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## Close-to-convexity of a class of harmonic mappings defined by a third-order differential inequality

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**Abstract:** In this paper, we consider a class of normalized harmonic functions in the unit disk satisfying a third-order differential inequality and we investigate several properties of this class such as close-to-convexity, coefficient bounds, growth estimates, sufficient coefficient condition, and convolution. Moreover, as an application, we construct harmonic polynomials involving Gaussian hypergeometric function which belong to the considered class. We also provide examples illustrating graphically with the help of Maple.

**Key words:** Harmonic, univalent, close-to-convex, hypergeometric function

### 1. Introduction

Let  $\mathcal{H}$  denote the class of complex valued harmonic functions  $f = h + \bar{g}$  defined in the open unit disk  $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ , and normalized by  $f(0) = f_z(0) - 1 = 0$ . Also, let  $\mathcal{H}^0 = \{f \in \mathcal{H} : f_{\bar{z}}(0) = 0\}$ . Each function  $f \in \mathcal{H}^0$  can be expressed as  $f = h + \bar{g}$ , where

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = \sum_{m=2}^{\infty} b_m z^m \quad (1.1)$$

are analytic in  $\mathcal{U}$ . We call  $h$  the analytic part and  $g$  the coanalytic part of  $f$ . A necessary and sufficient condition for  $f$  to be locally univalent and sense-preserving in  $\mathcal{U}$  is that  $|h'(z)| > |g'(z)|$  in  $\mathcal{U}$ . See [1, 3].

Denote by  $\mathcal{S}_H$  the class of functions  $f = h + \bar{g}$  that are harmonic, univalent, and sense-preserving in the unit disk  $\mathcal{U}$ . Furthermore, let  $\mathcal{S}_H^0 = \{f \in \mathcal{S}_H : f_{\bar{z}}(0) = 0\}$ . Note that, with  $g(z) = 0$ , the classical family  $\mathcal{S}$  of analytic univalent and normalized functions in  $\mathcal{U}$  is a subclass of  $\mathcal{S}_H^0$ , just as the family  $\mathcal{A}$  of analytic and normalized functions in  $\mathcal{U}$  is a subclass of  $\mathcal{H}^0$ . Let  $\mathcal{S}^*$ ,  $\mathcal{K}$  and  $\mathcal{C}$  be the subclasses of  $\mathcal{S}$  mapping  $\mathcal{U}$  onto starlike, convex, and close-to-convex domains, respectively, just as  $\mathcal{S}_H^{0,*}$ ,  $\mathcal{K}_H^0$ , and  $\mathcal{C}_H^0$  are the subclasses of  $\mathcal{S}_H^0$  mapping  $\mathcal{U}$  onto their respective domains.

In 2013, Ponnusamy et al. [15] introduced a class of functions  $f = h + \bar{g} \in \mathcal{H}^0$  satisfying the condition  $\operatorname{Re}[f_z(z)] > |f_{\bar{z}}(z)|$  for  $z \in \mathcal{U}$  and they proved that this class of functions are close-to-convex. Moreover, Li and Ponnusamy [12, 13] obtained univalence and convexity of the partial sums of the abovementioned class. Recently, Ghosh and Vasudevarao [5] defined a class of functions  $f = h + \bar{g} \in \mathcal{H}^0$  satisfying the condition

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$\operatorname{Re} \{h'(z) + \alpha zh''(z)\} > |g'(z) + \alpha zg''(z)|$  for  $z \in \mathcal{U}$  and they investigated coefficient bounds, growth estimates, convolution, and radius of convexity for the partial sums of members of their class. Other interesting studies of harmonic mappings that we have been inspired by in this work are [2, 9, 10, 14, 16].

Let  $\mathcal{R}_S(\lambda, \delta)$  denote a class of functions  $f \in \mathcal{A}$  such that

$$\operatorname{Re} \{f'(z) + \lambda zf''(z) + \delta z^2 f'''(z)\} > 0 \quad (\lambda \geq \delta \geq 0).$$

The class  $\mathcal{R}_S(\lambda, \delta)$  was studied by Rosihan et al. [18]. They obtained the form of  $f$  with a double integral operator by using usual subordination. Also, the class  $\mathcal{R}_S(\lambda, \frac{\lambda-1}{2})$  with  $\lambda \geq 1$  is a particular case of the class defined by Al-Refai [17].

Denote by  $\mathcal{R}_H^0(\lambda, \delta)$ , the class of functions  $f = h + \bar{g} \in \mathcal{H}^0$  and satisfy

$$\operatorname{Re}[h'(z) + \lambda zh''(z) + \delta z^2 h'''(z)] > |g'(z) + \lambda zg''(z) + \delta z^2 g'''(z)| \tag{1.2}$$

where  $\lambda \geq \delta \geq 0$ .

In this paper, we mainly deal with the functions  $f = h + \bar{g} \in \mathcal{H}^0$  of the class  $\mathcal{R}_H^0(\lambda, \delta)$  which is defined by the third-order differential inequality (1.2). In Section 2, we derive that the members of the class are close-to-convex and we obtain coefficient bounds, growth estimates, and sufficient coefficient condition of the class  $\mathcal{R}_H^0(\lambda, \delta)$ . We also provide examples, illustrating graphically with the help of Maple. In Section 3, we show that this class is closed under convex combination and convolution of its members. In Section 4, considering harmonic mappings involving Gaussian hypergeometric function, we examine sufficient conditions for such mappings to be in this class. Moreover, we provide an example of harmonic polynomial which belongs to the class  $\mathcal{R}_H^0(\lambda, \delta)$ .

### 2. Close-to-convexity, coefficient bounds, growth estimates

First, we give a result of Clunie and Sheil-Small [1] which derives a sufficient condition for  $f \in \mathcal{H}$  to be close-to-convex.

**Lemma 2.1** *Suppose  $h$  and  $g$  are analytic in  $\mathcal{U}$  with  $|g'(0)| < |h'(0)|$  and  $F_\mu = h + \mu g$  is close to convex for each  $\mu$  ( $|\mu| = 1$ ), then  $f = h + \bar{g}$  is close to convex in  $\mathcal{U}$ .*

**Theorem 2.2** *The harmonic mapping  $f = h + \bar{g} \in \mathcal{R}_H^0(\lambda, \delta)$  if and only if  $F_\mu = h + \mu g \in \mathcal{R}_S(\lambda, \delta)$  for each  $\mu$  ( $|\mu| = 1$ ).*

**Proof** Suppose  $f = h + \bar{g} \in \mathcal{R}_H^0(\lambda, \delta)$ . For each  $|\mu| = 1$ ,

$$\begin{aligned} & \operatorname{Re} \{F'_\mu(z) + \lambda z F''_\mu(z) + \delta z^2 F'''_\mu(z)\} \\ &= \operatorname{Re} \{h'(z) + \lambda zh''(z) + \delta z^2 h'''(z) + \mu (g'(z) + \lambda zg''(z) + \delta z^2 g'''(z))\} \\ &> \operatorname{Re} \{h'(z) + \lambda zh''(z) + \delta z^2 h'''(z)\} - |g'(z) + \lambda zg''(z) + \delta z^2 g'''(z)| > 0 \quad (z \in \mathcal{U}). \end{aligned}$$

