(e^{-s}) - e^{-u}\| = 0,$
- c)  $st_A^2 - \lim_{i,j} \|L_{i,j}(e^{-t}) - e^{-v}\| = 0,$
- d)  $st_A^2 - \lim_{i,j} \|L_{i,j}(e^{-2s} + e^{-2t}) - (e^{-2u} + e^{-2v})\| = 0.$

Before giving our main theorem, we prove the following lemma:

**Lemma 2.2** *Let  $(L_{i,j})$  be a double sequence of positive linear operators acting from  $C_*(D)$  into itself. Then, for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  and  $M > 0$  such that for every  $L_{i,j} : C_*(D) \rightarrow C_*(D)$ ,  $f \in C_*(D)$ , and  $(u, v) \in D$ , the following inequality holds*

$$\begin{aligned} & |L_{i,j}(f; u, v) - f(u, v)| \\ & \leq \varepsilon + \left( \varepsilon + M + \frac{4M}{\delta^2} \right) |L_{i,j}(1; u, v) - 1| \\ & \quad + \frac{4M}{\delta^2} |L_{i,j}(e^{-s}; u, v) - e^{-u}| + \frac{4M}{\delta^2} |L_{i,j}(e^{-t}; u, v) - e^{-v}| \\ & \quad + \frac{2M}{\delta^2} |L_{i,j}(e^{-2s} + e^{-2t}; u, v) - (e^{-2u} + e^{-2v})|. \end{aligned}$$

**Proof** Following the same lines as in the proof of Theorem 2.2 in [11], the proof is obtained. □

Now, we give the following, which is the main result in this section.

**Theorem 2.3** *Let  $A = (a_{i,j})$  be a nonnegative regular summability matrix and  $(L_{i,j})$  be a double sequence of positive linear operators acting from  $C_*(D)$  into itself. Then, for all  $f \in C_*(D)$*

$$st_A^T - \lim_i \|L_{i,j}(f) - f\| = 0$$

*iff the following statements hold:*

- a)  $st_A^T - \lim_i \|L_{i,j}(1) - 1\| = 0,$
- b)  $st_A^T - \lim_i \|L_{i,j}(e^{-s}) - e^{-u}\| = 0,$
- c)  $st_A^T - \lim_i \|L_{i,j}(e^{-t}) - e^{-v}\| = 0,$
- d)  $st_A^T - \lim_i \|L_{i,j}(e^{-2s} + e^{-2t}) - (e^{-2u} + e^{-2v})\| = 0.$

**Proof** Since the necessity is obvious, then it is enough to prove sufficiency. Suppose that the conditions (a), (b), (c), and (d) are satisfied. Let  $f \in C_*(D)$ . Then, from Lemma 2.2, for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  and  $M > 0$  such that for every  $(i, j) \in \mathbb{N}^2$  and  $(u, v) \in D$ , the following inequality holds

$$\begin{aligned} & |L_{i,j}(f; u, v) - f(u, v)| \\ & \leq \varepsilon + \left( \varepsilon + M + \frac{4M}{\delta^2} \right) |L_{i,j}(1; u, v) - 1| \\ & \quad + \frac{4M}{\delta^2} |L_{i,j}(e^{-s}; u, v) - e^{-u}| + \frac{4M}{\delta^2} |L_{i,j}(e^{-t}; u, v) - e^{-v}| \\ & \quad + \frac{2M}{\delta^2} |L_{i,j}(e^{-2s} + e^{-2t}; u, v) - (e^{-2u} + e^{-2v})|. \end{aligned}$$

Then taking the supremum over  $(u, v) \in D$ , we get

$$\begin{aligned} & \|L_{i,j}(f) - f\| \tag{2.1} \\ & \leq \varepsilon + K \{ \|L_{i,j}(1) - 1\| + \|L_{i,j}(e^{-s}) - e^{-u}\| \\ & \quad + \|L_{i,j}(e^{-t}) - e^{-v}\| + \|L_{i,j}(e^{-2s} + e^{-2t}) - (e^{-2u} + e^{-2v})\| \} \end{aligned}$$

where  $K := \max \{ \varepsilon + M + \frac{4M}{\delta^2}, \frac{4M}{\delta^2}, \frac{2M}{\delta^2} \}$ . For a given  $r > 0$  choose  $\varepsilon > 0$  such that  $\varepsilon < r$ . Define the following



sets:

