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Notes on multivalent Bazilević functions defined by higher order derivatives

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Abstract: In this paper we consider two subclasses $\mathcal{B}(p,q,\alpha,\beta)$ and $\mathcal{B}_1(p,q,\alpha,\beta)$ of $p$-valently Bazilević functions defined by higher order derivatives, and we defined and studied some properties of the images of the functions of these classes by the integral operators $I_{n,p}$ and $J_{n,p}$ for multivalent functions, defined by using higher order derivatives.

Key words: $p$-valent functions, $p$-valent starlike and convex functions, Bazilević functions, higher order derivatives, integral operator

1. Introduction

Let us denote by $\mathcal{A}(p)$, $p \in \mathbb{N} := \{1,2,\ldots\}$, the class of multivalent analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad z \in \mathbb{U} := \{z \in \mathbb{C} : |z| < 1\},$$

and let $\mathcal{A} := \mathcal{A}(1)$.

For $0 \leq \gamma < p - q$, $p > q$, $p \in \mathbb{N}$, and $q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we say that the function $f \in \mathcal{A}(p)$ is in the class $\mathcal{S}_{p,q}(\gamma)$ if it satisfies the inequality

$$\text{Re} \left( \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \right) > \gamma, \quad z \in \mathbb{U},$$

and is in the class $\mathcal{K}_{p,q}(\gamma)$ if it satisfies

$$\text{Re} \left( 1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right) > \gamma, \quad z \in \mathbb{U}.$$

The classes $\mathcal{S}_{p,q}^*(\gamma)$ and $\mathcal{K}_{p,q}(\gamma)$, were introduced and studied by Aouf [5, 7, 8]. Note that $\mathcal{S}_{p,0}^*(\gamma) =: \mathcal{S}_p^*(\gamma)$ and $\mathcal{K}_{p,0}(\gamma) =: \mathcal{K}_p(\gamma)$, which are, respectively, the classes of $p$-valent starlike and convex functions of order $\gamma$, with $0 \leq \gamma < p$ (see Owa [17] and Aouf [1, 2, 10]).

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Definition 1.1 (i) A function \( f \in \mathcal{A}(p) \) is said to be \( p \)-valently Bazilević functions defined by higher order derivative of type \( \alpha \) (\( \alpha > 0 \)) and order \( \beta \) (\( 0 \leq \beta < p - q, \ p > q \)), if there exists a function \( g \in S_{p,q}^*(0) =: S_{p,q}^* \) such that

\[
\Re \left[ \frac{z f^{(1+q)}(z)}{f^{(q)}(z)} \left( \frac{f^{(q)}(z)}{g^{(q)}(z)} \right)^\alpha \right] > \beta, \ z \in \mathbb{U},
\]

where the power is the principal one, and we denote by \( B(p,q,\alpha,\beta) \) to the class of such functions.

(ii) Further, let \( B_1(p,q,\alpha,\beta) \subseteq B(p,q,\alpha,\beta) \) the subclass of functions for which \( g \in \mathcal{A}(p) \), such that \( g^{(q)}(z) = \delta(p,q)z^{p-q} \), and therefore \( g \in S_{p,q}^* \), where

\[
\delta(p,q) = \frac{p^!}{(p-q)^!}, \ (p > q).
\]

Remark that for special choices of the parameters we obtain the following previously studied subclasses of \( B(p,q,\alpha,\beta) \) and \( B_1(p,q,\alpha,\beta) \):

(i) \( B(p,0,\alpha,\beta) := B(p,\alpha,\beta) \), the class of \( p \)-valently Bazilević functions of type \( \alpha \) (\( \alpha > 0 \)) and order \( \beta \) (\( 0 \leq \beta < p \)) (see Irmak et al. [14], Goswami and Bansal [13], Aouf [6] and Owa [19]);

(ii) \( B_1(p,0,\alpha,\beta) := B_1(p,\alpha,\beta) \) (see Owa [19] and Aouf [6]);

(iii) \( B(1,0,\alpha,\beta) := B(\alpha,\beta) \) and \( B_1(1,0,\alpha,\beta) := B_1(\alpha,\beta) \) (see Owa and Obradović [20]);

(iv) \( B(p,q,1,\beta) := C_{p,q}(\beta) = \left\{ f \in \mathcal{A}(p) : \Re \left[ \frac{z f^{(1+q)}(z)}{g^{(q)}(z)} \right] > \beta, \ z \in \mathbb{U}, \ g \in S_{p,q}^* \right\} \) (see Aouf [4]), and \( C_{p,0}(\beta) := C_p(\beta) \) (see Aouf [3, 9]).

