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On ordered Γ-hypersemigroups, minimal bi-ideals, and minimal left ideals

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Abstract: The definition of ordered Γ-hypersemigroups and the definitions of regular and intra-regular ordered Γ-hypersemigroups in the existing bibliography should be corrected. Care should be given to the definitions of bi-Γ-hyperideals and quasi-Γ-hyperideals as well. The main results are a characterization of minimal bi-ideals of an ordered Γ-hypersemigroup S in terms of $\mathcal{B}$-simple bi-ideals of S and a characterization of minimal left (resp. right) ideals of an ordered Γ-hypersemigroup S in terms of left (resp. right) simple subsemigroups of S.

Key words: Ordered Γ-hypersemigroup, regular, intra-regular, bi-Γ-hyperideal, quasi-Γ-hyperideal, $\mathcal{B}$-simple bi-ideal

1. Introduction

According to the existing bibliography, if S and Γ are nonempty sets such that every $\gamma \in \Gamma$ is a hyperoperation of S, i.e. $x\gamma y \subseteq S$ for every $x, y \in S$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$, we have $x\alpha(y\beta z) = (x\alpha y)\beta z$, then S is called a Γ-hypersemigroup. An ordered Γ-hypersemigroup is a Γ-hypersemigroup with an order relation "$\leq$" on S such that $a \leq b$ implies $a\gamma c \preceq b\gamma c$ and $c\gamma a \preceq c\gamma b$ for every $c \in S$ and every $\gamma \in \Gamma$ in the sense that for every $u \in a\gamma c$ there exists $v \in b\gamma c$ such that $u \leq v$ and for every $u \in c\gamma a$ there exists $v \in c\gamma b$ such that $u \leq v$.

Clearly, "$x\gamma y \subseteq S$ for every $x, y \in S$" is not enough for γ to be hyperoperation. A hyperoperation is actually an operation (called hyperoperation as it assigns to each couple of elements of S a subset (instead of an element) of S –it might be called operation as well. Therefore, as in any operation, the uniqueness condition is the main condition that defines the hyperoperation. Let us stay a little more on it to make everything clear.

γ is a hypeoperation on S means that γ is a mapping of $S \times S$ into the set $\mathcal{P}^*(S)$ of all nonempty subsets of S, also written as

$$\gamma : S \times S \to \mathcal{P}^*(S) \mid (x, y) \to \gamma(x, y) := x\gamma y.$$

That means that the following assertions are satisfied:

(1) $x\gamma y \subseteq S$ for every $x, y \in S$ and

(2) if $x, y, z, t \in S$ such that $x = z$ and $y = t$, then $x\gamma y = z\gamma t$.

So, it would be wrong to say "γ is a hyperoperation on S, i.e. $x\gamma y \subseteq S$ for every $x, y \in S$". Condition (2) (the uniqueness condition) is the essential, main condition that defines the hyperoperation and is missing from

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2010 *AMS Mathematics Subject Classification:* 06F99, 20N99, 06F05 the definition of Γ-hypersemigroup. Without this condition in an expression of the form $A\Gamma B\Gamma C\Gamma D$, we do not know where to put parentheses, and the investigation cannot go further (for details see [3]; the paper in [3] is on ordered Γ-semigroups but the same can be said for ordered Γ-hypersemigroups, as well). In addition, expressions of the form $x\alpha(y\beta z)$, $(x\alpha y)\beta z$ have no sense (see [7]).

According to the bibliography, an ordered Γ-hypersemigroup $(S,\Gamma,\le)$ is called *regular* if for every $a\in S$ there exist $x\in S$ and $\alpha,\beta\in\Gamma$, such that $a\le a\alpha x\beta a$ or if one of the following two equivalent conditions are satisfied: (α) $a\in(a\Gamma S\Gamma a]$ for every $a\in S$. (β) $A\subseteq(A\Gamma S\Gamma A]$ for every $A\subseteq S$.

In other words, the following are equivalent:

(1) For every $a\in S$ there exist $x\in S$ and $\alpha,\beta\in\Gamma$, such that $a\le a\alpha x\beta a$.

(2) $a\in(a\Gamma S\Gamma a]$ for every $a\in S$.

(3) $A\subseteq(A\Gamma S\Gamma A]$ for every $A\subseteq S$.

However, this is the definition of regular ordered Γ-semigroup [3] and not of regular ordered Γ-hypersemigroups. a being an element of S, $a\alpha x\beta a$ being a subset of S (after correction) and "$\le$" the order on S (that is relation between elements); the expression $a\le a\alpha x\beta a$ has no meaning. When the definition of regularity is wrong, there is no sense to speak about its equivalent definitions. We are not in a Γ-semigroup in which this is obvious. Even if we get the (equivalent definition) $A\subseteq(A\Gamma S\Gamma A]$ (or the $a\in(a\Gamma S\Gamma a]$) as the definition of regularity, what is the $A\Gamma S\Gamma A$? (or the $a\Gamma S\Gamma a$?) Thus, the equivalent definitions also cannot define the ordered Γ-hypersemigroup. Except the fact that in (3), the set A should be nonempty (we are not in a Γ-semigroup in which A can be empty).

The definition of an intra-regular ordered Γ-hypersemigroup is given as follows: A Γ-hypersemigroup $(S,\Gamma,\le)$ is called *intra-regular* if for every $a\in S$ there exist $x,y\in S$ and $\alpha,\beta,\gamma\in\Gamma$, such that $a\le x\alpha a\beta a\gamma y$ or if one of the following two conditions holds: (α) $a\in(S\Gamma a\Gamma a\Gamma S]$ for all $a\in S$. (β) $A\subseteq(S\Gamma A\Gamma A\Gamma S]$ for all $A\subseteq S$.

So, the following are equivalent:

(1) For every $a\in S$, there exist $x,y\in S$ and $\alpha,\beta,\gamma\in\Gamma$, such that $a\le x\alpha a\beta a\gamma y$.