$$\begin{aligned}
 S_i & : = \{ j \in \mathbb{N} : j \leq i, \|L_{i,j}(f) - f\| \geq r \}, \\
 S_i^1 & : = \left\{ j \in \mathbb{N} : j \leq i, \|L_{i,j}(1) - 1\| \geq \frac{r - \varepsilon}{4K} \right\}, \\
 S_i^2 & : = \left\{ j \in \mathbb{N} : j \leq i, \|L_{i,j}(e^{-s}) - e^{-u}\| \geq \frac{r - \varepsilon}{4K} \right\}, \\
 S_i^3 & : = \left\{ j \in \mathbb{N} : j \leq i, \|L_{i,j}(e^{-t}) - e^{-v}\| \geq \frac{r - \varepsilon}{4K} \right\}, \\
 S_i^4 & : = \left\{ j \in \mathbb{N} : j \leq i, \|L_{i,j}(e^{-2s} + e^{-2t}) - (e^{-2u} + e^{-2v})\| \geq \frac{r - \varepsilon}{4K} \right\}.
 \end{aligned}$$

In the view of (2.1), it can be easily seen that  $S_i \subset S_i^1 \cup S_i^2 \cup S_i^3 \cup S_i^4$ . Thus, for each  $j \in \mathbb{N}$ , we may write

$$\sum_{j \in S_i} a_{i,j} \leq \sum_{j \in S_i^1} a_{i,j} + \sum_{j \in S_i^2} a_{i,j} + \sum_{j \in S_i^3} a_{i,j} + \sum_{j \in S_i^4} a_{i,j}. \tag{2.2}$$

From (2.2), using (a), (b), (c), and (d), we get that

$$\lim_i \sum_{j \in S_i} a_{i,j} = 0$$

whence the result. □

Now, we give the following examples of double sequences of positive linear operators.

**Example 2.4** Consider the following Szász–Mirakyan operators (see [23])

$$S_{i,j}(f; u, v) = e^{-iu} e^{-jv} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} f\left(\frac{l}{i}, \frac{k}{j}\right) \frac{(iu)^l}{l!} \frac{(jv)^k}{k!}, \tag{2.3}$$

where  $(u, v) \in D$ ,  $f \in C_*(D)$ . Moreover, observe that

$$\begin{aligned}
 S_{i,j}(1; u, v) & = 1, \\
 S_{i,j}(e^{-s}; u, v) & = e^{-iu(1-e^{-\frac{1}{i}})}, \\
 S_{i,j}(e^{-t}; u, v) & = e^{-jv(1-e^{-\frac{1}{j}})}, \\
 S_{i,j}(e^{-2s} + e^{-2t}; u, v) & = e^{-iu(1-e^{-\frac{2}{i}})} + e^{-jv(1-e^{-\frac{2}{j}})}.
 \end{aligned}$$

Now we take  $A = C_1$  and define a double sequence  $(\alpha_{i,j})$  by

$$\alpha_{i,j} = \begin{cases} 1, & i = j = n^2, \\ \frac{n}{4(n+1)}, & i = 2n, j = 2(n-1), \\ \frac{n}{3(n+1)}, & i = 2n+1, j = 2n+3, \\ 0, & \text{otherwise.} \end{cases} \quad n \in \mathbb{N} \tag{2.4}$$

It is clear that

$$st_{C_1}^T - \lim_i \alpha_{i,j} = 0. \tag{2.5}$$

Now in the view of (2.3) and (2.4), we define the following positive linear operators on  $C_*(D)$  as follows:

$$L_{i,j}(f; u, v) = (1 + \alpha_{i,j}) S_{i,j}(f; u, v). \tag{2.6}$$

Therefore, by the Theorem 2.3 and (2.5), we see that

$$st_{C_1}^T - \lim_i \|L_{i,j}(f) - f\| = 0.$$

Moreover, since  $(\alpha_{i,j})$  is not  $P$ -convergent and statistical convergent, we can say that the Korovkin theorems in the Pringsheim's and statistical sense do not work for our operators defined by (2.6).