2. Integral operator \( I_{n,p}f^{(q)} \)

Unless stated otherwise, we assume that \( \alpha > 0, \ p \in \mathbb{N}, \ q \in \mathbb{N}_0, \ p > q, \ 0 \leq \beta < p - q, \ z = re^{i\theta} \in \mathbb{U}, \) and all the powers are the principal ones.

For \( f \in \mathcal{A}(p) \), we define the integral operator \( I_{n,p}f^{(q)} \) by

\[
I_{0,p}f^{(q)}(z) := \left( \frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} \right)^\alpha,
\]

and

\[
I_{n,p}f^{(q)}(z) := z^{-1} \int_0^1 I_{n-1,p}f^{(q)}(t)dt, \ n \in \mathbb{N}.
\]

Note that the integral operator \( I_{n,p}f^{(0)} := I_{n,p}f \) (\( f \in \mathcal{A}(p) \)) was studied by Owa [17, 18] and the integral operator \( I_{n,1}f := I_nf \) (\( f \in \mathcal{A} \)) was studied by Halenbeck [12], Thomas [25] and Halim and Thomas [11].

For \( f \in \mathcal{A}(p) \), Owa [19] proved the following result:

**Theorem A** If \( f \in B_1(p,0,\alpha,\beta) := B_1(p,\alpha,\beta) \) (\( p \in \mathbb{N}, \ \alpha > 0, \ 0 \leq \beta < p \)), then

\[
\Re I_{n,p}f(z) \geq \gamma_n(r) > \gamma_n(1), \ z \in \mathbb{U}, \ (n \in \mathbb{N}_0)
\]

where

\[
\beta \leq \frac{\beta}{p} < \frac{\beta}{p} + \left( 1 - \frac{\beta}{p} \right) \left( -1 + 2p\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}p^{k-1}}{k^n(k - 1 + p\alpha)} \right).
\]

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The equality in (2.1) is attained for the function $f$ given by

$$f(z) = \left\{ \alpha \int_0^z t^{p\alpha - 1} \left[ \beta + (p - \beta) \frac{1 - t}{1 + t} \right] dt \right\}^{\frac{1}{\alpha}}.$$ 

Also, for $f \in \mathcal{A}(p)$, Owa [18] proved that:

**Theorem B** If $f \in \mathcal{A}(p)$ satisfies

$$\text{Re} \left\{ \frac{zf'(z)}{pf(z)} \left( \frac{f(z)}{2^p} \right)^{\alpha} \right\} > 0, \quad z \in \mathbb{U}, \quad (\alpha > 0),$$

then

$$\text{Re} I_{n,p,f}(z) \geq \gamma_n(r) > \gamma_n(1), \quad z \in \mathbb{U}, \quad (n \in \mathbb{N}_0) \quad (2.2)$$

and

$$0 < \gamma_n(r) := -1 + 2p\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}r^{k-1}}{k^n(k - 1 + p\alpha)} < 1.$$ 

The equality in (2.2) is attained for the function $f$ given by

$$f(z) = \left( p\alpha \int_0^z t^{p\alpha - 1} \left( \frac{1 - t}{1 + t} \right) dt \right)^{\frac{1}{\alpha}}.$$ 

The main result regarding this integral operator is the next theorem:

**Theorem 2.1** If $f \in B_1(p,q,\alpha,\beta)$, then

$$\text{Re} I_{n,p,f^{(q)}}(z) \geq \gamma_{p,q}^n(r) > \gamma_{p,q}^n(1), \quad z \in \mathbb{U}, \quad r = |z|, \quad (n \in \mathbb{N}_0) \quad (2.3)$$

and

$$\frac{\beta}{p - q} < \gamma_{p,q}^n(r) := \frac{\beta}{p - q} + \left( 1 - \frac{\beta}{p - q} \right) \left( -1 + 2(p - q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}r^{k-1}}{k^n[k - 1 + (p - q)\alpha]} \right). \quad (2.4)$$