(2) $a\in(S\Gamma a\Gamma a\Gamma S]$ for every $a\in S$.

(3) $A\subseteq(S\Gamma A\Gamma A\Gamma S]$ for every subset A of S.

However, this is the definition of intra-regular ordered Γ-semigroup [3] and not of intra-regular ordered Γ-hypersemigroups. In case of an intra-regular ordered Γ-hypersemigroup, we have to define the $S\Gamma A\Gamma A\Gamma S$ and give an explanation that we have the right to write it without using parentheses.

In case of intra-regular ordered Γ-hypersemigroups, the set A in (3) should be nonempty.

The definition of bi-Γ-hyperideal for ordered Γ-hypersemigroups given by "(1) $B\Gamma S\Gamma B\subseteq B$ and (2) $x\in B$ and $y\in S$, such that $y\le x$ implies $y\in B$" without explaining what the $B\Gamma S\Gamma B\subseteq B$ means, defines the bi-ideal of ordered Γ-semigroups [3] (where the methodology is completely different), and cannot define the bi-Γ-hyperideal of ordered Γ-hypersemigroups. The fact that it is defined as subΓ-hypersemigroup in the bibliography does not play any role.

Examples given by a table of multiplication and an order, based on the above mentioned concepts should be also corrected.

It might be mentioned here that the paper in [3] has been written in an attempt to show the methodology we pass from ordered semigroups to ordered Γ-semigroups to conclude that many results on semigroups or ordered semigroups hold automatically (with the obvious change) for Γ-semigroups or ordered Γ-semigroups, respectively, and do not need any proof (see the Abstract in [3]). With exactly the same aim has been written the paper in [4]. We might refer to "Conclusion" on p. 566 in [5] and to [6] as well.

The combination of two elements of an ordered Γ-hypersemigroup S by a γ-hyperoperation, being a subset of S (and not an element of S -as in ordered Γ-semigroups), makes the investigation on ordered Γ-hypersemigroups different. Although the characterization of regular Γ-semigroups given in [3; p. 182] and [3; Theorems 3 and 5] hold without any change for ordered Γ-hypersemigroups when we use the (equivalent) definition of regularity for ordered Γ-hypersemigroups given by $A \subseteq (A\Gamma S\Gamma A]$, the notation $A \subseteq (A\Gamma S\Gamma A]$ between the two structures has completely different meaning. The same might be said for Proposition 4 and Propositions 6–10 in [3].

As an example, let us consider the Theorem 3 in [3]: An ordered Γ-semigroup S is regular if and only if the right ideals and the left ideals of S are idempotent and for every right ideal A and every left ideal B of S, $(A\Gamma B]$ is a quasi-ideal of S. If we replace the word "Γ-semigroup" by "Γ-hypersemigroup", the above mentioned theorem holds for regular ordered Γ-hypersemigroups as well and its proof is the same. But the concepts in each case is different. A nonempty subset Q of an ordered Γ-semigroup $(S, \Gamma, \leq)$ is called a quasi-ideal of S if (1) $(Q\Gamma S] \cap (S\Gamma Q] \subseteq Q$ and (2) if $a \in Q$ and $S \ni b \leq a$, then $b \in Q$. A nonempty subset Q of an ordered Γ-hypersemigroup $(S, \Gamma, \leq)$ is called a quasi-Γ-hyperideal of S if (1) $(Q\Gamma S] \cap (S\Gamma Q] \subseteq Q$ and (2) if $a \in Q$ and $S \ni b \leq a$, then $b \in Q$. Although the two definitions look like the same, in case of an ordered Γ-hypersemigroup, the $(Q\Gamma S] \cap (S\Gamma Q] \subseteq Q$ has the following meaning: If $x \leq t$ for some $t \in u\gamma s$, $u \in Q$, $\gamma \in \Gamma$, $s \in S$ and $x \leq h$ for some $h \in s'\mu v$, $s' \in S$, $\mu \in \Gamma$, $v \in Q$, then $x \in Q$.

The same might be said for Theorem 5 in [3]. The following theorem holds and its proof is the same with the proof of Theorem 5 in [3] if we use the equivalent definition of regularity mentioned above. This is a new theorem; the reader can easily prove it by taking into account the corresponding result on ordered Γ-semigroups.

Theorem: Let S be a regular ordered Γ-hypersemigroup. Then, B is a bi-Γ-hyperideal of S if and only if there exist a right Γ-hyperideal R and a left Γ-hyperideal L of S, such that $B = (R\Gamma L]$.

But, while for an ordered Γ-semigroup S, $(R\Gamma L]$ is the subset of S defined by

$$\{t \in S \mid t \leq u \text{ for some } u = x\gamma y;\, x \in R, \gamma \in \Gamma,\, y \in L\},$$

for an ordered Γ-hypersemigroup S, $(R\Gamma L]$ is the subset of S defined by

$$\{t \in S \mid t \leq u \text{ for some } u \in x\gamma y;\, x \in R, \gamma \in \Gamma,\, y \in L\}.$$

It has been proved by Yonglin Cao and Xinzhai Xu in [1] that a left ideal of an ordered semigroup S is minimal if and only if it is left simple. The proof is based on the fact that if L is a minimal left ideal of an ordered semigroup S and A is a left ideal of L, then the set $H = \{h \in A \mid h \leq ka \text{ for some } k \in L \text{ and } a \in A\}$ is a left ideal of S. According to the main theorem in [10], a bi-ideal of an ordered semigroup is minimal if and only if it is $\mathcal{B}$-simple. For the same theorem, in case of ordered Γ-semigroups, see [2]. In the following, we will examine these results for ordered Γ-hypersemigroups.

2. Main results

Using the definitions, notations, and some of the results in [9], let us characterize the minimal bi-ideals of an ordered Γ-hypersemigroup S in terms of $\mathcal{B}$-simple bi-ideals and the minimal left ideals of S in terms of left simple subsemigroups of S.