**Example 2.5** Let Szasz–Mirakyan operators be the same as in Example 2.4. Now take  $A = C(1, 1)$  and define a double sequence  $(\beta_{i,j})$  by

$$\beta_{i,j} = \begin{cases} \sqrt{i_j}, & \text{if } i \text{ and } j \text{ are squares,} \\ \frac{-1}{ij} & \text{otherwise.} \end{cases} \tag{2.7}$$

It is clear that

$$st_{C(1,1)}^2 - \lim_{i,j} \beta_{i,j} = 0. \tag{2.8}$$

Now in the view of (2.3) and (2.7), we define the following positive linear operators on  $C_*(D)$  as follows:

$$L_{i,j}(f; u, v) = (1 + \beta_{i,j}) S_{i,j}(f; u, v). \tag{2.9}$$

Therefore, by the Theorem 2.1 and (2.8), we see that

$$st_{C(1,1)}^2 - \lim_{i,j} \|L_{i,j}(f) - f\| = 0.$$

Moreover, since  $(\beta_{i,j})$  is not triangular statistical convergent, we can say that the Theorem 2.3 does not work for operators defined by (2.9).

### 3. The rate of approximation

In this section, we study the rate of convergence of a sequence of positive linear operators defined from  $C_*(D)$  into itself with the help of modulus of continuity. The earlier results proved in [5]. Now, we begin by recalling the following:

**Definition 3.1** [5] Let  $A = (a_{i,j})$  be a nonnegative regular summability matrix and let  $(\alpha_i)$  be a positive nonincreasing sequence. Then, a double sequence  $u = (u_{i,j})$  is said to be triangular  $A$ -statistically convergent to a number  $L$  with the rate of  $o_k(\alpha_i)$  iff for every  $\varepsilon > 0$ ,

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where

$$K_i(\varepsilon) := \{j \in \mathbb{N} : |u_{i,j} - L| \geq \varepsilon \alpha_j\}.$$

In this case, one can say that

$$u_{i,j} - L = st_A^T - o_k(\alpha_i) \text{ as } i \rightarrow \infty.$$

**Definition 3.2** [5] Let  $A = (a_{i,j})$  and  $(\alpha_i)$  be the same as in Definition 3.1. Then, a double sequence  $u = (u_{i,j})$  is said to be triangular  $A$ -statistically bounded with the rate of  $O_k(\alpha_i)$  iff for every  $\varepsilon > 0$ ,

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where

$$K_i(\varepsilon) := \{j \in \mathbb{N} : |u_{i,j}| \geq \varepsilon \alpha_j\}.$$

In this case, one can say that

$$u_{i,j} = st_A^T - O_k(\alpha_i) \text{ as } i \rightarrow \infty.$$

**Lemma 3.3** [5] Let  $(u_{i,j})$  and  $(v_{i,j})$  be double sequences. Suppose that  $A = (a_{i,j})$  is a nonnegative regular summability matrix. Moreover, let  $(\alpha_i)$  and  $(\beta_i)$  be positive nonincreasing sequences. If  $u_{i,j} - L_1 = st_A^T - o_k(\alpha_i)$  and  $v_{i,j} - L_2 = st_A^T - o_k(\beta_i)$ , then we have

$$(i) \quad (u_{i,j} - L_1) \mp (v_{i,j} - L_2) = st_A^T - o_k(\gamma_i) \text{ as } i \rightarrow \infty, \text{ where } \gamma_i := \max\{\alpha_i, \beta_i\} \text{ for each } i \in \mathbb{N},$$

$$(ii) \quad \lambda(u_{i,j} - L_1) = st_A^T - o_k(\alpha_i) \text{ as } i \rightarrow \infty \text{ for any real number } \lambda.$$

Furthermore, similar conclusions hold with the symbol “ $o_k$ ” replaced by “ $O_k$ ”.

Recall that for a continuous function  $f \in C(I)$  defined on an interval  $I \subset \mathbb{R}^2$ , the function  $\omega : C(I) \times [0, \infty) \rightarrow \mathbb{R} \cup \{\infty\}$ , defined as follows:

$$\omega(f; \delta) := \sup \left\{ |f(s, t) - f(u, v)| : (s, t), (u, v) \in D, \sqrt{(s - u)^2 + (t - v)^2} \leq \delta \right\}$$

is called its usual modulus of continuity.