Thus,  $F_\mu \in \mathcal{R}_S(\lambda, \delta)$  for each  $\mu$  ( $|\mu| = 1$ ). Conversely, let  $F_\mu = h + \mu g \in \mathcal{R}_S(\lambda, \delta)$  then

$$\operatorname{Re} \{h'(z) + \lambda zh''(z) + \delta z^2 h'''(z)\} > \operatorname{Re} \{-\mu (g'(z) + \lambda zg''(z) + \delta z^2 g'''(z))\} \quad (z \in \mathcal{U}).$$

Suitably choosing  $\mu$  ( $|\mu| = 1$ ), we obtain

$$\operatorname{Re} \{h'(z) + \lambda zh''(z) + \delta z^2 h'''(z)\} > |g'(z) + \lambda zg''(z) + \delta z^2 g'''(z)| \quad (z \in \mathcal{U});$$

hence,  $F_\mu \in \mathcal{R}_S(\lambda, \delta)$ . □

**Lemma 2.3** (Jack-Miller-Mocanu Lemma [7, 8]) Let  $\varphi$  defined by  $\varphi(z) = C_m z^m + C_{m+1} z^{m+1} + \dots$  be analytic in  $\mathcal{U}$ , with  $C_m \neq 0$ , and let  $z_0 \neq 0$ ,  $z_0 \in \mathcal{U}$  be a point of  $\mathcal{U}$  such that

$$|\varphi(z_0)| = \max_{|z| \leq |z_0|} |\varphi(z)|$$

then there is a real number  $n$ ,  $n \geq m \geq 1$ , such that

$$\frac{z_0 \varphi'(z_0)}{\varphi(z_0)} = n \quad \text{and} \quad \operatorname{Re} \left\{ 1 + \frac{z_0 \varphi''(z_0)}{\varphi'(z_0)} \right\} \geq n.$$

**Lemma 2.4** If  $F \in \mathcal{R}_S(\lambda, \delta)$  with  $\lambda \geq \delta \geq 0$  then  $\operatorname{Re}\{F'(z)\} > 0$ ; hence,  $F$  is close-to-convex in  $\mathcal{U}$ .

**Proof** Suppose  $F \in \mathcal{R}_S(\lambda, \delta)$  and  $F'(z) + \lambda z F''(z) + \delta z^2 F'''(z) = \Psi(z)$ . Then  $\operatorname{Re}\{\Psi(z)\} > 0$  for  $z \in \mathcal{U}$ . Consider an analytic function  $\varphi$  in  $\mathcal{U}$  with  $\varphi(0) = 0$  and

$$F'(z) = \frac{1 + \varphi(z)}{1 - \varphi(z)}, \quad \varphi(z) \neq 1.$$

We need to prove that  $|\varphi(z)| < 1$  for all  $z \in \mathcal{U}$ . Then we have

$$\begin{aligned} \Psi(z) &= F'(z) + \lambda z F''(z) + \delta z^2 F'''(z) \\ &= \frac{1 + \varphi(z)}{1 - \varphi(z)} + \frac{2\lambda z \varphi'(z)}{(1 - \varphi(z))^2} + \frac{2\delta z \varphi'(z)}{(1 - \varphi(z))^2} \frac{z \varphi''(z)}{\varphi'(z)} + \frac{4\delta (z \varphi'(z))^2}{(1 - \varphi(z))^3}. \end{aligned}$$

Since  $\varphi$  is analytic in  $\mathcal{U}$  and  $\varphi(0) = 0$ , if there is  $z_0 \in \mathcal{U}$  such that

$$\max_{|z| \leq |z_0|} |\varphi(z)| = |\varphi(z_0)| = 1,$$

then by Lemma 2.3, we can write

$$\varphi(z_0) = e^{i\theta}, \quad z_0 \varphi'(z_0) = n \varphi(z_0) = n e^{i\theta}, \quad (n \geq 1, 0 < \theta < 2\pi).$$

and

$$\operatorname{Re} \left\{ \frac{z_0 \varphi''(z_0)}{\varphi'(z_0)} \right\} \geq n - 1.$$

For such a point  $z_0 \in \mathcal{U}$ , we obtain

$$\begin{aligned} \operatorname{Re}\{\Psi(z_0)\} &= \operatorname{Re} \left\{ \frac{1 + e^{i\theta}}{1 - e^{i\theta}} + \frac{2\lambda n e^{i\theta}}{(1 - e^{i\theta})^2} + \frac{2\delta n e^{i\theta}}{(1 - e^{i\theta})^2} \frac{z_0 \varphi''(z_0)}{\varphi'(z_0)} + \frac{4\delta (n e^{i\theta})^2}{(1 - e^{i\theta})^3} \right\} \\ &= \frac{-\lambda n}{1 - \cos \theta} - \frac{\delta n}{1 - \cos \theta} \operatorname{Re} \left\{ \frac{z_0 \varphi''(z_0)}{\varphi'(z_0)} \right\} + \frac{\delta n^2}{1 - \cos \theta} \\ &\leq \frac{(\delta - \lambda)n}{1 - \cos \theta} \leq \frac{(\delta - \lambda)n}{2} \leq 0 \end{aligned}$$

which contradicts the hypothesis. Hence, there is no  $z_0 \in \mathcal{U}$  such that  $|\varphi(z_0)| = 1$ , which means that  $|\varphi(z)| < 1$  for all  $z \in \mathcal{U}$ . Therefore, we obtain that  $\operatorname{Re}\{F'(z)\} > 0$ .  $\square$

**Theorem 2.5** *The functions in the class  $\mathcal{R}_H^0(\lambda, \delta)$  are close-to-convex in  $\mathcal{U}$ .*

**Proof** Referring to Lemma 2.4, we derive that functions  $F_\mu = h + \mu g \in \mathcal{R}_S(\lambda, \delta)$  are close-to-convex in  $\mathcal{U}$  for each  $\mu (|\mu| = 1)$ . Now in view of Lemma 2.1 and Theorem 2.2, we obtain that functions in  $\mathcal{R}_H^0(\lambda, \delta)$  are close-to-convex in  $\mathcal{U}$ .  $\square$

**Theorem 2.6** *Let  $f = h + \bar{g} \in \mathcal{R}_H^0(\lambda, \delta)$  then for  $m \geq 2$ ,*

$$|b_m| \leq \frac{1}{m[1 + (m - 1)(\lambda + \delta(m - 2))]} \tag{2.1}$$

*The result is sharp and the equality is held for the function  $f(z) = z + \frac{1}{m[1+(m-1)(\lambda+\delta(m-2))]} \bar{z}^m$ .*