Let S be a Γ-hypersemigroup and "$\le$" be an order relation on S. Denoted by "$\preceq$" the reflexive and transitive relation (preorder) on the set of all nonempty subsets, $\mathcal{P}^*(S)$ of S defined by

$$A \preceq B \text{ if for every } a \in A \text{ there exists } b \in B \text{ such that } a \le b.$$

A Γ-hypersemigroup (S,Γ) is called an *ordered* Γ*-hypersemigroup* if there is an order relation "$\le$" on S such that, for every $c \in S$ and every $\gamma \in \Gamma$, $a \le b$ implies $c\gamma a \preceq c\gamma b$ and $a\gamma c \preceq b\gamma c$.

For nonempty subsets B and T of an ordered Γ-hypersemigroup S, the symbol $(B]_T$ denotes the subset of T defined by $(B]_T := \{t \in T \mid t \le b \text{ for some } b \in B\}$. In particular, for $T = S$, we write $(B]$ instead of $(B]_S$ and we have $(B] := \{t \in S \mid t \le b \text{ for some } b \in B\}$.

Clearly, $x \in (B]_T$ if and only if $x \in T$ and $x \in (B]$.

A nonempty subset B of an ordered Γ-hypersemigroup S is called a *bi-ideal* of S, if (1) $B\Gamma S\Gamma B \subseteq B$ and (2) if $a \in B$ and $S \ni b \le a$, then $b \in B$; that is, if $(B] = B$.

Lemma 1 *Let $(S,\Gamma,\le)$ be an ordered hypersemigroup and B a nonempty subset of S such that $(B] = B$. The following are equivalent:*

(1) B is a bi-ideal of S.

(2) if $a,b \in B$, $s \in S$ and $\gamma,\mu \in \Gamma$, then $(a\gamma s)\overline{\mu}\{b\} \subseteq B$.

Proof (1) $\Longrightarrow$ (2). Let $a,b \in B$, $s \in S$, $\gamma,\mu \in \Gamma$ and $x \in (a\gamma s)\overline{\mu}\{b\}$. We have $(a\gamma s)\overline{\mu}\{b\} \subseteq (a\gamma s)\Gamma\{b\}$ [9; Definition 3.3]. Since $a \in B$, $\gamma \in \Gamma$, $s \in S$, we have $a\gamma s \subseteq B\Gamma S$ [9; Lemma 3.7(2)]. Since $a\gamma s \subseteq B\Gamma S$ and $\{b\} \subseteq B$, we have $(a\gamma s)\Gamma\{b\} \subseteq (B\Gamma S)\Gamma B$ [9; Lemma 3.8]. We also have $(B\Gamma S)\Gamma B = B\Gamma S\Gamma B$ [9; Prop. 3.17]. Since B is a bi-ideal of S, $B\Gamma S\Gamma B \subseteq B$. Thus we get $x \in B$ and property (2) holds.
(2) $\Longrightarrow$ (1). Let $x \in B\Gamma S\Gamma B$. Then $x \in u\mu b$ for some $u \in B\Gamma S$, $\mu \in \Gamma$, $b \in B$ and $u \in a\gamma s$ for some $a \in B$, $\gamma \in \Gamma$, $s \in S$ [9; Lemma 3.7 and Prop. 3.17]. Thus we have

$$x \in u\mu b = \{u\}\overline{\mu}\{b\} \subseteq (a\gamma s)\overline{\mu}\{b\} \text{ (by [9; Lemmas 3.5, 3.6]).}$$

Since $a,b \in B$, $s \in S$, $\gamma,\mu \in \Gamma$, by (2), we have $(a\gamma s)\overline{\mu}\{b\} \subseteq B$. Then we have $x \in B$ and property (1) is satisfied. □

Lemma 2 *If S is a Γ-hypersemigroup then, for any nonempty subsets A,B,C of S and any $\gamma,\mu \in \Gamma$, we have*

$$(A\overline{\gamma}B)\overline{\mu}C = A\overline{\gamma}(B\overline{\mu}C).$$

Proof Let $x \in (A\overline{\gamma}B)\overline{\mu}C$. Then, $x \in u\mu c$ for some $u \in A\overline{\gamma}B$, $c \in C$ and $u \in a\gamma b$ for some $a \in A$, $b \in B$ (by

[9; Definition 3.2]). Then, we have

$$\begin{aligned} x \in u\mu c &= \{u\}\overline{\mu}\{c\} \subseteq (a\gamma b)\overline{\mu}\{c\} \text{ (by [9; Lemmas 3.5, 3.6])} \\ &= \{a\}\overline{\gamma}(b\mu c) \text{ (by [9; Def. 3.14])} \\ &\subseteq A\overline{\gamma}(B\overline{\mu}C) \text{ (by [9; Def. 3.2, Lemma 3.6]).} \end{aligned}$$

If $x \in A\overline{\gamma}(B\overline{\mu}C)$, then $x \in a\gamma u$ for some $a \in A$, $u \in B\overline{\mu}C$ and $u \in b\mu c$ for some $b \in B$, $c \in C$; thus we have

$$x \in \{a\}\overline{\gamma}\{u\} \subseteq \{a\}\overline{\gamma}(b\mu c) \subseteq A\overline{\gamma}(B\overline{\mu}C).$$

□

According to Lemma 2, for any nonempty subsets $A_1, A_2, ..., A_n$ of S we can write $A_1\overline{\gamma}_1A_2...\overline{\gamma}_nA_n$ without using parentheses.

Lemma 3 *Let S be an ordered Γ-hypersemigroup and A, B, C nonempty subsets of S. Then, we have the following:*

(1) If $A \preceq B$, then $A\overline{\gamma}C \preceq B\overline{\gamma}C$ and $C\overline{\gamma}A \preceq C\overline{\gamma}B$ for every $\gamma \in \Gamma$.

(2) If $A \preceq B \subseteq C$, then $A \preceq C$.

