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CANAN KÖROĞLU

AYHAN AYDIN

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Exact and nonstandard finite difference schemes for the Burgers equation $B(2,2)$

Canan KÖROĞLU[1,*], **Ayhan AYDIN**[2]
[1]Department of Mathematics, Faculty of Science, Hacettepe University, Ankara, Turkey
[2]Department of Mathematics, Faculty of Science, Atılım University, Ankara, Turkey

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Abstract: In this paper, we consider the Burgers equation $B(2,2)$. Exact and nonstandard finite difference schemes (NSFD) for the Burgers equation $B(2,2)$ are designed. First, two exact finite difference schemes for the Burgers equation $B(2,2)$ are proposed using traveling wave solution. Then, two NSFD schemes are represented for this equation. These two NSFD schemes are compared with a standard finite difference (SFD) scheme. Numerical results show that the NSFD schemes are accurate and efficient in the numerical simulation of the kink-wave solution of the $B(2,2)$ equation. We see that although the SFD scheme yields numerical instability for large step sizes, NSFD schemes provide reliable results for long time integration. Local truncation errors show that the NSFD schemes are consistent with the $B(2,2)$ equation.

Key words: Burgers $B(2,2)$ equation, exact finite difference scheme, nonstandard finite difference scheme, local truncation error, standard finite difference scheme

1. Introduction

Nonlinear partial differential equations (PDEs) play a prevalent role in many scientific and engineering phenomena.There are many researchers who studied to obtain the exact solution of nonlinear PDEs. If the exact solution is not possible to obtain, then numerical studies are essential to understand the behavior of the solution. It is important to be able to find accurate numerical solutions to the full nonlinear problem because nonlinear differential equations can exhibit very complicated behavior over extended time intervals. When the continuous model is discretized and transformed to discrete model, many numerical solvers fails to give accurate solutions, produce instabilities and fail to preserve the essential properties of the continuous model. Preservation of the qualitative properties of continuous model is one of the most important problems in numerical analysis. Numerical solutions of PDEs with finite difference approximation began in the early 1950s. It is well known that traditional schemes such as Runge–Kutta and others, sometimes fail generating oscillations in the numerical solution [11]. Recently, nonstandard finite difference (NSFD) methods are used for the numerical solution of PDEs to prevent numerical instabilities. Regarding the positivity and boundedness of solutions, NSFD schemes have a better performance over the standard finite difference (SFD) schemes. The following five rules are executed for the construction of discrete models that have the capability to match the properties of the NSFD solution by Mickens [12]:

- Rule 1: The orders of the discrete derivatives must be exactly equal to the orders of the corresponding derivatives of the differential equations.

*Correspondence: ckoroglu@hacettepe.edu.tr
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- Rule 2: Denominator functions for the discrete derivatives must, in general, be expressed in terms of more complicated functions of the step sizes than those conventionally used.
- Rule 3: Nonlinear terms should be approximated in a non-local way.
- Rule 4: Special solutions of differential equations should also be special discrete solutions of the finite difference models.
- Rule 5: The finite-difference equations should not have solutions that do not correspond exactly to solutions of the differential equations.

The most important advantages of NSFD scheme is that being able to choose a convenient denominator function instead of the step-size used in SFD scheme and discretizing the nonlinear terms locally [14]. This flexibility of the NSFD scheme yields better numerical results over the SFD scheme. If the step-size of SFD scheme is chosen small enough, there is no significant difference between a SFD and a NSFD scheme. On the other hand, if the step-size gets larger, than NSFD overcomes this shortcoming experiences. There are many numerical experiments that show the advantage of NSFD schemes. Patidar [16] reviewed many recent developments and further applications of NSFD methods encountered in the past decade. Dimitrov et al. [5] developed a new class of elementary stable NSFD schemes for general two-dimensional autonomous dynamical systems based on the standard Euler and second-order Runge–Kutta methods. Jodar et al. [8] constructed two implicit finite difference scheme based on NSFD methodology for a deterministic mathematical model describing the evolution of influenza A in human population. Since the Euler method and other well-known methods produce bad approximations simulating the influenza model for large time step sizes, they used a NSFD with arbitrarily large time step sizes, saving computational cost when integrating over long time periods. There are many other exact and NSFD schemes for numerical solution of ordinary differential equations (ODEs) (see [6, 16] and references therein). On the other hand, exact and NSFD methods for PDEs are rare and the theory is not complete in this area [13]. Chapwanya et al. [4] have investigated the performance of an exact scheme for the Michaelis–Menten equation with several NSFD schemes and designed several dynamically consistent NSFD schemes for related reaction-diffusion equations, advection-reaction equations, and advection-reaction-diffusion equations. Computationally, the power of NSFD schemes have been presented. In [23], authors have developed two exact finite difference schemes and an NSFD scheme for Burgers–Huxley equation. In [22] Zhang et al. have constructed a new version of exact finite difference scheme and proposed NSFD schemes for Burgers equation and Burgers-Fisher equation. Aydin et al. [2] have proposed and studied a linearly implicit NSFD method for the numerical solution of modified KdV equation. Koroglu et.al [9] have presented a NSFD scheme with theta method which includes the implicit Euler and a Crank–Nicolson type discretization for the numerical solution of the modified Korteweg-de Vries (MKdV) equation. In [10], the author have designed exact and consistent nonstandard finite difference schemes for the numerical solution of the KdVB$(2,1,2)$ equation. In these studies, exact finite difference schemes and NSFD schemes are obtained by means of traveling wave solution of the PDE under consideration.

Burgers' equation

$$u_t + \alpha \left(u^2\right)_x - \nu u_{xx} = 0, \tag{1.1}$$

is one of the important model used in many fields such as dispersive water, acoustics, shock waves, heat conduction, and turbulence [21]. Due to the form of the nonlinear convection term and the occurrence of a viscosity term, it can be considered as a simplified form of the Navier–Stokes equation. There are many analytical and numerical studied of Burger's equation in the literature (see [7, 22] and reference therein). Recently, the so called $B(n,n)$ equation

$$u_t + a\left(u^n\right)_x + b\left(u^n\right)_{xx} = 0, \tag{1.2}$$

is proposed as a generalization of the Burgers' equation (1.1) [20]. The equation (1.2) is a nonlinear dispersive $K(n,n)$ equation. It is a family of nonlinear Korteweg-de Vries (KdV) like equations. These equations support compact solitary traveling structures for $a > 0$. The study of compactons provides information about many scientific processes, such as the super deformed nuclei, preformation of cluster in hydrodynamic models, the fission of liquid drops and inertial fusion [19]. Traveling wave solution of (1.2) is given by [20]

$$u(x,t) = \frac{1}{\left\{\frac{a}{2c}\left(1+\tanh\left[\frac{a(n-1)}{2bn}(x-ct)\right]\right)\right\}^{\frac{1}{n-1}}}. \tag{1.3}$$

To the best of our knowledge, there are no theoretical studies about (1.2) other than the traveling wave solution and numerical schemes for the model (1.2) for $n = 2$ have not been studied in previous works. This motivates us to provide exact and NSFD schemes for the model (1.2). In this study, we will consider the $B(2,2)$ equation

$$u_t + a\left(u^2\right)_x + b\left(u^2\right)_{xx} = 0, \tag{1.4}$$

and develop two exact finite difference and two NSFD schemes which are never proposed and studied in the literature before.