**Proof** Suppose that  $f = h + \bar{g} \in \mathcal{R}_H^0(\lambda, \delta)$ . Using the series representation of  $g(z)$ , we derive

$$\begin{aligned} & r^{m-1} [m + \lambda m(m - 1) + \delta m(m - 1)(m - 2)] |b_m| \\ & \leq \frac{1}{2\pi} \int_0^{2\pi} |g'(re^{i\theta}) + \lambda r e^{i\theta} g''(re^{i\theta}) + \delta r^2 e^{2i\theta} g'''(re^{i\theta})| d\theta \\ & < \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \{h'(re^{i\theta}) + \lambda r e^{i\theta} h''(re^{i\theta}) + \delta r^2 e^{2i\theta} h'''(re^{i\theta})\} d\theta \\ & = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \left\{ 1 + \sum_{m=2}^{\infty} [m + \lambda m(m - 1) + \delta m(m - 1)(m - 2)] a_m r^{m-1} e^{i(m-1)\theta} \right\} d\theta \\ & = 1. \end{aligned}$$

The desired bound obtained by allowing  $r \rightarrow 1^-$ . Moreover, it is easy to verify that the equality is held for the function  $f(z) = z + \frac{1}{m[1+(m-1)(\lambda+\delta(m-2))]} \bar{z}^m$ .  $\square$

**Theorem 2.7** *Let  $f = h + \bar{g} \in \mathcal{R}_H^0(\lambda, \delta)$ . Then for  $m \geq 2$ , we have*

$$|a_m| + |b_m| \leq \frac{2}{m[1 + (m - 1)(\lambda + \delta(m - 2))]}$$

*This result is sharp and the equality is held for the function*

$$f(z) = z + \sum_{m=2}^{\infty} \frac{2}{m[1+(m-1)(\lambda+\delta(m-2))]} \bar{z}^m.$$

**Proof** Suppose that  $f = h + \bar{g} \in \mathcal{R}_H^0(\lambda, \delta)$ , then from Theorem 2.2,  $F_\mu = h + \mu g \in \mathcal{R}_S(\lambda, \delta)$  for  $\mu (|\mu| = 1)$ . Thus, for each  $|\mu| = 1$ , we have

$$\operatorname{Re} \{(h + \mu g)' + \lambda z(h + \mu g)'' + \delta z^2(h + \mu g)'''\} > 0$$

for  $z \in \mathcal{U}$ . Then there exists an analytic function  $p$  of the form  $p(z) = 1 + \sum_{m=1}^{\infty} p_m z^m$ , with  $\text{Re}p(z) > 0$  in  $\mathcal{U}$  such that

$$h'(z) + \lambda z h''(z) + \delta z^2 h'''(z) + \mu(g'(z) + \lambda z g''(z) + \delta z^2 g'''(z)) = p(z). \tag{2.2}$$

Comparing coefficients on both sides of (2.2) we have

$$m[1 + (m - 1)(\lambda + \delta(m - 2))] (a_m + \mu b_m) = p_{m-1} \text{ for } m \geq 2.$$

Since  $|p_m| \leq 2$  for  $m \geq 1$ , and  $\mu (|\mu| = 1)$  is arbitrary, proof is complete. The function  $f(z) = z + \sum_{m=2}^{\infty} \frac{2}{m[1+(m-1)(\lambda+\delta(m-2))]} z^m$  shows that the result is sharp.  $\square$

The following result gives a sufficient condition for a function to be in the class  $\mathcal{R}_H^0(\lambda, \delta)$ .

**Theorem 2.8** *Let  $f = h + \bar{g} \in \mathcal{H}^0$  with*

$$\sum_{m=2}^{\infty} m \{1 + (m - 1) [\lambda + \delta (m - 2)]\} (|a_m| + |b_m|) \leq 1, \tag{2.3}$$

*then  $f \in \mathcal{R}_H^0(\lambda, \delta)$ .*

**Proof** Suppose that  $f = h + \bar{g} \in \mathcal{H}^0$ . Then using (2.3),

$$\begin{aligned} & \text{Re} \{h'(z) + \lambda z h''(z) + \delta z^2 h'''(z)\} \\ &= \text{Re} \left\{ 1 + \sum_{m=2}^{\infty} m \{1 + (m - 1) [\lambda + \delta (m - 2)]\} a_m z^{m-1} \right\} \\ &> 1 - \sum_{m=2}^{\infty} m \{1 + (m - 1) [\lambda + \delta (m - 2)]\} |a_m| \\ &\geq \sum_{m=2}^{\infty} m \{1 + (m - 1) [\lambda + \delta (m - 2)]\} |b_m| \\ &> \left| \sum_{m=2}^{\infty} m \{1 + (m - 1) [\lambda + \delta (m - 2)]\} b_m z^{m-1} \right| \\ &= |g'(z) + \lambda z g''(z) + \delta z^2 g'''(z)|. \end{aligned}$$

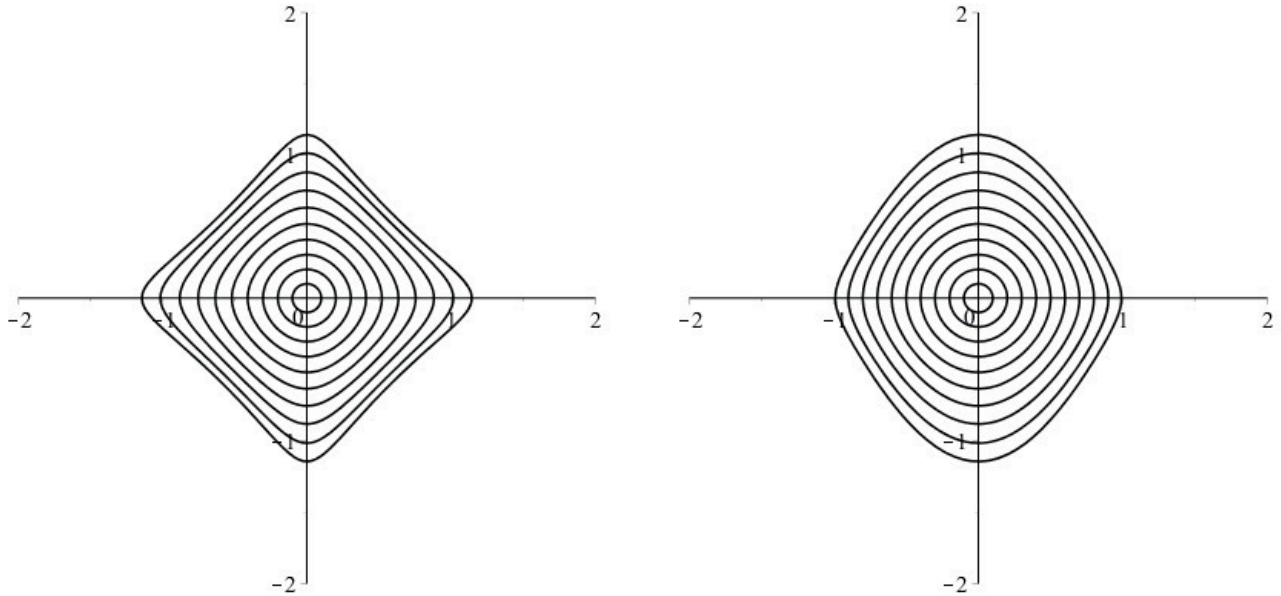
Hence,  $f \in \mathcal{R}_H^0(\lambda, \delta)$ .  $\square$

**Corollary 2.9** *Let  $f = h + \bar{g} \in \mathcal{H}^0$ . If*

$$\sum_{m=2}^{\infty} m^2 [3 - m + \lambda (m - 1)] (|a_m| + |b_m|) \leq 2,$$

*then  $f \in \mathcal{R}_H^0(\lambda, \frac{\lambda-1}{2})$  with  $\lambda \geq 1$ .*

**Example 2.10** (i) If we set  $\lambda = 0.5$  and  $\delta = 0.05$ , in view of Theorem 2.8, the harmonic polynomial  $f_1(z) = z + 0.15\bar{z}^3$  and  $f_2(z) = z - 0.079z^3 + 0.079\bar{z}^3$  belong to  $\mathcal{R}_H^0(0.5, 0.05)$ . The images of concentric circles inside the unit disk  $\mathcal{U}$  under  $f_1$  and  $f_2$  are shown in Figures 1a and 1b.