To estimate the rate of convergence of a double sequence of positive linear operators defined from  $C_*(D)$  into itself, we make use of the following modulus of continuity:

$$\omega^*(f; \delta) := \sup \left\{ |f(s, t) - f(u, v)| : (s, t), (u, v) \in D, \sqrt{(e^{-s} - e^{-u})^2 + (e^{-t} - e^{-v})^2} \leq \delta \right\}.$$

where  $f \in C_*(D)$  and  $\delta > 0$ . In fact, this modulus can be given the usual modulus of continuity by the relation:

$$\omega^*(f; \delta) = \omega(f^*; \delta)$$

where  $f^*$  is the continuous function defined on  $[0, 1] \times [0, 1]$  by

$$f^*(u, v) = \begin{cases} f(-\ln u, -\ln v), & (u, v) \in (0, 1] \times (0, 1], \\ \lim_{(s,t) \rightarrow (\infty, \infty)} f(s, t), & u = 0 \text{ or } v = 0. \end{cases}$$

With simple calculation, it can be seen that for any  $\delta > 0$

$$|f(s, t) - f(u, v)| \leq \left(1 + \frac{(e^{-s} - e^{-u})^2 + (e^{-t} - e^{-v})^2}{\delta^2}\right) \omega^*(f; \delta)$$

for all  $f \in C_*(D)$ .

Now we can give the following result.

**Theorem 3.4** *Let  $A = (a_{i,j})$  be a nonnegative regular summability matrix method and  $(L_{i,j})$  be a double sequence of positive linear operators from  $C_*(D)$  into itself. Suppose that the following conditions hold:*

- (i)  $\|L_{i,j}(1) - 1\| = st_A^T - o_k(\alpha_i)$  as  $i \rightarrow \infty$ ,
- (ii)  $\omega^*(f; \delta_{i,j}) = st_A^T - o_k(\beta_i)$  as  $i \rightarrow \infty$ , where  $\delta_{i,j} := \sqrt{\|L_{i,j}(\varphi_{(u,v)})\|}$  with  $\varphi_{(u,v)}(s, t) = (e^{-s} - e^{-u})^2 + (e^{-t} - e^{-v})^2$ . Then, for any  $f \in C_*(D)$

$$\|L_{i,j}(f) - f\| = st_A^T - o(\gamma_i), \text{ as } i \rightarrow \infty,$$

where  $\gamma_i := \max\{\alpha_i, \beta_i\}$  for each  $i \in \mathbb{N}$ . We can say that the similar results hold when the symbol “ $o_k$ ” is replaced by “ $O_k$ ”.

**Proof** Let  $f \in C_*(D)$  and  $(u, v) \in D$  be fixed. Using linearity and positivity of the  $L_{i,j}$ , we have, for any  $(i, j) \in \mathbb{N}^2$ ,

$$\begin{aligned} & |L_{i,j}(f; u, v) - f(u, v)| \\ &= |L_{i,j}(f(s, t) - f(u, v); u, v) - f(u, v)(L_{i,j}(1; u, v) - 1)| \\ &\leq L_{i,j}(|f(s, t) - f(u, v)|; u, v) + M|L_{i,j}(1; u, v) - 1| \\ &\leq L_{i,j}\left(\left(1 + \frac{(e^{-s} - e^{-u})^2 + (e^{-t} - e^{-v})^2}{\delta^2}\right) \omega^*(f; \delta); u, v\right) \\ &\quad + M|L_{i,j}(1; u, v) - 1| \\ &\leq \omega^*(f; \delta) L_{i,j}\left(\left(1 + \frac{(e^{-s} - e^{-u})^2 + (e^{-t} - e^{-v})^2}{\delta^2}\right); u, v\right) \\ &\quad + M|L_{i,j}(1; u, v) - 1| \\ &\leq \omega^*(f; \delta) |L_{i,j}(1; u, v) - 1| + \frac{\omega^*(f; \delta)}{\delta^2} L_{i,j}(\varphi_{(u,v)}; u, v) + \omega^*(f; \delta) \\ &\quad + M|L_{i,j}(1; u, v) - 1|, \end{aligned}$$

where  $M := \|f\|$ . Thus, taking supremum over  $(u, v) \in D$  on the both sides of the above inequality, we get, for any  $\delta > 0$ ,

$$\begin{aligned} \|L_{i,j}(f) - f\| &\leq \omega^*(f; \delta) \|L_{i,j}(1) - 1\| + \frac{\omega^*(f; \delta)}{\delta^2} \|L_{i,j}(\varphi_{(u,v)})\| \\ &\quad + \omega^*(f; \delta) + M \|L_{i,j}(1) - 1\| \end{aligned}$$