Proof (1) Let $x \in A\overline{\gamma}C$. By [9; Definition 3.2], $x \in a\gamma c$ for some $a \in A$, $c \in C$. Since $A \preceq B$ and $a \in A$, there exists $b \in B$ such that $a \leq b$. Since $a \leq b$, we have $a\gamma c \preceq b\gamma c$ and, since $x \in a\gamma c$, there exists $y \in b\gamma c$ such that $x \leq y$. By [9; Definition 3.2], $y \in b\gamma c \subseteq B\overline{\gamma}C$. For the element $y \in B\overline{\gamma}C$, we have $x \leq y$ and so $A\overline{\gamma}C \preceq B\overline{\gamma}C$.
(2) Let $A \preceq B \subseteq C$ and $a \in A$. Since $A \preceq B$ and $a \in A$, we have $a \leq b$ for some $b \in B$. Since $B \subseteq C$, we have $a \leq b$, where $b \in C$ and so $A \preceq C$. □

Definition 4 *Let S be an ordered Γ-hypersemigroup and B a bi-ideal of S. A nonempty subset A of B is called a bi-ideal of B if (1) $A\Gamma B\Gamma A \subseteq A$ and (2) if $a \in A$ and $B \ni b \leq a$, then $b \in A$.*

Here $A\Gamma B\Gamma A \subseteq A$ means that if $x \in (u\gamma b)\overline{\mu}\{a\}$ for some $u, a \in A$, $b \in B$ and $\gamma, \mu \in \Gamma$, then $x \in A$. For $B = S$, this is the definition of bi-ideal.

Definition 5 *Let S be an ordered Γ-hypersemigroup. A bi-ideal B of of S is called minimal (bi-ideal of S) if for every bi-ideal A of S such that $A \subseteq B$, we have $A = B$.*

A bi-ideal B of an ordered Γ-hypersemigroup S is called $\mathcal{B}$-simple if B is the only bi-ideal of B; that is, if A is a bi-ideal of B, then $A = B$.

Theorem 6 *Let S be an ordered Γ-hypersemigroup and B be a bi-ideal of S. Then B is a minimal bi-ideal of S if and only if B is $\mathcal{B}$-simple.*

Proof $\Longrightarrow$. Let B be a minimal bi-ideal of S and A be a bi-ideal of B. We have to prove that $A = B$.
We consider the set

$$H := \{h \in A \mid \{h\} \preceq (a_1\gamma b)\overline{\mu}\{a_2\} \text{ for some } a_1, a_2 \in A, b \in B, \gamma, \mu \in \Gamma\}.$$

It is enough to prove that H is a bi-ideal of S. Then, since $H \subseteq A \subseteq B$ and B is a minimal bi-ideal of S, we have $H = B$, then $A = B$ and the proof is complete.

The set H is a bi-ideal of S. In fact:

(1) H is a nonempty subset of S: Take $a \in A$, $b \in B$, $\gamma \in \Gamma$ $(A, B, \Gamma \neq \emptyset)$, and an element $h \in (a\gamma b)\overline{\gamma}\{a\}$. By [9; Definition 3.3], $(a\gamma b)\overline{\gamma}\{a\} \subseteq (a\gamma b)\Gamma\{a\}$. By [9; Lemma 3.7(2)], $a\gamma b \subseteq A\Gamma B$. By [9; Lemma 3.8 and Prop. 3.17], $(a\gamma b)\Gamma\{a\} \subseteq (A\Gamma B)\Gamma A = A\Gamma B\Gamma A$. Since A is a bi-ideal of B, we have $A\Gamma B\Gamma A \subseteq A$. Thus we have $h \in A$. Since $h \le h \in (a\gamma b)\overline{\gamma}\{a\}$, we have $\{h\} \preceq (a\gamma b)\overline{\gamma}\{a\}$. Since $h \in A$, $\{h\} \preceq (a\gamma b)\overline{\gamma}\{a\}$, $a \in A$, $b \in B$, $\gamma \in \Gamma$, we have $h \in H$ and the set H is a nonempty set.

(2) $H\Gamma S\Gamma H \subseteq H$, that is, if $h_1, h_2 \in H$, $s \in S$ and $\zeta, \omega \in \Gamma$, then $(h_1\zeta s)\overline{\omega}\{h_2\} \subseteq H$ (see Lemma 1). To prove it,

let $h_1, h_2 \in H$, $s \in S$, $\zeta, \omega \in \Gamma$ and $x \in (h_1\zeta s)\overline{\omega}\{h_2\}$. Then $x \in H$. Indeed: Since $h_1 \in H$, we have $h_1 \in A$ and $\{h_1\} \preceq (a_1\gamma_1 b_1)\overline{\mu}_1\{a_1'\}$ for some $a_1, a_1' \in A$, $b_1 \in B, \gamma_1, \mu_1 \in \Gamma$. Since $h_2 \in H$, we have $h_2 \in A$ and $\{h_2\} \preceq (a_2\gamma_2 b_2)\overline{\mu}_2\{a_2'\}$ for some $a_2, a_2' \in A$, $b_2 \in B, \gamma_2, \mu_2 \in \Gamma$. Then we have $h_1 \le t_1$ for some $t_1 \in (a_1\gamma_1 b_1)\overline{\mu}_1\{a_1'\}$ and $h_2 \le t_2$ for some $t_2 \in (a_2\gamma_2 b_2)\overline{\mu}_2\{a_2'\}$. Since $h_1 \le t_1$, we have $h_1\zeta s \preceq t_1\zeta s$. Then

$$\begin{aligned}
(h_1\zeta s)\overline{\mu}\{h_2\} &\preceq (t_1\zeta s)\overline{\mu}\{h_2\} \text{ (by Lemma 3(1))}\\
&= \Big(\{t_1\}\overline{\zeta}\{s\}\Big)\overline{\mu}\{h_2\} \text{ (by [9; Lemma 3.5])}\\
&= \{t_1\}\overline{\zeta}\{s\}\overline{\mu}\{h_2\} \text{ (by Lemma 2)}\\
&\subseteq \Big((a_1\gamma_1 b_1)\overline{\mu}_1\{a_1'\}\Big)\overline{\zeta}\{s\}\overline{\mu}\{h_2\}\\
&= \{a_1\}\overline{\gamma}_1\{b_1\}\overline{\mu}_1\{a_1'\}\overline{\zeta}\{s\}\overline{\mu}\{h_2\} \text{ (by [9; Lemma 3.5] and Lemma 2).}
\end{aligned}$$