**Figure 1.** (a) Image of  $f_1(\mathcal{U})$ , (b) image of  $f_2(\mathcal{U})$ .

(ii) Let  $\lambda = 3$  and  $\delta = 1$ , in view of Theorem 2.8, the harmonic polynomial  $f_3(z) = z - \frac{1}{16}\bar{z}^2 + \frac{1}{54}\bar{z}^3$  belong to  $\mathcal{R}_H^0(3, 1)$ . The images of concentric circles inside the unit disk  $\mathcal{U}$  under  $f_3$  is shown in Figure 2.

**Theorem 2.11** Let  $f = h + \bar{g} \in \mathcal{R}_H^0(\lambda, \delta)$  with  $\lambda \geq \delta \geq 0$ . Then

$$|z| + 2 \sum_{m=2}^{\infty} \frac{(-1)^{m-1} |z|^m}{\delta m^3 + (\lambda - 3\delta)m^2 + (1 - \lambda + 2\delta)m} \leq |f(z)|,$$

$$|f(z)| \leq |z| + 2 \sum_{m=2}^{\infty} \frac{|z|^m}{\delta m^3 + (\lambda - 3\delta)m^2 + (1 - \lambda + 2\delta)m}.$$

Inequalities are sharp for the function  $f(z) = z + \frac{1}{m[1+(m-1)(\lambda+\delta(m-2))]} \bar{z}^m$ .

**Proof** Let  $f = h + \bar{g} \in \mathcal{R}_H^0(\lambda, \delta)$ . Then using Theorem 2.2,  $F_\mu \in \mathcal{R}_S(\lambda, \delta)$  and for each  $|\mu| = 1$  we have  $\text{Re}\{\Phi(z)\} > 0$  where

$$\Phi(z) = F'_\mu(z) + \lambda z F''_\mu(z) + \delta z^2 F'''_\mu(z).$$

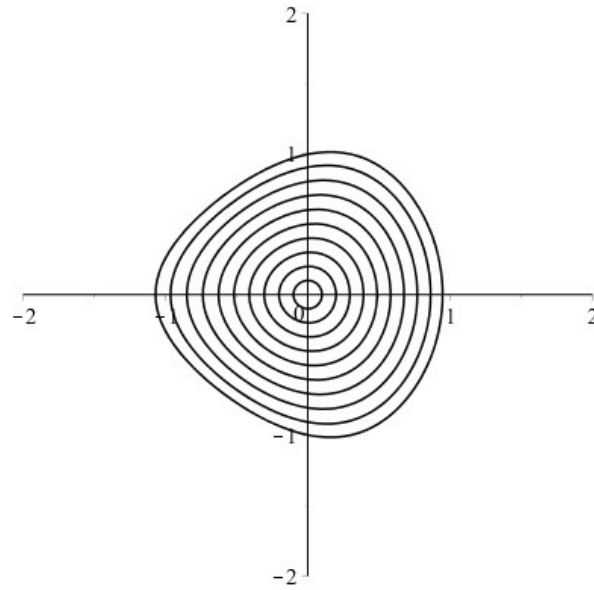


Figure 2. Image of  $f_3(\mathcal{U})$ .

Then using the method of Rosihan et al. [18, Sect. 2], we have

$$\begin{aligned}
 \Phi(z) &= F'_\mu(z) + (\rho(1 + \eta) + \eta)zF''_\mu(z) + \rho\eta z^2F'''_\mu(z) \\
 &= \eta z^{1-\frac{1}{\eta}} \left( \rho z^{1+\frac{1}{\eta}} F''_\mu(z) + z^{\frac{1}{\eta}} F'_\mu(z) \right)' \\
 &= \eta z^{1-\frac{1}{\eta}} \left[ \rho z^{1+\frac{1}{\eta}-\frac{1}{\rho}} (z^{\frac{1}{\rho}} F'_\mu(z))' \right]', \tag{2.4}
 \end{aligned}$$

where  $\rho = \frac{(\lambda-\delta)-\sqrt{(\lambda-\delta)^2-4\delta}}{2}$ ,  $\eta + \rho = \lambda - \delta$ ,  $\rho\eta = \delta$ ,  $\text{Re}\rho \geq 0$ ,  $\text{Re}\eta \geq 0$ .

Then integrating (2.4) gives

$$\rho z^{1+\frac{1}{\eta}-\frac{1}{\rho}} (z^{\frac{1}{\rho}} F'_\mu(z))' = \frac{1}{\eta} \int_0^z \vartheta^{\frac{1}{\eta}-1} \Phi(\vartheta) d\vartheta.$$

Making substitution  $\vartheta = u^\eta z$  and simplifying yields

$$\left( z^{\frac{1}{\rho}} F'_\mu(z) \right)' = \frac{1}{\rho} z^{\frac{1}{\rho}-1} \int_0^1 \Phi(u^\eta z) du. \tag{2.5}$$

Now, integrating (2.5) and making substitution  $\xi = v^\rho z$  gets

$$z^{\frac{1}{\rho}} F'_\mu(z) = \frac{1}{\rho} \int_0^z \xi^{\frac{1}{\rho}-1} \int_0^1 \Phi(u^\eta \xi) du d\xi = z^{\frac{1}{\rho}} \int_0^1 \int_0^1 \Phi(u^\eta v^\rho z) dudv$$



which simplifies to

$$F'_\mu(z) = \int_0^1 \int_0^1 \Phi(u^\eta v^\rho z) dudv. \tag{2.6}$$

On the other hand, since  $\text{Re}\{\Phi(z)\} > 0$  then  $\Phi(z) \prec \frac{1+z}{1-z}$  where  $\prec$  denotes the subordination [7, 8].