The equality in (2.3) is attained for the function $f \in \mathcal{A}(p)$ given by

$$f^{(q)}(z) = \delta(p,q) \left\{ \alpha \int_0^z t^{(p-q)\alpha - 1} \left[ \beta + (p - q - \beta) \frac{1 - t}{1 + t} \right] dt \right\}^{\frac{1}{\alpha}}.$$ 

**Proof** Since $f \in B_1(p,q,\alpha,\beta)$, then we have

$$\text{Re} h(z) > \frac{\beta}{p - q}, \quad z \in \mathbb{U},$$

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where the function $h$ is defined by

$$h(z) = \frac{f^{(1+q)}(z)}{\delta(p, q + 1)z^{p-q-1}} \left( \frac{f^{(q)}(z)}{\delta(p, q)z^{p-q}} \right)^{\alpha-1}, \quad z \in \mathbb{U},$$

and $h(0) = 1$. Thus, it is easy to check that

$$\left( \frac{f^{(q)}(z)}{\delta(p, q)z^{p-q}} \right)^{\alpha} = \frac{(p-q)\alpha}{z^{(p-q)\alpha}} \int_{0}^{z} t^{(p-q)\alpha-1} h(t)dt, \quad z \in \mathbb{U},$$

that is

$$\text{Re} \int_{0}^{r} f^{(q)}(z) = \text{Re} \left( \frac{f^{(q)}(z)}{\delta(p, q)z^{p-q}} \right)^{\alpha} = \text{Re} \left[ \frac{(p-q)\alpha}{z^{(p-q)\alpha}} \int_{0}^{z} t^{(p-q)\alpha-1} h(t)dt \right], \quad z = re^{i\theta}. \quad (2.5)$$

Substituting $t = re^{i\theta}$ in (2.5), we have

$$\text{Re} \int_{0}^{r} f^{(q)}(z) = \frac{(p-q)\alpha}{r^{(p-q)\alpha}} \int_{0}^{r} r^{(p-q)\alpha-1} \text{Re} h (re^{i\theta}) d\rho, \quad z = re^{i\theta}. \quad (2.6)$$

It is well-known that for $q \in \mathcal{A}$, with $\text{Re} q(z) > 0$ for all $z \in \mathbb{U}$, (see [16, p. 532]) the next inequality holds:

$$\text{Re} q(z) \geq \frac{1-r}{1+r}, \quad |z| = r < 1, \quad (2.7)$$

therefore

$$\text{Re} h(z) \geq \frac{\beta}{p-q} + \left( 1 - \frac{\beta}{p-q} \right) \frac{1-r}{1+r}, \quad |z| = r < 1. \quad (2.8)$$

From (2.6) and (2.8) we obtain

$$\text{Re} \int_{0}^{r} f^{(q)}(z) \geq \frac{(p-q)\alpha}{r^{(p-q)\alpha}} \int_{0}^{r} r^{(p-q)\alpha-1} \left[ \frac{\beta}{p-q} + \left( 1 - \frac{\beta}{p-q} \right) \frac{1-r}{1+r} \right] d\rho$$

$$= \frac{\beta}{p-q} + \left( 1 - \frac{\beta}{p-q} \right) \left[ -1 + \frac{2(p-q)\alpha}{r^{(p-q)\alpha}} \int_{0}^{r} r^{(p-q)\alpha-1} d\rho \right], \quad |z| = r < 1. \quad (2.9)$$

Taking $\rho = r\varphi$ in (2.9) we deduce

$$\text{Re} \int_{0}^{r} f^{(q)}(z) \geq \frac{\beta}{p-q} + \left( 1 - \frac{\beta}{p-q} \right) \left[ -1 + 2(p-q)\alpha \int_{0}^{1} \frac{\varphi^{(p-q)\alpha-1}}{1 + r\varphi} d\varphi \right], \quad |z| = r < 1,$$

and using that

$$\int_{0}^{1} \frac{\varphi^{(p-q)\alpha-1}}{1 + r\varphi} d\varphi = \int_{0}^{1} \left[ \varphi^{(p-q)\alpha-1} \sum_{s=0}^{\infty} (-1)^{s} r^{s} \varphi^{s} \right] d\varphi = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{(p-q)\alpha + k - 1},$$

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we have
\[
\text{Re} I_{n,p,f(q)}(z) \geq \frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right) \left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}r^{k-1}}{k-1 + (p-q)\alpha}\right] = \gamma_{p,q}^0(r), \ |z| = r < 1.
\]