The outline of the paper is as follows: In Section 2, an explicit and a fully implicit exact finite difference schemes are presented. In Section 3, two consistent linearly implicit NSFD schemes are constructed for the B(2,2) equation. Local truncation errors for two NSFD schemes are studied. Some numerical results that show the capability of the NSFD schemes are given Section 4. Numerical results obtained by the NSFD schemes compared with a standard finite difference (SFD) scheme. Finally, the conclusion is given in Section 5.

2. Exact finite difference schemes for $B(2,2)$ equation

In [12], an exact finite difference scheme is defined as a finite difference model for which the solution to the difference equation has the same general solution as the associated differential equation. Recently, there is an increasing interest to find exact finite difference models for particular PDEs, because these finite difference models do not exhibit numerical instabilities (see [15, 17, 22]). However, not every PDE has exact finite difference model. In this section, we will construct two exact finite difference schemes for the $B(2,2)$ equation (1.4). We start with the traveling wave solution (1.3) of the $B(2,2)$ equation

$$u(x,t) = \frac{c}{a}\left(1 + e^{-\frac{a}{2b}(x-ct)}\right). \tag{2.1}$$

If we choose $h = c\Delta t$, then we get

$$u(x+h,t) = u(x,t-\Delta t), \qquad u(x-h,t) = u(x,t+\Delta t). \tag{2.2}$$

Then, based on (2.1) we have the following formulas

$$\begin{aligned} u(x-h,t) - u(x,t) &= \left(u(x,t) - \frac{c}{a}\right)\left(e^{\frac{a}{2b}h} - 1\right) \\ u(x+h,t) - u(x,t) &= \left(\frac{c}{a} - u(x,t)\right)\left(1 - e^{\frac{-a}{2b}h}\right). \end{aligned} \tag{2.3}$$

Let the step functions are $\Psi_1 = \frac{\left(e^{\frac{a}{2b}h}-1\right)}{\frac{a}{2b}}$, $\Psi_2 = \frac{\left(1-e^{\frac{-a}{2b}h}\right)}{\frac{a}{2b}}$, $\Phi_1 = \frac{\left(e^{\frac{a}{2b}c\Delta t}-1\right)}{\frac{a}{2b}c}$, and $\Phi_2 = \frac{\left(1-e^{\frac{-a}{2b}c\Delta t}\right)}{\frac{a}{2b}c}$. Then $\Psi_1 = c\Phi_1 =$ and $\Psi_2 = c\Phi_2$. Using the relations (2.3), we have the following forward and backward difference operators

$$\begin{aligned} \partial_x u &= \frac{u(x+h,t)-u(x,t)}{\Psi_2} \\ &= \tfrac{a}{2b}\left(\tfrac{c}{a}-u(x,t)\right) \\ \overline{\partial}_x u &= \frac{u(x,t)-u(x-h,t)}{\Psi_1} \\ &= \tfrac{a}{2b}\left(\tfrac{c}{a}-u(x,t)\right) \end{aligned} \tag{2.4}$$

We select $u_{xx} = \partial_x \overline{\partial}_x u(x,t)$ and using (2.4) we can write

$$\begin{aligned} \partial_x \overline{\partial}_x u(x,t) &= \frac{u(x+h,t)-u(x,t)}{\Psi_1\Psi_2} - \frac{u(x,t)-u(x-h,t)}{\Psi_1\Psi_2}, \\ &= \frac{1}{\Psi_1}\left[\frac{a}{2b}\left(\frac{c}{a}-u(x,t)\right) - \frac{a}{2b}\left(\frac{c}{a}-u(x-h,t)\right)\right], \\ &= -\frac{a}{2b}\left(\frac{u(x,t)-u(x+h,t)}{\Psi_1}\right) - \frac{a}{b}\left(\frac{u(x+h,t)-u(x-h,t)}{2\Psi_1}\right). \end{aligned} \tag{2.5}$$

Now, we add and subtract the terms

$$\frac{1}{u(x,t)}\frac{u(x+h,t)-u(x,t)}{\Psi_2}\frac{u(x+h,t)-u(x,t)}{\Psi_1}, \frac{1}{u(x,t)}\frac{c}{2b}\frac{u(x+h,t)-u(x,t)}{\Psi_1}$$

to the equation (2.5) and using $\Psi_1 = c\Phi_1$, we get

$$\begin{aligned} \partial_x \overline{\partial}_x u(x,t) &= \frac{-1}{2b}\frac{1}{u}\frac{u(x,t)-u(x,t-\Delta t)}{\Phi_1} - \frac{a}{b}\frac{u(x+h,t)-u(x-h,t)}{2\Psi_1} \\ &\quad -\frac{1}{u}\frac{u(x+h,t)-u(x,t)}{\Psi_2}\frac{u(x+h,t)-u(x,t)}{\Psi_1}. \end{aligned} \tag{2.6}$$

Now, instead of $u_{xx} = \partial_x \overline{\partial}_x u(x,t)$, we use $u_{xx} = \overline{\partial}_x \partial_x u(x,t)$ and using (2.4) we can write

$$\begin{aligned} \overline{\partial}_x \partial_x u(x,t) &= \frac{u(x+h,t)-u(x,t)}{\Psi_1\Psi_2} - \frac{u(x,t)-u(x-h,t)}{\Psi_1\Psi_2}, \\ &= \frac{1}{\Psi_2}\left[\frac{a}{2b}\left(\frac{c}{a}-u(x+h,t)\right) - \frac{a}{2b}\left(\frac{c}{a}-u(x,t)\right)\right], \\ &= -\frac{a}{2b}\left(\frac{u(x+h,t)-u(x-h,t)}{\Psi_2}\right) - \frac{a}{b}\left(\frac{u(x+h,t)-u(x-h,t)}{2\Psi_2}\right). \end{aligned} \tag{2.7}$$