Let

$$\varphi(z) = 1 + \sum_{m=1}^\infty \frac{z^m}{(1 + \eta m)(1 + \rho m)} = \int_0^1 \int_0^1 \frac{dvdu}{1 - u^\eta v^\rho z}$$

and  $Q(z) = \frac{1+z}{1-z} = \sum_{m=0}^\infty 2z^m$ . Then, from (2.6) we have

$$\begin{aligned} F'_\mu(z) &\prec (\varphi * Q)(z) \\ &= \left(1 + \sum_{m=1}^\infty \frac{z^m}{(1 + \eta m)(1 + \rho m)}\right) * \left(1 + \sum_{m=1}^\infty 2z^m\right) \\ &= 1 + \sum_{m=1}^\infty \frac{2}{1 + (\lambda - \delta)m + \delta m^2} z^m. \end{aligned}$$

Since

$$\begin{aligned} |F'_\mu(z)| &= |h'(z) + \mu g'(z)| \\ &\leq 1 + 2 \sum_{m=1}^\infty \frac{|z|^m}{1 + (\lambda - \delta)m + \delta m^2} \end{aligned}$$

and

$$\begin{aligned} |F'_\mu(z)| &= |h'(z) + \mu g'(z)| \\ &\geq 1 + 2 \sum_{m=1}^\infty \frac{(-1)^m |z|^m}{1 + (\lambda - \delta)m + \delta m^2}, \end{aligned}$$

in particular we have

$$|h'(z)| + |g'(z)| \leq 1 + 2 \sum_{m=1}^\infty \frac{|z|^m}{1 + (\lambda - \delta)m + \delta m^2}$$

and

$$|h'(z)| - |g'(z)| \geq 1 + 2 \sum_{m=1}^\infty \frac{(-1)^m |z|^m}{1 + (\lambda - \delta)m + \delta m^2}.$$

Let  $\Gamma$  be the radial segment from 0 to  $z$ , then

$$\begin{aligned} |f(z)| &= \left| \int_{\Gamma} \frac{\partial f}{\partial \varsigma} d\varsigma + \frac{\partial f}{\partial \bar{\varsigma}} d\bar{\varsigma} \right| \leq \int_{\Gamma} (|h'(\varsigma)| + |g'(\varsigma)|) |d\varsigma| \\ &\leq \int_0^{|z|} \left( 1 + 2 \sum_{m=1}^{\infty} \frac{|s|^m}{1 + (\lambda - \delta)m + \delta m^2} \right) ds \\ &= |z| + 2 \sum_{m=1}^{\infty} \frac{|z|^{m+1}}{(m+1)[1 + (\lambda - \delta)m + \delta m^2]} \\ &= |z| + 2 \sum_{m=2}^{\infty} \frac{|z|^m}{\delta m^3 + (\lambda - 3\delta)m^2 + (1 - \lambda + 2\delta)m} \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq \int_{\Gamma} (|h'(\varsigma)| - |g'(\varsigma)|) |d\varsigma| \\ &\geq \int_0^{|z|} \left( 1 + 2 \sum_{m=1}^{\infty} \frac{(-1)^m |s|^m}{1 + (\lambda - \delta)m + \delta m^2} ds \right) \\ &= |z| + 2 \sum_{m=2}^{\infty} \frac{(-1)^{m-1} |z|^m}{\delta m^3 + (\lambda - 3\delta)m^2 + (1 - \lambda + 2\delta)m}. \end{aligned}$$

□

**Remark 2.12** (i) Our results, obtained in this section, yield the results of the classes  $\mathcal{R}_H^0(0, 0)$  and  $\mathcal{R}_H^0(\lambda, 0)$  which are defined and studied in [12, 13, 15] and [5], respectively.

(ii) The class  $\mathcal{R}_S(\lambda, \frac{\lambda-1}{2})$  with  $\lambda \geq 1$  is a particular case of the class defined by Al-Refai [17] and our results yield the results of [17].

### 3. Convex combinations and convolutions

In this section, we prove that the class  $\mathcal{R}_H^0(\lambda, \delta)$  is closed under convex combinations and convolutions of its members.

**Theorem 3.1** The class  $\mathcal{R}_H^0(\lambda, \delta)$  is closed under convex combinations.

**Proof** Suppose  $f_i = h_i + \bar{g}_i \in \mathcal{R}_H^0(\lambda, \delta)$  for  $i = 1, 2, \dots, m$  and  $\sum_{i=1}^m t_i = 1$  ( $0 \leq t_i \leq 1$ ). The convex combination of functions  $f_i$  ( $i = 1, 2, \dots, m$ ) may be written as

$$f(z) = \sum_{i=1}^m t_i f_i(z) = h(z) + \overline{g(z)},$$

where

$$h(z) = \sum_{i=1}^m t_i h_i(z) \quad \text{and} \quad g(z) = \sum_{i=1}^m t_i g_i(z).$$

Then both  $h$  and  $g$  are analytic in  $\mathcal{U}$  with  $h(0) = g(0) = h'(0) - 1 = g'(0) = 0$  and

$$\begin{aligned} \operatorname{Re}\{h'(z) + \lambda z h''(z) + \delta z^2 h'''(z)\} &= \operatorname{Re}\left\{\sum_{i=1}^m t_i (h'_i(z) + \lambda z h''_i(z) + \delta z^2 h'''_i(z))\right\} \\ &> \sum_{i=1}^m t_i |g'_i(z) + \lambda z g''_i(z) + \delta z^2 g'''_i(z)| \\ &\geq |g'(z) + \lambda z g''(z) + \delta z^2 g'''(z)| \end{aligned}$$

showing that  $f \in \mathcal{R}_H^0(\lambda, \delta)$ . □

A sequence  $\{q_m\}_{m=0}^\infty$  of nonnegative real numbers is said to be a convex null sequence, if  $q_m \rightarrow 0$  as  $m \rightarrow \infty$ , and  $q_0 - q_1 \geq q_1 - q_2 \geq q_2 - q_3 \geq \dots \geq q_{m-1} - q_m \geq \dots \geq 0$ . To prove results for convolution, we shall need the following Lemma 3.2 and Lemma 3.3.

**Lemma 3.2** [4] *If  $\{q_m\}_{m=0}^\infty$  is a convex null sequence, then function*

$$q(z) = \frac{q_0}{2} + \sum_{m=1}^\infty q_m z^m$$

*is analytic and  $\operatorname{Re}\{q(z)\} > 0$  in  $\mathcal{U}$ .*

**Lemma 3.3** [19] *Let the function  $p$  be analytic in  $\mathcal{U}$  with  $p(0) = 1$  and  $\operatorname{Re}\{p(z)\} > 1/2$  in  $\mathcal{U}$ . Then for any analytic function  $F$  in  $\mathcal{U}$ , the function  $p * F$  takes values in the convex hull of the image of  $\mathcal{U}$  under  $F$ .*

**Lemma 3.4** *Let  $F \in \mathcal{R}_S(\lambda, \delta)$ , then  $\operatorname{Re}\left\{\frac{F(z)}{z}\right\} > \frac{1}{2}$ .*

**Proof** Suppose  $F \in \mathcal{R}_S(\lambda, \delta)$  be given by  $F(z) = z + \sum_{m=2}^\infty A_m z^m$ , then

$$\operatorname{Re}\left\{1 + \sum_{m=2}^\infty m [1 + (m-1)(\lambda + \delta(m-2))] A_m z^{m-1}\right\} > 0 \quad (z \in \mathcal{U}),$$

which is equivalent to  $\operatorname{Re}\{p(z)\} > \frac{1}{2}$  in  $\mathcal{U}$ , where

$$p(z) = 1 + \frac{1}{2} \sum_{m=2}^\infty m [1 + (m-1)(\lambda + \delta(m-2))] A_m z^{m-1}.$$