Now if  $\delta := \delta_{i,j} := \sqrt{\|L_{i,j}(\varphi_{(u,v)})\|}$ , then we can write

$$\|L_{i,j}(f) - f\| \leq \omega^*(f; \delta) \|L_{i,j}(1) - 1\| + 2\omega^*(f; \delta) + M \|L_{i,j}(1) - 1\|$$

and hence

$$\|L_{i,j}(f) - f\| \leq N \{\omega^*(f; \delta) \|L_{i,j}(1) - 1\| + \omega^*(f; \delta) + \|L_{i,j}(1) - 1\|\} \tag{3.1}$$

where  $N = \max\{2, M\}$ . Now, for a given  $r > 0$ , we define the following sets:

$$\begin{aligned} U_i & : = \{j \in \mathbb{N} : j \leq i, \|L_{i,j}(f) - f\| \geq r\gamma_i\}, \\ U_i^1 & : = \left\{j \in \mathbb{N} : j \leq i, \omega^*(f; \delta) \|L_{i,j}(1) - 1\| \geq \frac{r}{3N}\beta_i\right\}, \\ U_i^2 & : = \left\{j \in \mathbb{N} : j \leq i, \omega^*(f; \delta) \geq \frac{r}{3N}\beta_i\right\}, \\ U_i^3 & : = \left\{j \in \mathbb{N} : j \leq i, \|L_{i,j}(1) - 1\| \geq \frac{r}{3N}\alpha_i\right\}. \end{aligned}$$

In the view of (3.1) that

$$U_i \subset U_i^1 \cup U_i^2 \cup U_i^3.$$

Moreover, define the sets:

$$\begin{aligned} U_i^4 & : = \left\{j \in \mathbb{N} : j \leq i, \omega^*(f; \delta) \geq \sqrt{\frac{r}{3N}}\beta_i\right\}, \\ U_i^5 & : = \left\{j \in \mathbb{N} : j \leq i, \|L_{i,j}(1) - 1\| \geq \sqrt{\frac{r}{3N}}\alpha_i\right\}. \end{aligned}$$

Then, observe that  $U_i^1 \subset U_i^4 \cup U_i^5$ . Therefore, we have  $U_i \subset U_i^2 \cup U_i^3 \cup U_i^4 \cup U_i^5$ . Now, since  $\gamma_i := \max\{\alpha_i, \beta_i\}$  for each  $i \in \mathbb{N}$ , we get

$$\begin{aligned} \sum_{j \in U_i} a_{i,j} & \leq \sum_{j \in U_i^2} a_{i,j} + \sum_{j \in U_i^3} a_{i,j} \\ & \quad + \sum_{j \in U_i^4} a_{i,j} + \sum_{j \in U_i^5} a_{i,j}. \end{aligned}$$

Letting  $i \rightarrow \infty$  and from (i) and (ii), we have

$$\lim_i \sum_{j \in U_i} a_{i,j} = 0$$

whence the result. □

#### 4. Possible further developments

Here we recall the notion of  $A$ -statistical type convergence, includes the triangular  $A$ -statistical convergence as a particular case, given by Bardaro et al. ([5, 6]).

Let  $A = (a_{i,j})$  be a nonnegative regular summability matrix,  $\Psi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$  a fixed function. A sequence  $u = (u_{i,j})$  is said to be  $\Psi - A$ -statistically convergent to a real number  $L$  provided that for every  $\varepsilon > 0$

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where  $K_i(\varepsilon) = \{j \in \mathbb{N} : \Psi(i, j) \geq 0, |u_{i,j} - L| \geq \varepsilon\}$  (See [5, 6]).

Note that, if  $\Psi(i, j) = i - j$  then, the triangular  $A$ -statistical convergence is the special case of  $\Psi - A$ -statistical convergence. If  $\Psi(i, j) = \beta(i) - j$ , where  $\beta : \mathbb{N} \rightarrow \mathbb{N}$  is a suitable increasing function, then we get other interesting results. Hence, all the theory given in the previous sections can be carried on also in the setting of  $\Psi - A$ -statistical convergence.