Now, we add and subtract the terms

$$\frac{1}{u(x,t)}\frac{u(x-h,t)-u(x,t)}{\Psi_2}\frac{u(x-h,t)-u(x,t)}{\Psi_1}, \frac{1}{u(x,t)}\frac{c}{2b}\frac{u(x-h,t)-u(x,t)}{\Psi_2}$$

to the above equation (2.7) and using $\Psi_2 = c\Phi_2$, we get

$$\begin{aligned} \overline{\partial}_x \partial_x u(x,t) &= \frac{-1}{2b}\frac{1}{u}\frac{u(x,t+\Delta t)-u(x,t)}{\Phi_2} - \frac{a}{b}\frac{u(x+h,t)-u(x-h,t)}{2\Psi_2} \\ &\quad -\frac{1}{u}\frac{u(x,t)-u(x-h,t)}{\Psi_2}\frac{u(x,t)-u(x-h,t)}{\Psi_1}. \end{aligned} \tag{2.8}$$

Now we consider the solution (2.1) at the discrete point (x_j, t_n)

$$U_j^n = u(x_j, t_n) = \frac{c}{a}\left(1 + e^{-\frac{a}{2b}(x_j - ct_n)}\right). \tag{2.9}$$

Then we can write an implicit exact finite difference scheme by means of (2.6)

$$\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Psi_1\Psi_2} = \frac{-1}{2b}\frac{1}{U_j^{n+1}}\frac{U_j^{n+1} - U_j^n}{\Phi_1} - \frac{a}{b}\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Psi_1} - \frac{1}{U_j^{n+1}}\frac{\left(U_{j+1}^{n+1} - U_j^{n+1}\right)^2}{\Psi_1\Psi_2}, \tag{2.10}$$

and we can write an explicit exact finite difference scheme by means of (2.8)

$$\frac{U_{j+1}^{n} - 2U_j^{n} + U_{j-1}^{n}}{\Psi_1\Psi_2} = \frac{-1}{2b}\frac{1}{U_j^{n}}\frac{U_j^{n+1} - U_j^n}{\Phi_2} - \frac{a}{b}\frac{U_{j+1}^{n} - U_{j-1}^{n}}{2\Psi_2} - \frac{1}{U_j^{n}}\frac{\left(U_{j}^{n} - U_{j-1}^{n}\right)^2}{\Psi_2\Psi_1}. \tag{2.11}$$

Thus, we get the following theorem.

Theorem 2.1 *For the $B(2,2)$ equation*

$$u_t + a\left(u^2\right)_x + b\left(u^2\right)_{xx} = 0$$

an implicit exact finite difference scheme and an explicit exact finite difference scheme are given by (2.10) and (2.11), respectively. The space stepsizes ψ_1 and ψ_2 and temporal stepsizes Φ_1 and Φ_2 satisfy

$$\psi_1 = \frac{\left(e^{\frac{a}{2b}h} - 1\right)}{\frac{a}{2b}}, \quad \psi_2 = \frac{\left(1 - e^{\frac{-a}{2b}h}\right)}{\frac{a}{2b}}$$

$$\Phi_1 = \frac{\left(e^{\frac{a}{2b}c\Delta t} - 1\right)}{\frac{a}{2b}c}, \textit{and} \quad \Phi_2 = \frac{\left(1 - e^{\frac{-a}{2b}c\Delta t}\right)}{\frac{a}{2b}c}$$

where $h = c\Delta t$.

In this section, two exact finite difference schemes are proposed for the $B(2,2)$ equation. Notice that, the step-size for exact schemes must satisfy some fixed conditions. In the following sections, we release these conditions and propose two NSFD schemes for the $B(2,2)$ equation. Generally, exact finite difference scheme seems to be standard. But it is different from approach in [22]. First of all, travelling wave solution of $B(2,2)$, equation (2.1) is different from the travelling wave solution of Zhang et al. [22]. In addition, the nonlinear terms $(u_x)^2$ and uu_{xx}, which are not appear in [22], play important roles in computing of exact finite difference scheme.

3. NSFD schemes for the $B(2,2)$ equation

In this section, we will propose two NSFD schemes for the numerical solution of the $B(2,2)$ equation (1.4). We start with the derivation of the first NSFD scheme which will be named as NSFD1 scheme. For this purpose, we consider the exact travelling wave solution of (2.1), [20].

A standard finite difference (SFD) scheme for the B$(2,2)$ equation (1.4) can be

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+a\frac{(u^2)_{j+1}^n-(u^2)_j^n}{h}+b\frac{(u^2)_{j-1}^n-2(u^2)_j^n+(u^2)_{j+1}^n}{h^2}=0, \tag{3.1}$$

where Δt is the temporal step-size and $h=\Delta x$ is the spatial step-size. We re-write the $B(2,2)$ equation (1.4) as

$$u_t+2auu_x+2b(u_x)^2+2buu_{xx}=0 \tag{3.2}$$

and propose the following NSFD discretization

$$\frac{U_j^{n+1}-U_j^n}{\Phi_1}+2aU_j^{n+1}\frac{U_j^n-U_{j-1}^n}{\Gamma}+2b\left(\frac{U_{j+1}^n-U_j^n}{\Gamma}\right)\left(\frac{U_j^n-U_{j-1}^n}{\Gamma}\right)+2bU_j^{n+1}\frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{\Gamma^2}=0 \tag{3.3}$$

where Φ_1 and Γ are time-step and space-step functions, respectively.