Now consider a sequence  $\{q_m\}_{m=0}^\infty$  defined by

$$q_0 = 1 \quad \text{and} \quad q_{m-1} = \frac{2}{m [1 + (m-1)(\lambda + \delta(m-2))]} \quad \text{for } m \geq 2.$$

It can be easily seen that the sequence  $\{q_m\}_{m=0}^\infty$  is convex null sequence and using Lemma 3.2, the function

$$q(z) = 1 + \sum_{m=2}^\infty \frac{2}{m [1 + (m - 1) (\lambda + \delta(m - 2))]} z^{m-1}$$

is analytic and  $\operatorname{Re}\{q(z)\} > \frac{1}{2}$  in  $\mathcal{U}$ . Writing

$$\frac{F(z)}{z} = p(z) * \left( 1 + \sum_{m=2}^\infty \frac{2}{m [1 + (m - 1) (\lambda + \delta(m - 2))]} z^{m-1} \right),$$

and making use of Lemma 3.3 gives that  $\operatorname{Re} \left\{ \frac{F(z)}{z} \right\} > \frac{1}{2}$  for  $z \in \mathcal{U}$ . □

**Lemma 3.5** *Let  $F_i \in \mathcal{R}_S(\lambda, \delta)$  for  $i = 1, 2$ . Then  $F_1 * F_2 \in \mathcal{R}_S(\lambda, \delta)$ .*

**Proof** Suppose  $F_1(z) = z + \sum_{m=2}^\infty A_m z^m$  and  $F_2(z) = z + \sum_{m=2}^\infty B_m z^m$ . Then the convolution of  $F_1(z)$  and  $F_2(z)$  is defined by

$$F(z) = (F_1 * F_2)(z) = z + \sum_{m=2}^\infty A_m B_m z^m.$$

Since  $F'(z) = F_1'(z) * \frac{F_2(z)}{z}$ ,  $zF''(z) = zF_1''(z) * \frac{F_2(z)}{z}$  and  $z^2F'''(z) = z^2F_1'''(z) * \frac{F_2(z)}{z}$ , then we have

$$F'(z) + \lambda zF''(z) + \delta z^2F'''(z) = (F_1'(z) + \lambda zF_1''(z) + \delta z^2F_1'''(z)) * \frac{F_2(z)}{z}. \tag{3.1}$$

Since  $F_1 \in \mathcal{R}_S(\lambda, \delta)$ ,

$$\operatorname{Re} \{ F_1'(z) + \lambda zF_1''(z) + \delta z^2F_1'''(z) \} > 0 \quad (z \in \mathcal{U})$$

and using Lemma 3.4,  $\operatorname{Re} \left\{ \frac{F_2(z)}{z} \right\} > \frac{1}{2}$  in  $\mathcal{U}$ . Now making use of Lemma 3.3 to (3.1) yields  $\operatorname{Re} (F'(z) + \lambda zF''(z) + \delta z^2F'''(z)) > 0$  in  $\mathcal{U}$ . Thus,  $F = F_1 * F_2 \in \mathcal{R}_S(\lambda, \delta)$ . □

Now using Lemma 3.5, we prove that the class  $\mathcal{R}_H^0(\lambda, \delta)$  is closed under convolutions of its members.

**Theorem 3.6** *Let  $f_i \in \mathcal{R}_H^0(\lambda, \delta)$  for  $i = 1, 2$ . Then  $f_1 * f_2 \in \mathcal{R}_H^0(\lambda, \delta)$ .*

**Proof** Suppose  $f_i = h_i + \overline{g_i} \in \mathcal{R}_H^0(\lambda, \delta)$  ( $i = 1, 2$ ). Then the convolution of  $f_1$  and  $f_2$  is defined as  $f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2}$ . In order to prove that  $f_1 * f_2 \in \mathcal{R}_H^0(\lambda, \delta)$ , we need to prove that  $F_\mu = h_1 * h_2 + \mu(g_1 * g_2) \in \mathcal{R}_S(\lambda, \delta)$  for each  $\mu(|\mu| = 1)$ . By Lemma 3.5, the class  $\mathcal{R}_S(\lambda, \delta)$  is closed under convolutions for each  $\mu(|\mu| = 1)$ ,  $h_i + \mu g_i \in \mathcal{R}_S(\lambda, \delta)$  for  $i = 1, 2$ . Then both  $F_1$  and  $F_2$  given by

$$F_1 = (h_1 - g_1) * (h_2 - \mu g_2) \quad \text{and} \quad F_2 = (h_1 + g_1) * (h_2 + \mu g_2),$$

belong to  $\mathcal{R}_S(\lambda, \delta)$ . Since  $\mathcal{R}_S(\lambda, \delta)$  is closed under convex combinations, then the function

$$F = \frac{1}{2}(F_1 + F_2) = h_1 * h_2 + \mu(g_1 * g_2)$$

belongs to  $\mathcal{R}_S(\lambda, \delta)$ . Hence,  $\mathcal{R}_H^0(\lambda, \delta)$  is closed under convolution. □

Now we consider the Hadamard product of a harmonic function with an analytic function which was defined by Goodloe [6] as

$$f \tilde{*} \psi = h * \psi + \overline{g * \psi},$$

where  $f = h + \bar{g}$  is harmonic function and  $\psi$  is an analytic function in  $\mathcal{U}$ .

**Theorem 3.7** *Let  $f \in \mathcal{R}_H^0(\lambda, \delta)$  and  $\psi \in \mathcal{A}$  be such that  $\operatorname{Re}\left(\frac{\psi(z)}{z}\right) > \frac{1}{2}$  for  $z \in \mathcal{U}$ , then  $f \tilde{*} \psi \in \mathcal{R}_H^0(\lambda, \delta)$ .*

**Proof** Suppose that  $f = h + \bar{g} \in \mathcal{R}_H^0(\lambda, \delta)$ , then  $F_\mu = h + \mu g \in \mathcal{R}_S(\lambda, \delta)$  for each  $\mu (|\mu| = 1)$ . By Theorem 2.2, in order to show that  $f \tilde{*} \psi \in \mathcal{R}_H^0(\lambda, \delta)$ , we need to show that  $G = h * \psi + \mu(g * \psi) \in \mathcal{R}_S(\lambda, \delta)$  for each  $\mu (|\mu| = 1)$ . Write  $G$  as  $G = F_\mu * \psi$ , and

$$G'(z) + \lambda z G''(z) + \delta z^2 G'''(z) = (F'_\mu(z) + \lambda z F''_\mu(z) + \delta z^2 F'''_\mu(z)) * \frac{\psi(z)}{z}.$$

Since  $\operatorname{Re}\left(\frac{\psi(z)}{z}\right) > \frac{1}{2}$  and  $\operatorname{Re}\{F'_\mu(z) + \lambda z F''_\mu(z) + \delta z^2 F'''_\mu(z)\} > 0$  in  $\mathcal{U}$ , Lemma 3.3 proves that  $G \in \mathcal{R}_S(\lambda, \delta)$ . □

**Corollary 3.8** *Let  $f \in \mathcal{R}_H^0(\lambda, \delta)$  and  $\psi \in \mathcal{K}$ , then  $f \tilde{*} \psi \in \mathcal{R}_H^0(\lambda, \delta)$ .*