We write, for short, $A := \{a_1\}\overline{\gamma}_1\{b_1\}\overline{\mu}_1\{a_1'\}\overline{\zeta}\{s\}$ and, by Lemma 3(2), we have

$$(h_1\zeta s)\overline{\mu}\{h_2\} \preceq A\overline{\mu}\{h_2\} \tag{A}$$

Since $\{h_2\} \preceq \{a_2\}\overline{\gamma}_2\{b_2\}\overline{\mu}_2\{a_2'\}$, by Lemma 3(1), we have

$$A\overline{\mu}\{h_2\} \preceq A\overline{\mu}\Big(\{a_2\}\overline{\gamma}_2\{b_2\}\overline{\mu}_2\{a_2'\}\Big).$$

We put $C := \{b_2\}\overline{\mu}_2\{a_2'\}$, and we have

$$\begin{aligned}
A\overline{\mu}\{h_2\} &\preceq A\overline{\mu}\Big(\{a_2\}\overline{\gamma}_2 C\Big) = A\overline{\mu}\{a_2\}\overline{\gamma}_2 C \text{ (by Lemma 2)}\\
&= A\overline{\mu}\{a_2\}\overline{\gamma}_2\Big(\{b_2\}\overline{\mu}_2\{a_2'\}\Big) = A\overline{\mu}\{a_2\}\overline{\gamma}_2\{b_2\}\overline{\mu}_2\{a_2'\} \text{ (by Lemma 2 as } \emptyset \neq A\overline{\mu}\{a_2\} \subseteq S) \qquad \text{(B)}
\end{aligned}$$

Since "$\preceq$" is a transitive relation on S, by (A) and (B), we obtain

$$(h_1\zeta s)\overline{\mu}\{h_2\} \preceq \{a_1\}\overline{\gamma}_1\{b_1\}\overline{\mu}_1\{a_1'\}\overline{\zeta}\{s\}\overline{\mu}\{a_2\}\overline{\gamma}_2\{b_2\}\overline{\mu}_2\{a_2'\}.$$

On the other hand, $\{b_1\}\overline{\mu}_1\{a_1'\}\overline{\zeta}\{s\}\overline{\mu}\{a_2\}\overline{\gamma}_2\{b_2\} \subseteq B\Gamma S\Gamma B \subseteq B$. Thus we have,

$$\begin{aligned}
x \in (h_1\zeta s)\overline{\mu}\{h_2\} &\preceq \{a_1\}\overline{\gamma}_1\{b_1\}\overline{\mu}_1\{a_1'\}\overline{\zeta}\{s\}\overline{\mu}\{a_2\}\overline{\gamma}_2\{b_2\}\overline{\mu}_2\{a_2'\} \subseteq \{a_1\}\overline{\gamma}_1 B\overline{\mu}_2\{a_2'\}\\
&\subseteq A\Gamma B\Gamma A \subseteq A \qquad \text{(C)}
\end{aligned}$$

By (C), we have $x \le y$ for some $y \in A$; in addition,

$$x \in (h_1\zeta s)\overline{\omega}\{h_2\} = \{h_1\}\overline{\zeta}\{s\}\overline{\omega}\{h_2\} \subseteq A\Gamma S\Gamma A \subseteq B\Gamma S\Gamma B \subseteq B.$$

Since $B \ni x \leq y \in A$ and A is a bi-ideal of B, we have $x \in A$.

Again by (C), $x \leq y$ for some $y \in \{a_1\}\overline{\gamma}_1 B\overline{\mu}_2\{a_2'\}$. Since $y \in \Big(\{a_1\}\overline{\gamma}_1 B\Big)\overline{\mu}_2\{a_2'\}$, we have $y \in u\mu_2 a_2'$ for some $u \in \{a_1\}\overline{\gamma}_1 B$ and $u \in a_1\gamma_1 t$ for some $t \in B$ (by [9; Definition 3.2]). Thus, we have

$$x \leq y \in u\mu_2 a_2' = \{u\}\overline{\mu}_2\{a_2'\} \subseteq (a_1\gamma_1 t)\overline{\mu}_2\{a_2'\},$$

and so $\{x\} \preceq (a_1\gamma_1 t)\overline{\mu}_2\{a_2'\}$. Since $x \in A$ and $\{x\} \preceq (a_1\gamma_1 t)\overline{\mu}_2\{a_2'\}$, where $a_1, a_2 \in A$, $t \in B$ and $\gamma_1, \mu_2 \in \Gamma$, we have $x \in H$.

(3) Let $h \in H$ and $S \ni t \leq h$. Then $t \in H$. Indeed: Since $h \in H$, we have $h \in A$ and $\{h\} \preceq (a_1\gamma b)\overline{\mu}\{a_2\}$ for some $a_1, a_2 \in A$, $b \in B, \gamma, \mu \in \Gamma$. Then $h \leq z$ for some $z \in (a_1\gamma b)\overline{\mu}\{a_2\}$; in addition, by [9; Lemma 3.7(2)], we have $a_1\gamma b \subseteq A\Gamma B$, by [9; Definition 3.3], we have $(a_1\gamma b)\overline{\mu}\{a_2\} \subseteq (a_1\gamma b)\Gamma\{a_2\}$ and by [9; Lemma 3.8], we have $(a_1\gamma b)\Gamma\{a_2\} \subseteq (A\Gamma B)\Gamma A$ and so $z \in A\Gamma B\Gamma A \subseteq A \subseteq B$. Since $t \leq z \in B$ and B is a bi-ideal of S, we have $t \in B$. Since $B \ni t \leq h \in A$ and A is a bi-ideal of B, we have $t \in A$. Since $t \leq h$, we have $\{t\} \preceq \{h\}$. Since the relation "$\preceq$" is transitive, we get $\{t\} \preceq (a_1\gamma b)\overline{\mu}\{a_2\}$. We have $t \in A$, $\{t\} \preceq (a_1\gamma b)\overline{\mu}\{a_2\}$, $a_1, a_2 \in A$, $b \in B$, $\gamma, \mu \in \Gamma$ and so $t \in H$.