According to (3.3) equation, we can get

$$\Phi_1=\frac{\left(U_j^{n+1}-U_j^n\right)\Gamma^2}{-2a\Gamma U_j^{n+1}\left(U_j^n-U_{j-1}^n\right)-2b\left(U_{j+1}^n-U_j^n\right)\left(U_j^n-U_{j-1}^n\right)-2bU_j^{n+1}\left(U_{j+1}^n-2U_j^n+U_{j-1}^n\right)}. \tag{3.4}$$

We solve for Φ_1 and define $s_j^n=e^{-\frac{a}{2b}(x_j-ct_n)}$. After tedious calculations, we have

$$\Phi_1=\frac{e^{\frac{a}{2b}c\Delta t}-1}{\frac{a}{2b}c}\left(\frac{1}{2(1+se^{\frac{a}{2b}c\Delta t})-se^{-\frac{a}{2b}h}-e^{-\frac{a}{2b}h}(1+se^{\frac{a}{2b}c\Delta t})}\right). \tag{3.5}$$

If we choose

$$\Gamma=\frac{\left(e^{\frac{a}{2b}h}-1\right)}{\frac{a}{2b}}=h+\mathcal{O}(h^2) \tag{3.6}$$

then the denumerator function Φ_1 in (3.3) can be written in a simple form

$$\Phi_1=\frac{\left(e^{\frac{a}{2b}c\Delta t}-1\right)}{\frac{a}{2b}c}=\Delta t+\mathcal{O}(\Delta t^2). \tag{3.7}$$

In the following sections, we call the NSFD scheme (3.3) as NSFD1 for which the denumerator functions are given in (3.6) and (3.7).

Now, we discuss the consistency of the NSFD1 with the $B(2,2)$ equation (1.4). Setting $u_j^n=u(x_j,t_n)$, we have Taylor's formula for the solution of equation (3.2) with appropriate $\overline{x}_j\in(x_j,x_{j+1})$, $\bar{t}_n\in(t_n,t_{n+1})$. For this purpose, for functions defined on the grid we introduce the difference quotients

$$\frac{\partial u_j^n}{\partial t}=\frac{U_j^{n+1}-U_j^n}{\Phi_1},\ \frac{\overline{\partial}u_j^n}{\partial x}=\frac{U_j^n-U_{j-1}^n}{\Gamma},\ \frac{\partial u_j^n}{\partial x}=\frac{U_{j+1}^n-U_j^n}{\Gamma},\ \frac{\partial\overline{\partial}u_j^n}{\partial x^2}=\frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{\Gamma^2}. \tag{3.8}$$

and analyze the local truncation error of the NSFD1 scheme (3.3). We define the residual and using Taylor's series expansion about (x_j,t_n) we get

$$
\begin{aligned}
\tau_j^n &= \frac{\partial u_j^n}{\partial t} + 2au_j^{n+1}\frac{\overline{\partial} u_j^n}{\partial x} + 2b\frac{\partial u_j^n}{\partial x}\frac{\overline{\partial} u_j^n}{\partial x} + 2bu_j^{n+1}\frac{\partial\overline{\partial} u_j^n}{\partial x^2} \\
&= \left(\frac{\partial u_j^n}{\partial t} - u_t(x_j,t_n)\right) + 2a\left(u_j^{n+1}\frac{\overline{\partial} u_j^n}{\partial x} - u(x_j,t_n)u_x(x_j,t_n)\right) + 2b\left(\frac{\partial u_j^n}{\partial x}\frac{\overline{\partial} u_j^n}{\partial x} - u_x(x_j,t_n)u_x(x_j,t_n)\right) \\
&+2b\left(u_j^{n+1}\frac{\partial\overline{\partial} u_j^n}{\partial x^2} - u(x_j,t_n)u_{xx}(x_j,t_n)\right) \\
&= (\frac{\Delta t}{\Phi_1} - 1)u_t(x_j,t_n) + \frac{\Delta t^2}{2\Phi_1}u_{tt}(x_j,t_n) + \frac{\Delta t^3}{6\Phi_1}u_{ttt}(x_j,\overline{t}_n) + 2a[(\frac{\Delta x}{\Gamma} - 1)u(x_j,t_n)u_x(x_j,t_n) \\
&- \frac{\Delta x^2}{2\Gamma}u(x_j,t_n)u_{xx}(x_j,t_n) + \frac{\Delta x^3}{6\Gamma}u(x_j,t_n)u_{xxx}(\overline{x}_j,t_n) + \frac{\Delta x\Delta t}{\Gamma}u_x(x_j,t_n)u_t(x_j,t_n) \\
&- \frac{\Delta x^2\Delta t}{2\Gamma}u_t(x_j,t_n)u_{xx}(x_j,t_n) + \frac{\Delta x^3\Delta t}{6\Gamma}u_t(x_j,t_n)u_{xxx}(\overline{x}_j,t_n) + \frac{\Delta x\Delta t^2}{2\Gamma}u_{tt}(x_j,t_n)u_x(x_j,t_n) \\
&- \frac{\Delta x^2\Delta t^2}{4\Gamma}u_{tt}(x_j,t_n)u_{xx}(x_j,t_n) + \frac{\Delta x^3\Delta t^2}{12\Gamma}u_{tt}(x_j,t_n)u_{xxx}(\overline{x}_j,t_n)] + 2b[(\frac{\Delta x^2}{\Gamma^2} - 1)u_x(x_j,t_n)u_x(x_j,t_n) \\
&+ \frac{\Delta x^4}{3\Gamma^2}u_x(x_j,t_n)u_{xxx}(\overline{x}_j,t_n) - \frac{\Delta x^4}{4\Gamma^2}u_{xx}(x_j,t_n)u_{xx}(x_j,t_n) + \frac{\Delta x^6}{36\Gamma^2}u_{xxx}(\overline{x}_j,t_n)u_{xxx}(\overline{x}_j,t_n)] \\
&+2b[(\frac{\Delta x^2}{\Psi_1} - 1)u(x_j,t_n)u_{xx}(x_j,t_n) + \frac{\Delta x^2\Delta t}{\Psi_1}u_t(x_j,t_n)u_{xx}(x_j,t_n) + \frac{\Delta x^2\Delta t^2}{2\Psi_1}u_{tt}(x_j,t_n)u_{xx}(x_j,t_n)]
\end{aligned}
$$

where $\Psi_1 = \Gamma^2$.

If we choose Δt and Δx small enough, we know that $\Phi_1 \approx \Delta t$ and $\Gamma \approx \Delta x$. After the tedious computations, we conclude that $\tau_j^n = \mathcal{O}(\Delta t + \Delta x)$. It is consistent with the B$(2,2)$ equation (1.4) since $\tau_j^n \longrightarrow 0$ as $(\Delta t, \Delta x) \longrightarrow (0,0)$.