**Proof** Suppose  $\psi \in \mathcal{K}$ , then  $\operatorname{Re}\left(\frac{\psi(z)}{z}\right) > \frac{1}{2}$  for  $z \in \mathcal{U}$ . Theorem 3.7 concludes that  $f \tilde{*} \psi \in \mathcal{R}_H^0(\lambda, \delta)$ . □

#### 4. Applications

In this section, we consider the harmonic mappings whose coanalytic part involves the Gaussian hypergeometric function  ${}_2F_1(a, b, c; z)$ , which is defined by

$${}_2F_1(a, b, c; z) = F(a, b, c; z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} z^m \quad (z \in \mathcal{U}), \tag{4.1}$$

where  $a, b, c \in \mathbb{C}, c \neq 0, -1, -2, \dots$  and  $(a)_m$  is Pochhammer Symbol defined by  $(a)_m = a(a+1)(a+2)\dots(a+m-1)$  and  $(a)_0 = 1$  for  $m \in \mathbb{N}$ . The series (4.1) is absolutely convergent in  $\mathcal{U}$ . Moreover, if  $\operatorname{Re}(c - a - b) > 0$ , then the series (4.1) is convergent in  $|z| \leq 1$ . Furthermore, for  $z = 1$ , we have the following well-known Gauss formula [20]

$$F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} < \infty. \tag{4.2}$$

To prove our result we need the following lemma.

**Lemma 4.1** [11] *Let  $a, b \in \mathbb{C} \setminus \{0\}$ ,  $c > 0$ . Then the following holds:*

(i) *For  $a, b > 0$ ,  $c > a + b + 1$ ,*

$$\sum_{m=0}^{\infty} (m + 1) \frac{(a)_m (b)_m}{(c)_m (1)_m} = \frac{\Gamma(c)\Gamma(c - a - b - 1)}{\Gamma(c - a)\Gamma(c - b)} (ab + c - a - b - 1).$$

(ii) For  $a, b > 0, c > a + b + 2,$

$$\sum_{m=0}^{\infty} (m + 1)^2 \frac{(a)_m (b)_m}{(c)_m (1)_m} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left( \frac{(a)_2 (b)_2}{(c-a-b-2)_2} + \frac{3ab}{c-a-b-1} + 1 \right).$$

(iii) For  $a, b > 0, c > a + b + 3,$

$$\sum_{m=0}^{\infty} (m + 1)^3 \frac{(a)_m (b)_m}{(c)_m (1)_m} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ \frac{(a)_3 (b)_3}{(c-a-b-3)_3} + \frac{6(a)_2 (b)_2}{(c-a-b-2)_2} + \frac{7ab}{c-a-b-1} + 1 \right].$$

(iv) For  $a, b, c \neq 1,$  with  $c > \max\{0, a + b - 1\},$

$$\sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m (1)_{m+1}} = \frac{1}{(a-1)(b-1)} \left[ \frac{\Gamma(c)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)} - (c-1) \right].$$

**Theorem 4.2** Suppose  $t_1(z) = z + \overline{z^2 F(a, b, c; z)},$   $t_2(z) = z + \overline{z(F(a, b, c; z) - 1)}$  and  $t_3(z) = z + \overline{z \int_0^z F(a, b, c; \varsigma) d\varsigma},$  where  $c$  is a positive real number and either  $a, b \in (-1, \infty)$  with  $ab > 0$  or  $a, b \in \mathbb{C} \setminus \{0\}$  with  $b = \bar{a}.$

(i) If  $c > \operatorname{Re}(a + b) + 3$  and

$$\left[ 2(1 + \lambda) + (1 + 4\lambda + 6\delta) \frac{ab}{c - a - b - 1} + (\lambda + 6\delta) \frac{(a)_2 (b)_2}{(c - a - b - 2)_2} + \delta \frac{(a)_3 (b)_3}{(c - a - b - 3)_3} \right] F(a, b, c; 1) \leq 1, \tag{4.3}$$

then  $t_1 \in \mathcal{R}_H^0(\lambda, \delta).$

(ii) If  $c > \operatorname{Re}(a + b) + 3$  and

$$\left[ 1 + (1 + 2\lambda) \frac{ab}{c - a - b - 1} + (\lambda + 3\delta) \frac{(a)_2 (b)_2}{(c - a - b - 2)_2} + \delta \frac{(a)_3 (b)_3}{(c - a - b - 3)_3} \right] F(a, b, c; 1) \leq 2, \tag{4.4}$$

then  $t_2 \in \mathcal{R}_H^0(\lambda, \delta).$

(iii) If  $a, b, c \neq 1$  and  $c > \operatorname{Re}(a + b) + 2$

$$\left[ \frac{c - a - b}{(a - 1)(b - 1)} + 1 + 2\lambda + (\lambda + 3\delta) \frac{ab}{c - a - b - 1} + \delta \frac{(a)_2 (b)_2}{(c - a - b - 2)_2} \right] F(a, b, c; 1) \leq 1 + \frac{c - 1}{(a - 1)(b - 1)}, \tag{4.5}$$

then  $t_3 \in \mathcal{R}_H^0(\lambda, \delta).$

**Proof** (i) Let  $t_1(z) = z + \overline{z F(a, b, c; z)} = z + \sum_{m=2}^{\infty} Q_{1,m} z^m$  where

$$Q_{1,m} = \frac{(a)_{m-2} (b)_{m-2}}{(c)_{m-2} (m-1)!} \text{ for } m \geq 2.$$

In view of Theorem 2.8, to prove that  $t_1 \in \mathcal{R}_H^0(\lambda, \delta)$ , we need to show that

$$\sum_{m=2}^{\infty} m \{1 + (m - 1)[\lambda + \delta(m - 2)]\} |Q_{1,m}| \leq 1.$$

Then, using Lemma 4.1 and the Gauss formula (4.2) we observe that

$$\begin{aligned} & \sum_{m=2}^{\infty} m \{1 + (m - 1)[\lambda + \delta(m - 2)]\} |Q_{1,m}| \\ = & \sum_{m=2}^{\infty} m \{1 + (m - 1)[\lambda + \delta(m - 2)]\} \frac{(a)_{m-2}(b)_{m-2}}{(c)_{m-2}(m - 2)!} \\ = & \sum_{m=0}^{\infty} (m + 2)[1 + (m + 1)(\lambda + \delta m)] \frac{(a)_m(b)_m}{(c)_m m!} \\ = & \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m m!} + (1 + \lambda - \delta) \sum_{m=0}^{\infty} (m + 1) \frac{(a)_m(b)_m}{(c)_m m!} \\ & + \lambda \sum_{m=0}^{\infty} (m + 1)^2 \frac{(a)_m(b)_m}{(c)_m m!} + \delta \sum_{m=0}^{\infty} (m + 1)^3 \frac{(a)_m(b)_m}{(c)_m m!} \\ = & F(a, b, c; 1) + (1 + \lambda - \delta) \left[ \frac{ab}{c - a - b - 1} + 1 \right] F(a, b, c; 1) \\ & + \lambda \left[ \frac{(a)_2(b)_2}{(c - a - b - 2)_2} + \frac{3ab}{c - a - b - 1} + 1 \right] F(a, b, c; 1) \\ & + \delta \left[ \frac{(a)_3(b)_3}{(c - a - b - 3)_3} + \frac{6(a)_2(b)_2}{(c - a - b - 2)_2} + \frac{7ab}{c - a - b - 1} + 1 \right] F(a, b, c; 1). \end{aligned}$$

Hence, the result follows from the given condition (4.3).