$\Longleftarrow$. Let C be a bi-ideal of S such that $C \subseteq B$. Then $C = B$. Indeed: Let $b \in B$. Fix an element $c \in C$ and consider the set

$$A := \{t \in B \mid \{t\} \preceq (c\gamma b_1)\overline{\mu}\{c\} \text{ for some } b_1 \in B,\ \gamma, \mu \in \Gamma\}.$$

The set A is a bi-ideal of B. In fact:

(1) A is nonempty: Take $\gamma \in \Gamma$ and $x \in (c\gamma b)\overline{\gamma}\{c\}$. By [9; Definition 3.3], $(c\gamma b)\overline{\gamma}\{c\} \subseteq (c\gamma b)\Gamma\{c\}$. By [9; Lemma 3.7(2)], $c\gamma b \subseteq C\Gamma B$. By [9; Lemma 3.8],

$$(c\gamma b)\Gamma\{c\} \subseteq C\Gamma B\Gamma C \subseteq B\Gamma S\Gamma B \subseteq B.$$

Thus we have $x \in B$. Since $x \leq x \in (c\gamma b)\overline{\gamma}\{c\}$, we have $\{x\} \preceq (c\gamma b)\overline{\gamma}\{c\}$. Since $x \in B$, $\{x\} \preceq (c\gamma b)\overline{\gamma}\{c\}$, $b \in B$ and $\gamma \in \Gamma$, we have $x \in A$ and so $A \neq \emptyset$.

(2) $A\Gamma B\Gamma A \subseteq A$, that is, if $x, y \in A$, $z \in B$ and $\zeta, \omega \in \Gamma$, then $(x\zeta z)\overline{\omega}\{y\} \subseteq A$. To prove it,

let $x, y \in A$, $z \in B$, $\zeta, \omega \in \Gamma$ and $h \in (x\zeta z)\overline{\omega}\{y\}$. Then $h \in A$. Indeed:

Since $x \in A$, we have $x \in B$ and $\{x\} \preceq (c\gamma_1 b_1)\overline{\mu}_1\{c\}$ for some $b_1 \in B$, $\gamma_1, \mu_1 \in \Gamma$. Since $y \in A$, we have $y \in B$ and $\{y\} \preceq (c\gamma_2 b_2)\overline{\mu}_2\{c\}$ for some $b_2 \in B$, $\gamma_2, \mu_2 \in \Gamma$. Since $\{x\} \preceq \{c\}\overline{\gamma}_1\{b_1\}\overline{\mu}_1\{c\}$, by Lemma 3(1), we have

$$(x\zeta z)\overline{\omega}\{y\} = \{x\}\overline{\zeta}\{z\}\overline{\omega}\{y\} \preceq \{c\}\overline{\gamma}_1\{b_1\}\overline{\mu}_1\{c\}\overline{\zeta}\{z\}\overline{\omega}\{y\}.$$

We put $F := \{c\}\overline{\gamma}_1\{b_1\}\overline{\mu}_1\{c\}\overline{\zeta}\{z\}$ and we have $(x\zeta z)\overline{\omega}\{y\} \preceq F\overline{\omega}\{y\}$.

Since $\{y\} \preceq \{c\}\overline{\gamma}_2\{b_2\}\overline{\mu}_2\{c\}$, by Lemma 3, we have $F\overline{\omega}\{y\} \preceq F\overline{\omega}\Big(\{c\}\overline{\gamma}_2\{b_2\}\overline{\mu}_2\{c\}\Big)$.

We put $K := \{b_2\}\overline{\mu}_2\{c\}$ and we get

$$\begin{aligned} F\overline{\omega}\{y\} &\preceq F\overline{\omega}\Big(\{c\}\overline{\gamma}_2 K\Big) = F\overline{\omega}\{c\}\overline{\gamma}_2 K \text{ (by Lemma 2)} \\ &= F\overline{\omega}\{c\}\overline{\gamma}_2\Big(\{b_2\}\overline{\mu}_2\{c\}\Big) \\ &= F\overline{\omega}\{c\}\overline{\gamma}_2\{b_2\}\overline{\mu}_2\{c\} \text{ (by Lemma 2 as } \emptyset \neq F\overline{\omega}\{c\} \subseteq S). \end{aligned}$$

Thus we have $(x\zeta z)\overline{\omega}\{y\} \preceq \{c\}\overline{\gamma}_1\{b_1\}\overline{\mu}_1\{c\}\overline{\zeta}\{z\}\overline{\omega}\{c\}\overline{\gamma}_2\{b_2\}\overline{\mu}_2\{c\}$. Since

$$\{b_1\}\overline{\mu}_1\{c\}\overline{\zeta}\{z\}\overline{\omega}\{c\}\overline{\gamma}_2\{b_2\} \subseteq B\Gamma S\Gamma B \subseteq B,$$

we have

$$h \in (x\zeta z)\overline{\omega}\{y\} \preceq \{c\}\overline{\gamma}_1\{b_1\}\overline{\mu}_1\{c\}\overline{\zeta}\{z\}\overline{\omega}\{c\}\overline{\gamma}_2\{b_2\}\overline{\mu}_2\{c\} \subseteq \{c\}\overline{\gamma}_1 B\overline{\mu}_2\{c\} \subseteq B\Gamma S\Gamma B \subseteq B \qquad \text{(D)}$$