In addition, the stability region is derived by using the freezing of coefficient method and the von Neumann stability analysis ([1, 3]). The equation of the amplification factor is given as

$$\xi = 1 - aU_{max}r(1 - e^{-i\beta h}) - bU_{max}r\frac{e^{i\beta h} - 2 - e^{-i\beta h}}{\Gamma}, \tag{3.9}$$

where $r = \frac{\Phi_1}{\Gamma}$. Hence we obtain

$$r \leq \frac{4}{U_{max}}\frac{\left(a - b\frac{2}{\Gamma}\right)}{5a^2 - \frac{16}{\Gamma} + b^2\frac{16}{\Gamma^2}}.$$

Now, we construct the following second NSFD scheme for the numerical solution of the $B(2,2)$ equation (1.4) which will be called NSFD2 scheme. We consider the difference scheme

$$\frac{U_j^{n+1} - U_j^n}{\Phi_2} + 2aU_j^{n+1}\frac{U_j^n - U_{j-1}^n}{\sqrt{\Psi_2}} + 2b\left(\frac{U_{j+1}^n - U_j^n}{\sqrt{\Psi_2}}\right)\left(\frac{U_j^n - U_{j-1}^n}{\sqrt{\Psi_2}}\right) + 2bU_j^{n+1}\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Psi_2} = 0 \tag{3.10}$$

where $\Phi_2 = \frac{\left(e^{\frac{a}{2b}c\Delta t} - 1\right)}{\frac{a}{2b}c}$ and $\Psi_2 = \psi_1\psi_2 = \frac{\left(e^{\frac{a}{2b}h} - 1\right)}{\frac{a}{2b}}\frac{\left(1 - e^{\frac{-a}{2b}h}\right)}{\frac{a}{2b}}$. Note that when we compare the NSFD1 (3.3) and NSFD2 (3.10), we see that the denominator functions of spatial derivatives Γ in (3.3) and $\sqrt{\Psi_2}$ in (3.10) are different. We can find that $\Phi_2 \longrightarrow \Delta t$, $\Psi_2 \longrightarrow \Delta x^2$ as Δt and Δx approach zero. The difference quotients to analyze the local truncation error of the NSFD2 scheme (3.10) is given as follows:

$$\frac{\partial u_j^n}{\partial t} = \frac{U_j^{n+1} - U_j^n}{\Phi_2}, \frac{\overline{\partial} u_j^n}{\partial x} = \frac{U_j^n - U_{j-1}^n}{\sqrt{\Psi_2}}, \frac{\partial u_j^n}{\partial x} = \frac{U_{j+1}^n - U_j^n}{\sqrt{\Psi_2}}, \frac{\partial\overline{\partial} u_j^n}{\partial x^2} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Psi_2}. \tag{3.11}$$

The local truncation error τ_j^n for the NSFD2 (3.10) can be analysed similarly. We note that

$$\begin{aligned}
\tau_j^n &= \frac{\partial u_j^n}{\partial t} + 2au_j^{n+1}\frac{\overline{\partial} u_j^n}{\partial x} + 2b\frac{\partial u_j^n}{\partial x}\frac{\overline{\partial} u_j^n}{\partial x} + 2bu_j^{n+1}\frac{\partial\overline{\partial} u_j^n}{\partial x^2} \\
&= \left(\frac{\partial u_j^n}{\partial t} - u_t(x_j,t_n)\right) + 2a\left(u_j^{n+1}\frac{\overline{\partial} u_j^n}{\partial x} - u(x_j,t_n)u_x(x_j,t_n)\right) + 2b\left(\frac{\partial u_j^n}{\partial x}\frac{\overline{\partial} u_j^n}{\partial x} - u_x(x_j,t_n)u_x(x_j,t_n)\right) \\
&+2b\left(u_j^{n+1}\frac{\partial\overline{\partial} u_j^n}{\partial x^2} - u(x_j,t_n)u_{xx}(x_j,t_n)\right) \\
&= (\frac{\Delta t}{\Phi_2} - 1)u_t(x_j,t_n) + \frac{\Delta t^2}{2\Phi_2}u_{tt}(x_j,t_n) + \frac{\Delta t^3}{6\Phi_2}u_{ttt}(x_j,\overline{t}_n) + 2a[(\frac{\Delta x}{\sqrt{\Psi_2}} - 1)u(x_j,t_n)u_x(x_j,t_n) \\
&- \frac{\Delta x^2}{2\sqrt{\Psi_2}}u(x_j,t_n)u_{xx}(x_j,t_n) + \frac{\Delta x^3}{6\sqrt{\Psi_2}}u(x_j,t_n)u_{xxx}(\overline{x}_j,t_n) + \frac{\Delta x\Delta t}{\sqrt{\Psi_2}}u_x(x_j,t_n)u_t(x_j,t_n) \\
&- \frac{\Delta x^2\Delta t}{2\sqrt{\Psi_2}}u_t(x_j,t_n)u_{xx}(x_j,t_n) + \frac{\Delta x^3\Delta t}{6\sqrt{\Psi_2}}u_t(x_j,t_n)u_{xxx}(\overline{x}_j,t_n) + \frac{\Delta x\Delta t^2}{2\sqrt{\Psi_2}}u_{tt}(x_j,t_n)u_x(x_j,t_n) \\
&- \frac{\Delta x^2\Delta t^2}{4\sqrt{\Psi_2}}u_{tt}(x_j,t_n)u_{xx}(x_j,t_n) + \frac{\Delta x^3\Delta t^2}{12\sqrt{\Psi_2}}u_{tt}(x_j,t_n)u_{xxx}(\overline{x}_j,t_n)] + 2b[(\frac{\Delta x^2}{\Psi_2} - 1)u_x(x_j,t_n)u_x(x_j,t_n) \\
&+ \frac{\Delta x^4}{3\Psi_2}u_x(x_j,t_n)u_{xxx}(\overline{x}_j,t_n) - \frac{\Delta x^4}{4\Psi_2}u_{xx}(x_j,t_n)u_{xx}(x_j,t_n) + \frac{\Delta x^6}{36\Psi_2}u_{xxx}(\overline{x}_j,t_n)u_{xxx}(\overline{x}_j,t_n)] \\
&+2b[(\frac{\Delta x^2}{\Psi_2} - 1)u(x_j,t_n)u_{xx}(x_j,t_n) + \frac{\Delta x^2\Delta t}{\Psi_2}u_t(x_j,t_n)u_{xx}(x_j,t_n) + \frac{\Delta x^2\Delta t^2}{2\Psi_2}u_{tt}(x_j,t_n)u_{xx}(x_j,t_n)]
\end{aligned}$$

Similar to the local truncation error of NSFD1 (3) , if Δt and Δx small enough, we know that $\Phi_2 \longrightarrow \Delta t$ and $\Psi_2 \longrightarrow \Delta x^2$. After long computations, we conclude that $\tau_j^n = \mathcal{O}(\Delta t + \Delta x)$. It is consistent with the B$(2,2)$ equation (1.4) since $\tau_j^n \longrightarrow 0$ as $(\Delta t, \Delta x) \longrightarrow (0,0)$. As in the NSFD1, the linear stability of the NSFD2 scheme is

$$r \le \frac{4}{U_{max}}\frac{\left(a - b\frac{2}{\sqrt{\Psi_2}}\right)}{5a^2 - \frac{16}{\sqrt{\Psi_2}} + b^2\frac{16}{\Psi_2}}$$

where $r = \frac{\Phi_2}{\sqrt{\Psi_2}}$.