(ii) Let  $t_2(z) = z + \overline{z(F(a, b, c; z) - 1)} = z + \sum_{m=2}^{\infty} Q_{2,m} z^m$ , where

$$Q_{2,m} = \frac{(a)_{m-1}(b)_{m-1}}{(c)_{m-1}(m - 1)!} \text{ for } m \geq 2.$$

In view of Theorem 2.8, to prove that  $t_2 \in \mathcal{R}_H^0(\lambda, \delta)$ , we need to show that

$$\sum_{m=2}^{\infty} m \{1 + (m - 1)[\lambda + \delta(m - 2)]\} |Q_{2,m}| \leq 1.$$

Then, using Lemma 4.1 and the Gauss formula (4.2) we observe that

$$\begin{aligned} & \sum_{m=2}^{\infty} m \{1 + (m - 1)[\lambda + \delta(m - 2)]\} \frac{(a)_{m-1}(b)_{m-1}}{(c)_{m-1}(m - 1)!} \\ = & \sum_{m=0}^{\infty} \frac{(a)_{m+1}(b)_{m+1}}{(c)_{m+1}(m + 1)!} + (1 + 2\lambda) \sum_{m=0}^{\infty} \frac{(a)_{m+1}(b)_{m+1}}{(c)_{m+1}m!} \\ & + (\lambda + 3\delta) \sum_{m=1}^{\infty} \frac{(a)_{m+1}(b)_{m+1}}{(c)_{m+1}(m - 1)!} + \delta \sum_{m=2}^{\infty} \frac{(a)_{m+1}(b)_{m+1}}{(c)_{m+1}(m - 2)!} \\ = & F(a, b, c; 1) - 1 + (1 + 2\lambda) \frac{ab}{c} F(a + 1, b + 1, c + 1; 1) \\ & + (\lambda + 3\delta) \frac{(a)_2(b)_2}{(c)_2} F(a + 2, b + 2, c + 2; 1) \\ & + \delta \frac{(a)_3(b)_3}{(c)_3} F(a + 3, b + 3, c + 3; 1). \end{aligned}$$

Condition (4.4) concludes that  $t_2 \in \mathcal{R}_H^0(\lambda, \delta)$ .

(iii) Let

$$t_3(z) = z + z \int_0^z \overline{F(a, b, c; \varsigma)} d\varsigma = z + \sum_{m=2}^{\infty} Q_{3,m} z^m,$$

where

$$Q_{3,m} = \frac{(a)_{m-2}(b)_{m-2}}{(c)_{m-2}(m - 1)!} \text{ for } m \geq 2.$$

Using a similar approach to the proof of (ii), we have

$$\begin{aligned} & \sum_{m=2}^{\infty} m \{1 + (m - 1)[\lambda + \delta(m - 2)]\} \frac{(a)_{m-2}(b)_{m-2}}{(c)_{m-2}(m - 1)!} \\ = & \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m(m + 1)!} + (1 + 2\lambda) \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m m!} \\ & + (\lambda + 3\delta) \sum_{m=1}^{\infty} \frac{(a)_m(b)_m}{(c)_m(m - 1)!} + \delta \sum_{m=2}^{\infty} \frac{(a)_m(b)_m}{(c)_m(m - 2)!} \\ = & \frac{c - 1}{(a - 1)(b - 1)} [F(a - 1, b - 1, c - 1; 1) - 1] + (1 + 2\lambda) F(a, b, c; 1) \\ & + (\lambda + 3\delta) \frac{ab}{c - a - b - 1} F(a, b, c; 1) + \delta \frac{(a)_2(b)_2}{(c - a - b - 2)_2} F(a, b, c; 1). \end{aligned}$$

Condition (4.5) completes the proof. □

Note that for  $\varrho \in \mathbb{C} \setminus \{-1, -2, \dots\}$  and  $m \in \mathbb{N} \cup \{0\}$ , we have

$$\frac{(-1)^m (-\varrho)_m}{m!} = \binom{\varrho}{m} = \frac{\Gamma(\varrho + 1)}{m! \Gamma(\varrho - m + 1)}.$$



In particular, when  $\varrho = l$  ( $l \in \mathbb{N}$ ,  $l \geq m$ ), we have

$$(-l)_m = \frac{(-1)^m l!}{(l-m)!}.$$

Now, using the above relation in Theorem 4.2, we construct the following harmonic polynomial which is in the class  $\mathcal{R}_H^0(\lambda, \delta)$ .

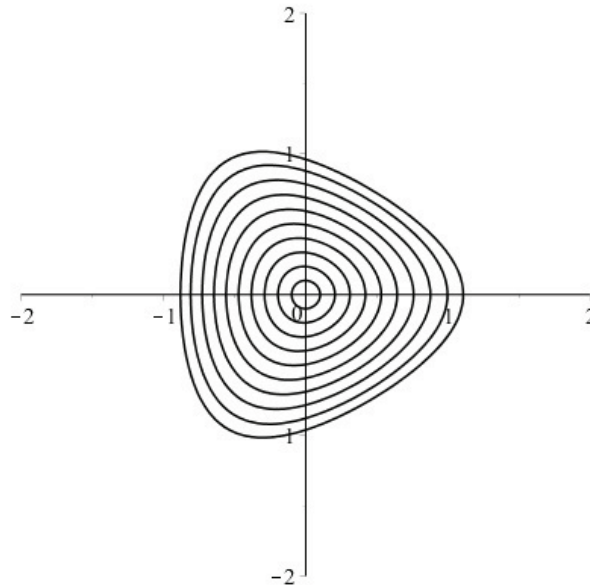
**Example 4.3** Let  $a = b = -3$  and  $c$  be a positive real number. Then using the above relation, Theorem 4.2 (ii) yields the following: If

$$-c^3 + (18\lambda + 15)c^2 + (162\lambda + 108\delta + 106)c + (324\lambda + 360\delta + 168) \leq 0 \tag{4.6}$$

then

$$\begin{aligned} T_2(z) &= z + \overline{\sum_{m=1}^3 \binom{3}{m} \frac{(3-m+1)_m}{(c)_m} z^{m+1}} \\ &= z + \frac{9}{c} \bar{z}^2 + \frac{18}{c(c+1)} \bar{z}^3 + \frac{6}{c(c+1)(c+2)} \bar{z}^4 \in \mathcal{R}_H^0(\lambda, \delta). \end{aligned}$$

For instance, setting  $\lambda = 3$ ,  $\delta = 1$ , and  $c = 78.2$ , condition (4.6) is satisfied and the images of concentric circles inside the unit disk  $\mathcal{U}$  under  $T_2$  with  $c = 78.2$  are shown in Figure 3.



**Figure 3.** Image of  $T_2(\mathcal{U})$  with  $c = 78.2$ .

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