It easy to see that
\[
\text{Re} I_{n+1,p,f(q)}(z) = \text{Re} \left[\frac{1}{z} \int_0^r I_{n,p,f(q)}(t)dt\right] = \frac{1}{r} \int_0^r \text{Re} I_{n,p,f(q)}(\rho e^{i\theta})d\rho
\]
\[
\geq \frac{1}{r} \int_0^r \left\{ \frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right) \left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}r^{k-1}}{k-1 + (p-q)\alpha}\right]\right\} d\rho
\]
\[
= \frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right) \left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}r^{k-1}}{k(k-1 + (p-q)\alpha)}\right] = \gamma_{p,q}^{n+1}(r), \ |z| = r < 1.
\]

and by mathematical induction, we conclude that
\[
\text{Re} I_{n+1,p,f(q)}(z) = \text{Re} \left[\frac{1}{z} \int_0^r I_{n,p,f(q)}(t)dt\right] = \frac{1}{r} \int_0^r \text{Re} I_{n,p,f(q)}(\rho e^{i\theta})d\rho
\]
\[
\geq \frac{1}{r} \int_0^r \left\{ \frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right) \left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}r^{k-1}}{k(k-1 + (p-q)\alpha)}\right]\right\} d\rho
\]
\[
= \frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right) \left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}r^{k-1}}{k(k-1 + (p-q)\alpha)}\right] = \gamma_{p,q}^{n+1}(r), \ |z| = r < 1.
\]

If we define the function \(\Phi_{p,q}^{n,\alpha}\) by
\[
\Phi_{p,q}^{n,\alpha}(r) = (p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}r^{k-1}}{k(k-1 + (p-q)\alpha)}, \ 0 < r < 1,
\]
according to the result of Thomas [25, page 20] we get \(\frac{1}{2} < \Phi_{p,q}^{n,\alpha}(r) < 1\), and this inequality implies our conclusion (2.4). Moreover,
\[
r\Phi_{p,q}^{n,\alpha}(r) = \int_0^r \Phi_{p,q}^{n-1,\alpha}(\rho)d\rho, \ n \in \mathbb{N},
\]
thus \((\Phi_{p,q}^{n,\alpha}(r))^\prime < 0\) and \(\gamma_{p,q}^{n}(r)\) decreases with \(r\) as \(r \to 1\) for fixed \(n\), and increases to 1 when \(n \to \infty\) for fixed \(r\), which completes our proof.

Remark 2.2 (i) Taking \(q = 0\) in Theorem 2.1 we obtain Theorem A of Owa [19];

(ii) Putting \(\beta = q = 0\) in Theorem 2.1 we obtain Theorem B due to Owa [18];

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Taking $\beta = q = 0$ and $p = 1$, in Theorem 2.1 we obtain the result of Thomas [25] and Halim and Thomas [11];

For $\beta = q = 0$ and $p = 1$, Theorem 2.1 reduces to the result of Hallenbeck [12];

Our result of Theorem 2.1 with (i) $q = 0$, (ii) $q = \beta = 0$, (iii) $q = \beta = 0$ and $\alpha = p^{-1}$ ($p \in \mathbb{N}$) improve the results of Owa [19, Lemma 4, Corollaries 3 and 4, respectively].

Putting $q = 0$ and $\alpha = 1$ in Theorem 2.1 we get the following special case:

**Corollary 2.3** If $f \in \mathcal{A}(p)$ satisfies

$$\text{Re} \frac{f'(z)}{z^{p-1}} > \beta, \ z \in \mathbb{U}, \ (0 \leq \beta < p)$$

then

$$\text{Re} I_{n,p}f(z) \geq \gamma_n^p(r) > \gamma_n^p(1), \ z \in \mathbb{U}, \ r = |z|, \ (n \in \mathbb{N}_0) \quad (2.10)$$

and

$$\frac{\beta}{p} < \gamma_n^p(r) = \frac{\beta}{p} + \left(1 - \frac{\beta}{p}\right) \left(-1 + 2p \sum_{k=1}^{\infty} \frac{(-1)^{k+1}r^{k-1}}{k^n(k-1+p)}\right).$$

The equality in (2.10) is attained for the function

$$f(z) = z^p + 2(p - \beta) \sum_{k=1}^{\infty} (-1)^k \frac{z^{p+k}}{p+k}.$$

**Remark 2.4** Our result of Corollary 2.3 is an improvement of the result of Saitoh [24, Theorem 1, with $j = 1$ and Corollary 2], and of Aouf [6, Theorem 2, with $\alpha = n = 1$]