By (D), $h \le d$ for some $d \in B$. Since B is a bi-ideal of S, we have $h \in B$. Again by (D), $h \le d$ for some $d \in \Big(\{c\}\overline{\gamma}_1 B\Big)\overline{\mu}_2\{c\}$. By [9; Definition 3.2], $d \in u\mu_2 c$ for some $u \in \{c\}\overline{\gamma}_1 B$ and $u \in c\gamma_1 m$ for some $m \in B$. Then we have $d \in \{u\}\overline{\mu}_2\{c\} \subseteq (c\gamma_1 m)\overline{\mu}_2\{c\}$. Since $h \le d \in (c\gamma_1 m)\overline{\mu}_2\{c\}$, we have $\{h\} \preceq (c\gamma_1 m)\overline{\mu}_2\{c\}$. Since $h \in B$, $\{h\} \preceq (c\gamma_1 m)\overline{\mu}_2\{c\}$, $m \in B$ and $\gamma_1, \mu_2 \in \Gamma$, we have $h \in A$.

(3) Let $x \in A$ and $B \ni y \le x$. Then $y \in A$. Indeed: Since $x \in A$, we have $x \in B$ and $\{x\} \preceq (c\gamma b_1)\overline{\mu}\{c\}$ for some $b_1 \in B$, $\gamma, \mu \in \Gamma$. Since $y \le x$, we have $\{y\} \preceq \{x\}$. Since "$\preceq$" is a transitive relation, we have $\{y\} \preceq (c\gamma b_1)\overline{\mu}\{c\}$. Since $y \in B$, $\{y\} \preceq (c\gamma b_1)\overline{\mu}\{c\}$, $b_1 \in B$, $\gamma, \mu \in \Gamma$, we have $y \in A$.

By (1), (2) and (3), A is a bi-ideal of B. Since B is $\mathcal{B}$-simple, we have $A = B$. Since $b \in B$, we have $b \in A$. Since $b \in A$, we have $\{b\} \preceq (c\gamma t)\overline{\mu}\{c\}$ for some $t \in B$, $\gamma, \mu \in \Gamma$. By [9; Remark 3.4], $c\gamma t \subseteq C\Gamma B$. By [9; Lemma 3.6], $(c\gamma t)\overline{\mu}\{c\} \subseteq (C\Gamma B)\overline{\mu}\{c\}$. By [9; Definition 3.3], $(C\Gamma B)\overline{\mu}\{c\} \subseteq C\Gamma B\Gamma\{c\}$. Since $c \in C$, by [9; Lemma 3.8], $C\Gamma B\Gamma\{c\} \subseteq (C\Gamma B)\Gamma C \subseteq C\Gamma S\Gamma C$. Since C is a bi-ideal of S, $C\Gamma S\Gamma C \subseteq C$. Since $\{b\} \preceq (c\gamma t)\overline{\mu}\{c\} \subseteq C$, by Lemma 3(2), $\{b\} \preceq C$. Then there exists $d \in C$ such that $b \le d$. Since $S \ni b \le d \in C$ and C is a bi-ideal of S, we have $b \in C$. Thus we have $B \subseteq C$ and so $C = B$ and the proof is complete. □

In the following part, we will give a characterization of minimal left ideals of an ordered Γ-hypersemigroup analogous to that one given for bi-ideals in Theorem 6.

Definition 7 *Let S be an ordered Γ-hypersemigroup and L a left ideal of S. A nonempty subset A of L is called a left ideal of L if (1) $L\Gamma A \subseteq A$ and (2) if $a \in A$ and $L \ni b \le a$, then $b \in A$.*

Definition 8 *Let S be an ordered Γ-hypersemigroup. A left ideal L of S is called minimal (left ideal of S) if for every left ideal A of S such that $A \subseteq L$, we have $A = L$.*

A left ideal L of an ordered Γ-hypersemigroup S is called left simple if L is the only left ideal of L; that is, if A is a left ideal of L, then $A = L$.

Theorem 9 *Let S be an ordered Γ-hypersemigroup and L a left ideal of S. Then, L is minimal left ideal of S if and only if L is a left simple.*

Proof $\Longrightarrow$. Let A be a left ideal of L. Then $A = L$. Indeed: We consider the set

$$T := \{h \in A \mid \{h\} \preceq k\gamma a \text{ for some } k \in L, \gamma \in \Gamma, a \in A\}.$$

Take an element $c \in L$, an element $d \in A$, an element $\gamma \in \Gamma$, and an element $u \in c\gamma d$ $(c\gamma d \neq \emptyset)$. We have

$$u \in c\gamma d = \{c\}\overline{\gamma}\{d\} \subseteq \{c\}\Gamma\{d\} \subseteq L\Gamma A \subseteq A.$$

Since $u \le u \in c\gamma d$, we get $\{u\} \preceq c\gamma d$. Since $u \in A$ and $\{u\} \preceq c\gamma d$ where $c \in L$, $\gamma \in \Gamma$, $d \in A$, we have $u \in T$ and so T is a nonempty set. Moreover $T \subseteq A \subseteq L$. We prove that T is a left ideal of S. Then, since $T \subseteq L$, T is a left ideal of S and L is a minimal left ideal of S, we have $T = L$; then $A = L$ and the proof is complete.