4. Numerical results

In this section we will consider the $B(2,2)$ equation (1.4)

$$u_t + a\left(u^2\right)_x + b\left(u^2\right)_{xx} = 0, \tag{4.1}$$

with initial condition

$$u(x,0) = \frac{c}{a}\left(1 + e^{-\frac{ax}{2b}}\right). \tag{4.2}$$

and boundary conditions

$$\begin{aligned} u(x_L,t) &= u(x_L,t) = \frac{c}{a}\left(1+e^{-\frac{a}{2b}(x_L-ct)}\right), \\ u(x_R,t) &= u(x_R,t) = \frac{c}{a}\left(1+e^{-\frac{a}{2b}(x_R-ct)}\right). \end{aligned} \tag{4.3}$$

We note that the initial condition (4.2) and the boundary conditions (4.3) are taken from the traveling solution (2.1). We solved the above problem on the space time domain $(x,t) = [x_L, x_R] \times [0,T]$. We consider the equally spaced mesh points $x_j = x_L + jh$, and $t_n = n\Delta t$, $j = 1,2,\cdots,M+1$, $n = 0,1,2,\cdots,N$ with spatial mesh size $h = (x_R - x_L)/M$ and temporal mesh size $\Delta t = T/N$. We measured the accuracy of the numerical solution by using the L_∞ and L_2 erros

$$L_\infty = \max_{0\le j\le M} |u(x_j,t_n) - U_j^n|,$$

$$L_2 = \left(h\sum_{j=1}^{M} |u(x_j,t_n) - U_j^n|^2\right)^{1/2}$$

at the final time $t = T$ and the absolute error

$$Abs.Err = |u(x_j,t_n) - U_j^n|$$

at the mesh points (x_j,t_n). Here, $u(x_j,t_n)$ is the exact solution obtained from the traveling wave solution (2.1) and U_j^n is the numerical solution obtained from the NSFD schemes (3.3), (3.10) or the standard FD scheme (3.1). We choose $c = 0.001$ in all computations.

Table 1 represents the L_∞ and L_2 errors of the NSFD schemes (NSFD1 and NSFD2) and SFD scheme for $h = 0.1$ and $\Delta t = 0.01$ on the spatial domain $x \in [0,1]$ for $0 \le t \le 5$. From Table 1 we can deduce that the accuracy of the NSFD2 scheme is better than the other methods. Table 1 also shows that the numerical results of our suggested nonstandard methods have a precise agreement with the exact solution. We give some additional tables for the accuracy of the proposed nonstandard method for different set of parameters. Table 2 represents the L_∞ and L_2 errors for the numerical solution of the equation (1.2) with $a = b = 0.01$ and $c = 0.001$. Table 3 represents the L_∞ and L_2 errors for the numerical solution of the equation (1.2) with $a = b = 0.01$ and $c = 0.001$. These tables shows the effect of the parameters a, b, c to the amplitude of the wave c/a (4.2) and ac/b in the denominator functions Ψ_i, and Φ_i, $i = 1,2$. We see that nonstandard scheme shows excellent performance for large step size. In addition, errors increase when the amplitude of the wave increase.

Table 1. Equation (1.2) with $a = b = 1$, $c = 0.001$. L_∞ and L_2 errors of the nonstandard and standard methods for $h = 0.1$, $\Delta t = 0.01$.

T	NSFD1		NSFD2		SFD	
	L_∞	L_2	L_∞	L_2	L_∞	L_2
1	8.0825×10^{-8}	1.5932×10^{-8}	5.6144×10^{-8}	1.1380×10^{-8}	8.7789×10^{-8}	1.6425×10^{-8}
2	2.1512×10^{-7}	3.4788×10^{-8}	1.5200×10^{-7}	2.4929×10^{-8}	2.5400×10^{-7}	3.7483×10^{-8}
3	4.7768×10^{-7}	6.3145×10^{-8}	3.4811×10^{-7}	4.6099×10^{-8}	6.4588×10^{-7}	7.9153×10^{-8}
4	1.0774×10^{-6}	1.2966×10^{-7}	8.2373×10^{-7}	1.0039×10^{-7}	1.7634×10^{-6}	2.2151×10^{-7}
5	2.6292×10^{-6}	3.3972×10^{-7}	2.1404×10^{-6}	2.8621×10^{-7}	5.4079×10^{-6}	7.9416×10^{-7}

Table 2. Equation (1.2) with $a = b = 0.01$, $c = 0.001$. L_∞ and L_2 errors of the nonstandard and standard methods for $h = 0.1$, $\Delta t = 0.005$.

T	NSFD1		NSFD2		SFD	
	L_∞	L_2	L_∞	L_2	L_∞	L_2
1	8.2189×10^{-6}	1.6008×10^{-6}	5.7572×10^{-6}	1.1458×10^{-6}	8.7967×10^{-6}	1.6435×10^{-6}
2	2.1981×10^{-5}	3.5082×10^{-5}	1.5703×10^{-5}	2.5247×10^{-6}	2.5501×10^{-6}	3.7545×10^{-6}
3	4.9129×10^{-5}	6.4321×10^{-6}	3.6324×10^{-5}	4.7454×10^{-6}	6.5070×10^{-5}	7.9670×10^{-5}
4	1.1171×10^{-4}	1.3454×10^{-5}	8.6983×10^{-5}	1.0634×10^{-5}	1.7855×10^{-4}	2.2491×10^{-5}
5	2.7498×10^{-4}	3.5838×10^{-5}	2.2877×10^{-4}	3.0983×10^{-5}	5.5093×10^{-4}	8.1271×10^{-5}

Table 3. Equation (1.2) with $a = 0.01$, $b = c = 0.001$. L_∞ and L_2 errors of the nonstandard and standard methods for $h = 0.1$, $\Delta t = 0.005$.