### References

- [1] Altomare F. Korovkin-type theorems and approximation by positive linear operators. *Surveys in Approximation Theory* 2010; 5: 92-164. arxiv.org/abs/1009.2601
- [2] Anastassiou GA, Duman O. *Towards Intelligent Modeling: Statistical Approximation Theory*. Intelligent Systems Reference Library, vol. 14. Berlin, Germany: Springer-Verlag, 2011.
- [3] Aslan İ, Duman O. Approximation by nonlinear integral operators via summability process. *Mathematische Nachrichten* 2020; 293 (3): 430-448. doi: 10.1002/mana.201800187
- [4] Atlıhan ÖG, Orhan C. Summation process of positive linear operators. *Computers and Mathematics with Applications* 2008; 56 (5): 1188-1195. doi: 10.1016/j.camwa.2008.02.020
- [5] Bardaro C, Boccuto A, Demirci K, Mantellini I, Orhan S. Triangular  $A$ -statistical approximation by double sequences of positive linear operators. *Results in Mathematics* 2015; 68: 271-291. doi: 10.1007/s00025-015-0433-7
- [6] Bardaro C, Boccuto A, Demirci K, Mantellini I, Orhan S. Korovkin-type theorems for modular  $\Psi - A$ -statistical convergence. *Journal of Function Spaces* 2015; 201: 1-11. doi: 10.1155/2015/160401
- [7] Bojanic R, Khan MK. Summability of Hermite-Fejér interpolation for functions of bounded variation. *Journal of Natural Sciences and Mathematics* 1992; 32: 5-10.
- [8] Boos J. *Classical and Modern Methods in Summability*. New York, NY, USA: Oxford University Press, 2000.
- [9] Boyanov BD, Veselinov VM. A Note on the approximation of functions in an infinite interval by linear positive operators. *Bulletin mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie* 1970; 14 (62)1: 9-13.
- [10] Devore RA. *The Approximation of continuous functions by positive linear operators*. Lecture Notes in Mathematics, Vol. 293, Berlin, Germany: Springer-Verlag, 1972.
- [11] Demirci K, Karakuş S.  $A$ -statistical Korovkin-type approximation theorem for of two variables on an infinite interval. *Acta Mathematica Universitatis Comenianae* 2012; 81 (2): 151-157.
- [12] Duman O, Khan MK, Orhan C.  $A$ -Statistical convergence of approximating operators. *Mathematical Inequalities and Applications* 2003; 6: 689-699. doi: 10.7153/mia-06-62
- [13] Gadjiev AD, Orhan C. Some approximation theorems via statistical convergence. *The Rocky Mountain Journal of Mathematics* 2002; 32 (1): 129-138.
- [14] Guessab A, Schmeisser G. Two Korovkin-type theorems in multivariate approximation. *Banach Journal of Mathematical Analysis* 2008; 2 (2): 121-128. doi: 10.15352/bjma/1240336298
- [15] Hamilton HJ. Transformations of multiple sequences. *Duke Mathematical Journal* 1936; 2: 29-60. doi: 10.1215/S0012-7094-36-00204-1

- [16] Karakuş S, Demirci K, Duman O. Equi-statistical convergence of positive linear operators. *Journal of Mathematical Analysis and Applications* 2008; 339: 1065-1072. doi: 10.1016/j.jmaa.2007.07.050
- [17] Korovkin PP. *Linear operators and approximation theory*. New Delhi, India: Hindustan Publishing Company, 1960.
- [18] Moricz F. Statistical convergence of multiple sequences. *Archiv der Mathematik (Basel)* 2004; 81: 82-89. doi: 10.1007/s00013-003-0506-9
- [19] Pringsheim A. Zur theorie der zweifach unendlichen zahlenfolgen. *Mathematische Annalen* 1900; 53: 289-321 (in German). doi: 10.1007/BF01448977
- [20] Robison GM. Divergent double sequences and series. *Transactions of the American Mathematical Society* 1926; 28: 50-73. doi: 10.2307/1989172
- [21] Sakaolu İ, Orhan C. Strong summation process in  $L_p$  spaces. *Nonlinear Analysis: Theory, Methods and Applications* 2013; 86: 89-94. doi: 10.1016/j.na.2013.03.010
- [22] Şahin Bayram N, Orhan C.  $A$ -Summation process in the space of locally integrable functions. *Studia Universitatis Babeş-Bolyai Mathematica* 2020; 65 (2): 255-268. doi: 10.24193/subbmath.2020.2.07
- [23] Totik V. Uniform approximation by Szasz-Mirakjan type operators. *Acta Mathematica Hungarica* 1983; 41 (3/4): 291-307. doi: 10.1007/BF01961317