The following assertions are satisfied:

(1) $S\Gamma T \subseteq T$. Indeed: Let $u \in S\Gamma T$. By [9; Remark 3.4], $u \in m\rho h$ for some $m \in S$, $\rho \in \Gamma$, $h \in T$. Since $h \in T$, we have $h \in A$ and $\{h\} \preceq k\gamma a$ for some $k \in L$, $\gamma \in \Gamma$, $a \in A$. Then, by Lemma 3(1), we have

$$u \in m\rho h = \{m\}\overline{\rho}\{h\} \preceq \{m\}\overline{\rho}(k\gamma a) = (m\rho k)\overline{\gamma}\{a\}.$$

Then there exists $v \in (m\rho k)\overline{\gamma}\{a\}$ such that $u \leq v$. Since $v \in (m\rho k)\overline{\gamma}\{a\}$, we have $v \in w\gamma a$ for some $w \in m\rho k$. Then we have

$$\begin{aligned} v \in w\gamma a &= \{w\}\overline{\gamma}\{a\} \subseteq (m\rho k)\overline{\gamma}\{a\} \subseteq (m\rho k)\Gamma\{a\} \text{ (by [9; Def. 3.3])} \\ &\subseteq (S\Gamma L)\Gamma A \subseteq L\Gamma A \subseteq A. \end{aligned}$$

On the other hand, $u \in S\Gamma T \subseteq S\Gamma A \subseteq S\Gamma L \subseteq L$ and so $u \in L$. Since $L \ni u \leq v \in A$ and A is a left ideal of L, we have $u \in A$. Since $v \in L\Gamma A$, we have $v \in t\omega b$ for some $t \in L$, $\omega \in \Gamma$, $b \in A$. Since $u \leq v \in t\omega b$, we have $\{u\} \preceq t\omega b$. Since $u \in A$ and $\{u\} \preceq t\omega b$, where $t \in L$, $\omega \in \Gamma$, $b \in A$, we have $u \in T$.

(2) Let $h \in T$ and $S \ni m \leq h$. Then $m \in T$. Indeed: Since $h \in T$, we have $h \in A$ and $\{h\} \preceq k\gamma a$ for some $k \in L$, $\gamma \in \Gamma$, $a \in A$. Since $h \in A$, we have $h \in L$. Since $S \ni m \leq h \in L$ and L is a left ideal of S, we have $m \in L$. Since $L \ni m \leq h \in A$ and A is a left ideal of L, we have $m \in A$. Since $\{h\} \preceq k\gamma a$, we have $h \leq u$ for some $u \in k\gamma a$. Since $m \leq u$ and $u \in k\gamma a$, we have $\{m\} \preceq k\gamma a$. We have $m \in A$ and $\{m\} \preceq k\gamma a$, where $k \in L$, $\gamma \in \Gamma$, $a \in A$ and so $m \in T$.

By (1) and (2), T is a left ideal of S.

$\Longleftarrow$. Let A be a left ideal of S such that $A \subseteq L$. Then $A = L$.
In fact, the following assertions are satisfied:

(1) $L\Gamma(L\Gamma A]_L \subseteq (L\Gamma A]_L$. Indeed: Let $x \in L\Gamma(L\Gamma A]_L$. Then $x \in y\gamma t$ for some $y \in L$, $\gamma \in \Gamma$, $t \in (L\Gamma A]_L$, $t \in L$, $t \leq z$ for some $z \in L\Gamma A$. Since $t \leq z$, we have $y\gamma t \preceq y\gamma z$ and since $x \in y\gamma t$, there exists $k \in y\gamma z$ such that $x \leq k$. We have $x \in y\gamma t \subseteq L\Gamma L \subseteq S\Gamma L \subseteq L$ and so $x \in L$. In addition, $x \leq k \in y\gamma z \subseteq L\Gamma(L\Gamma A) \subseteq (S\Gamma L)\Gamma A \subseteq L\Gamma A$ and so $x \in (L\Gamma A]$. We have $x \in L$ and $x \in (L\Gamma A]$ and so $x \in (L\Gamma A]_L$.

(2) If $x \in (L\Gamma A]_L$ and $L \ni y \leq x$, then $y \in (L\Gamma A]_L$. Indeed: Since $x \in (L\Gamma A]_L$, we have $x \in L$ and $x \leq z$ for some $z \in L\Gamma A$. We have $y \in L$ and $y \leq z$ for some $z \in L\Gamma A$ and so $y \in (L\Gamma A]_L$.

By (1) and (2), $(L\Gamma A]_L$ is a left ideal of L. Since L is left simple, we have

$$L = (L\Gamma A]_L \subseteq (S\Gamma A]_L \subseteq (A]_L \subseteq (A] = A \text{ (since } A \text{ is a left ideal of } S)$$

and so $A = L$. □

Definition 10 *Let S be an ordered* Γ*-hypersemigroup and R a right ideal of S. A nonempty subset* A *of* R *is called a right ideal of R if (1)* $A\Gamma R \subseteq A$ *and (2) if* $a \in A$ *and* $R \ni b \leq a$*, then* $b \in A$*.*

Definition 11 *Let S be an ordered* Γ*-hypersemigroup. A right ideal R of of S is called minimal (right ideal of S) if for every right ideal* A *of S such that* $A \subseteq R$*, we have* $A = R$*.*

A right ideal R *of an ordered* Γ*-hypersemigroup* S *is called right simple if for every right ideal* A *of* R*, we have* $A = R$*.*

The right analogue of Theorem 9 also holds and we have the following theorem.

Theorem 12 *Let S be an ordered Γ-hypersemigroup and R a right ideal of S. Then R is minimal right ideal of S if and only if R is a left simple.*

For the $\Rightarrow$-part it is enough to prove that the set

$$T := \{h \in A \mid \{h\} \preceq a\gamma k \text{ for some } a \in A, \gamma \in \Gamma, k \in R\}$$

is a right ideal of S and for the $\Leftarrow$-part that the set $(A\Gamma R]_R$ is a right ideal of R.

The definition of Γ-hypersemigroups [9] corrects the definitions of ordered Γ-hypersemigroups and related structures like the definition of the LA-Γ-hypersemigroup etc. (see [8]).

As there are still people interested in examples for ordered Γ-hypersemigroups, it might be noted the following: For a given example, a computer program to generate it and another one to check its validation is needed. As we have already said in [9], even for the much more simpler structure of that of ordered semigroup, it is impossible to write an example given by a table of multiplication and a figure by hand. For somebody who is no expert on the subject, it is very difficult to check the examples on ordered semigroups of order 5, for example, by hand if he does not know the Light's associativity test and its extended form for ordered semigroups.

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