For the special case $\alpha = \frac{1}{p-q}$; ($p > q$) Theorem 2.1 reduces to the next special case:

**Corollary 2.5** If $f \in \mathcal{A}(p)$ satisfies

$$\text{Re} \left[ \frac{f^{(1+q)}(z)}{f^{(q)}(z)} \left(\frac{f^{(q)}(z)}{\delta(p,q)}\right)^{\frac{1}{p-q}} \right] > \beta, \ z \in \mathbb{U}, \ (0 \leq \beta < p-q)$$

then

$$\text{Re} I_{n,p}f^{(q)}(z) \geq \gamma_{p,q}^n(r) > \gamma_{p,q}^n(1), \ z \in \mathbb{U}, \ r = |z|, \ (n \in \mathbb{N}_0) \quad (2.11)$$

and

$$\frac{\beta}{p-q} < \gamma_{p,q}^n(r) = \frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right) \left(-1 + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}r^{k-1}}{k^{n+1}}\right) < 1.$$

The equality in (2.11) is attained for the function $f \in \mathcal{A}(p)$ given by

$$f^{(q)}(z) = \delta(p,q) \left\{ \left(\frac{2\beta}{p-q} - 1\right) z + 2 \left(1 - \frac{\beta}{p-q}\right) \log(1 + z) \right\}^{p-q}.$$
Remark 2.6 For the special case \( q = 0 \), the result of Corollary 2.5 is an improvement of the result due to Owa [19, Corollary 7].

Putting \( p = 1 \) and \( q = 0 \) in Corollary 2.5 we get:

Corollary 2.7 If \( f \in \mathcal{A} \) satisfies
\[
\text{Re} f'(z) > \beta, \ z \in \mathbb{U}, \ (0 \leq \beta < 1)
\]
then
\[
\text{Re} I_n f(z) \geq \gamma_n(r) > \gamma_n(1), \ z \in \mathbb{U}, \ r = |z|, \ (n \in \mathbb{N}_0)
\]
and
\[
\beta < \gamma_n(r) = \beta + (1 - \beta) \left( -1 + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^{n+1}} \right).
\]
The equality in (2.12) is attained for the function
\[
f(z) = (2\beta - 1)z + 2(1 - \beta) \log(1 + z).
\]

Remark 2.8 (i) The result of Corollary 2.7 was also obtained by Owa [19, Corollary 8], Hallenbeck [12, with \( n = \beta = 0 \)], Ling et al. [15, Corollary 3], and Patel and Rout [21, Corollary 3];

(ii) The above corollary improve the results of Owa and Obradović [20, Theorem 4 with \( \alpha = 1 \) and Corollary 4], Saitoh [23, Corollary 3], Saitoh [24, Corollary 8 with \( \lambda = 1 \)], and Ponnusamy and Karunakran [22, with \( k = m = 1 \)].

3. Integral operator \( J_n f^{(q)} \)

For \( f \in \mathcal{A}(p) \), we define the integral operator
\[
J_0 f^{(q)}(z) := \frac{f^{(q)}(z)}{\delta(p, q)z^{p-q}},
\]
and
\[
J_n f^{(q)}(z) := \frac{a + 1}{z^{a+1}} \int_0^z t^a J_{n-1} f^{(q)}(t) dt, \ (a > -1, \ n \in \mathbb{N}).
\]

For the operator \( J_n f^{(q)} \) we obtained the next result:

Theorem 3.1 If \( f \in \mathcal{A}(p) \) satisfies
\[
\text{Re} \frac{f^{(q)}(z)}{\delta(p, q)z^{p-q}} > \alpha, \ z \in \mathbb{U}, \ (\alpha < 1),
\]
then
\[
\text{Re} J_n f^{(q)}(z) \geq \gamma_n(r) > \gamma_n(1), \ z \in \mathbb{U}, \ r = |z|, \ (n \in \mathbb{N}_0)
\]
and

\[ 0 < \gamma_n(r) := 1 + 2(a + 1)^n(1 - \alpha) \sum_{k=1}^{\infty} \frac{(-r)^k}{(k + a + 1)^n} < 1. \]