T	NSFD1		NSFD2		SFD	
	L_∞	L_2	L_∞	L_2	L_∞	L_2
1	2.6630×10^{-4}	3.0465×10^{-5}	2.8059×10^{-4}	3.2944×10^{-5}	3.1480×10^{-4}	3.5651×10^{-5}
2	5.3709×10^{-4}	6.1459×10^{-5}	5.6906×10^{-4}	6.6709×10^{-5}	6.5579×10^{-4}	7.3577×10^{-5}
3	8.1245×10^{-4}	9.2993×10^{-5}	8.6570×10^{-4}	1.0132×10^{-4}	1.0258×10^{-3}	1.1403×10^{-4}
4	1.0925×10^{-3}	1.2507×10^{-4}	1.1708×10^{-3}	1.3681×10^{-4}	1.4279×10^{-3}	1.5729×10^{-4}
5	1.3772×10^{-3}	1.5772×10^{-4}	1.4847×10^{-3}	1.7320×10^{-4}	1.8658×10^{-3}	2.0369×10^{-4}

Table 4 shows the errors of the nonstandard methods (3.3), (3.10) and standard method (3.1) with $h = 0.1$ at $T = 2$ for different temporal step size Δt. We see that errors of the proposed NSFD schemes (3.3) and (3.10) are decreasing as Δt values become larger and larger; which shows the convergence for large values of Δt. Moreover, Table 5 shows the relative errors of the proposed method at different values of the final time T and some disjoint points of x. This comparison shows that the NSFD2 method gives more accurate results than the other methods for short time of integration. On the other hand for long time integration NSFD1 gives slightly better results than NSFD2.

Table 4. Convergence of nonstandard methods (3.3), (3.10) and standard method (3.1) with $h = 0.1$ at $T = 2$.

Δt	NSFD1		NSFD2		SFD	
	L_∞	L_2	L_∞	L_2	L_∞	L_2
0.5	1.4577×10^{-7}	2.7818×10^{-8}	1.1336×10^{-7}	2.1864×10^{-8}	2.5028×10^{-7}	4.4885×10^{-8}
0.05	1.8063×10^{-7}	3.2797×10^{-8}	1.1524×10^{-7}	2.2920×10^{-8}	2.4638×10^{-7}	3.7109×10^{-8}
0.005	2.1981×10^{-7}	1.5703×10^{-7}	2.5501×10^{-7}	3.5082×10^{-8}	2.5247×10^{-8}	3.7545×10^{-8}

Figures 1, 2, 3, 4 represent the surface of the traveling wave solution of the exact solution (1.3), SFD solution (3.1), the NSFD1 solution (3.3) and NSFD2 solution(3.10) on the space-time domain $(x, t) = [0, 1] \times [0, 2]$. Relative errors of the three schemes for different set of parameters a, b, c are displayed in Figure 5. From the figure, we see that the NSFD methods (3.3) and (3.10) have smaller errors than the SFD method (3.1) in all three cases. This shows the advantage of the nonstandard schemes over the standard one. Moreover, among all three methods, NSFD2 method (3.10) is more accurate than the other methods which confirms the Table 1–5.

Table 5. Comparison of the relative errors between the proposed methods with $\Delta t = 0.01$, $h = 0.1$.

x		$T = 1$	$T = 5$	$T = 10$
	NSFD1	3.1766×10^{-5}	8.0176×10^{-4}	0.2720
0.2	NSFD2	2.8767×10^{-5}	7.2638×10^{-4}	0.2952
	SFD	3.1375×10^{-5}	0.0024	4.5586
	NSFD1	2.73209×10^{-5}	1.9413×10^{-4}	0.1559
0.5	NSFD2	1.9718×10^{-5}	1.5649×10^{-4}	0.2119
	SFD	2.7586×10^{-5}	3.8209×10^{-4}	1.2773
	NSFD1	2.2626×10^{-5}	1.4856×10^{-4}	0.0654
0.8	NSFD2	1.6768×10^{-5}	1.5657×10^{-4}	0.0730
	SFD	2.2561×10^{-5}	5.1169×10^{-4}	0.4061

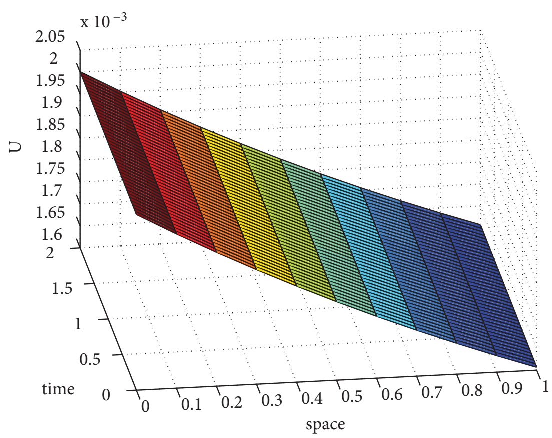


Figure 1. Exact :Surface of the wave for $a = 1$, $b = 1$, $c = 0.001$, $h = 0.1$, $\Delta t = 0.005$.

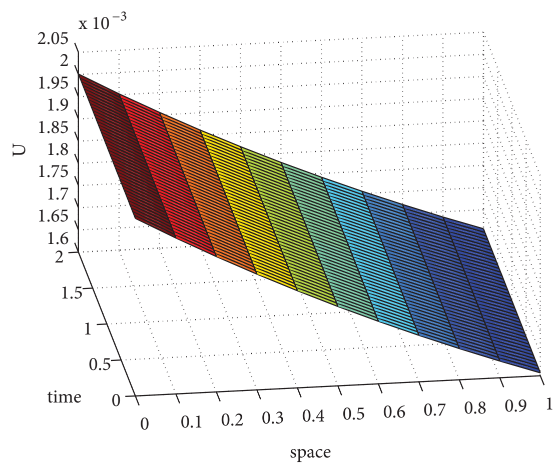


Figure 2. SFD :Surface of the wave for $a = 1$, $b = 1$, $c = 0.001$, $h = 0.1$, $\Delta t = 0.005$.

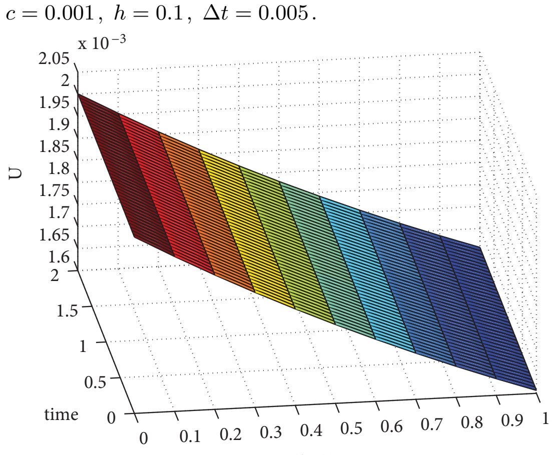


Figure 3. NSFD1 :Surface of the wave for $a = 1$, $b = 1$, $c = 0.001$, $h = 0.1$, $\Delta t = 0.005$.

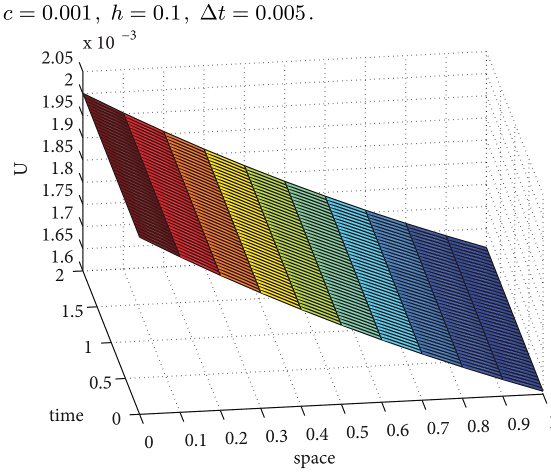
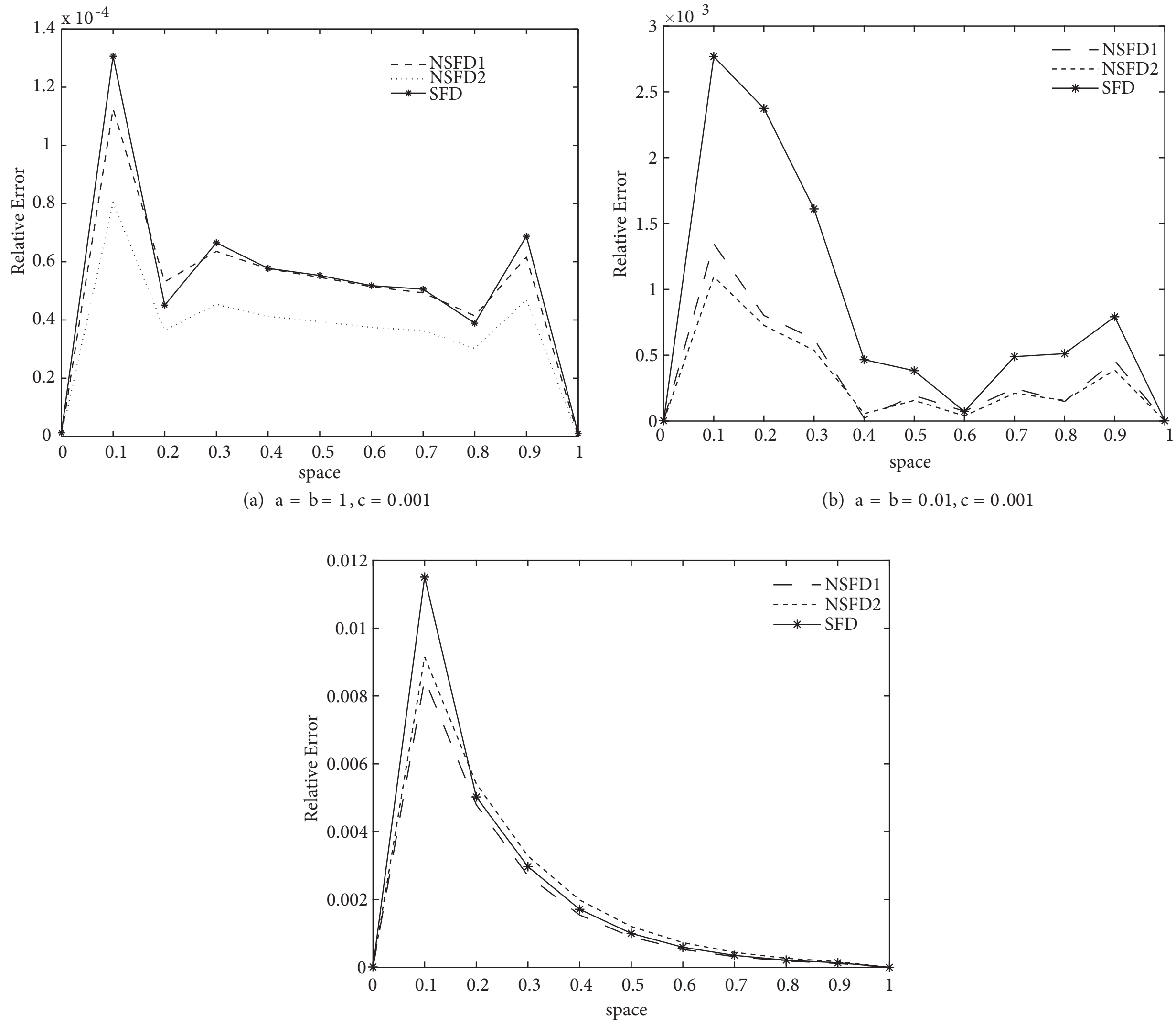


Figure 4. NSFD2 :Surface of the wave for $a = 1$, $b = 1$, $c = 0.001$, $h = 0.1$, $\Delta t = 0.005$.



(a) a = b = 1, c = 0.001

(b) a = b = 0.01, c = 0.001

(c) a = 0.01, b = c = 0.001

Figure 5. Relative errors for $h = 0.1$, $\Delta t = 0.005$ with different set of parameters.

5. Conclusion

In this paper, we present two exact finite difference schemes for the $B(2,2)$ equation in terms of travelling wave solution. Two nonstandard finite difference (NSFD) schemes named as NSFD1 and NSFD2 for the $B(2,2)$ equation are constructed. The numerical results obtained by the NSFD schemes are compared to the exact solution and a standard finite difference (SFD) scheme. The results presented graphically reveal that NSFD schemes show better performance than the SFD scheme. In addition, we have shown that NSFD2 scheme is more accurate than the NSFD1 scheme. We conclude that NSFD schemes are robust tools for nonlinear evolution equation $B(2,2)$. The method described in this paper can be extended to many other PDEs involving nonlinear derivative terms to obtain exact and NSFD scheme.

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