The equality in (3.1) is attained for the function \( f \in A(p) \) given by

\[ f^{(q)}(z) = \delta(p, q)z^{p-q} \left[ \alpha + (1 - \alpha) \frac{1 - z}{1 + z} \right]. \]

**Proof** For \( n = 0 \) the implication is trivial. For \( n = 1 \), if we denote

\[ g(z) = \frac{1}{\alpha} \left[ \frac{f^{(q)}(z)}{\delta(p, q)z^{p-q}} - \alpha \right], \quad z \in \mathbb{U}, \]

then, from our assumption we have \( \text{Re} g(z) > 0, \quad z \in \mathbb{U}, \) and \( g(0) = 1 \). Using the inequality (2.7) for the function \( g \) and letting \( z = re^{i\theta} \) and \( t = \rho e^{i\theta} \), for \( a > -1 \) we get

\[
\text{Re} J_{n+1} f^{(q)}(z) = \text{Re} \left( \frac{a+1}{\rho^{a+1}} \int_0^r \frac{t^a J_n f^{(q)}(t) dt}{\rho^{a+1}} \right) \geq \frac{a+1}{\rho^{a+1}} \int_0^r \rho^a \left[ \alpha + (1 - \alpha) \frac{1 - \rho}{1 + \rho} \right] d\rho \\
= \frac{a+1}{\rho^{a+1}} \int_0^r \rho^a \left[ 1 + 2(1 - \alpha) \sum_{k=1}^{\infty} (-\rho)^k \right] d\rho = 1 + \frac{2(a+1)(1-\alpha)}{r^{a+1}} \int_0^r \sum_{k=1}^{\infty} (-1)^k \rho^{k+a} d\rho \\
= 1 + 2(a + 1)(1 - \alpha) \sum_{k=1}^{\infty} \frac{(-r)^k}{k + a + 1}, \quad |z| = r,
\]

thus (3.1) holds for \( n = 1 \). Further, assuming that (3.1) holds for a fixed \( n \in \mathbb{N} \), we have

\[
\text{Re} J_{n+1} f^{(q)}(z) = \text{Re} \left( \frac{a+1}{\rho^{a+1}} \int_0^r \frac{t^a J_n f^{(q)}(t) dt}{\rho^{a+1}} \right) = \frac{a+1}{\rho^{a+1}} \int_0^r \rho^a \text{Re} J_n f^{(q)} (\rho e^{i\theta}) d\rho \\
\geq \frac{a+1}{\rho^{a+1}} \int_0^r \rho^a + 2(a + 1)^n(1 - \alpha) \sum_{k=1}^{\infty} \frac{(-1)^k \rho^{k+a}}{(k + a + 1)^n} d\rho \\
= 1 + 2(a + 1)^{n+1}(1 - \alpha) \sum_{k=1}^{\infty} \frac{(-r)^k}{(k + a + 1)^{n+1}} \gamma_{n+1}(r), \quad |z| = r.
\]

Moreover, it is easy to see that \( 0 < \gamma_n < 1 \), which completes our proof. \( \square \)

Taking \( \alpha = \frac{p-q}{p-q+\beta} \), \( p > q, \beta > 0 \) in the above theorem we get the next special case:

**Corollary 3.2** If \( f \in A(p) \) satisfies

\[ \text{Re} \frac{f^{(q)}(z)}{\delta(p, q)z^{p-q}} = \frac{p-q}{p-q+\beta}, \quad z \in \mathbb{U}, \quad (\beta > 0), \]

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then
\[ \text{Re} J_n f^{(q)}(z) \geq \gamma_n(r) > \gamma_n(1), \quad z \in \mathbb{U}, \quad r = |z|, \quad (n \in \mathbb{N}_0) \tag{3.2} \]
and
\[ 0 < \gamma_n(r) := 1 + \frac{2\beta(a+1)^n}{p - q + \beta} \sum_{k=1}^{\infty} \frac{(-r)^k}{(k+a+1)^n} < 1. \]

The equality in (3.2) is attained for the function \( f \in \mathcal{A}(p) \) given by
\[ f^{(q)}(z) = \frac{\delta(p,q)z^{p-q} - 1 + \beta + (1 - \beta)z}{1 + z}. \]

**Remark 3.3** Putting \( q = 0 \) in Theorem 3.1 and in Corollary 3.2 we obtain the results of Owa [18, Theorem 2 and Corollary 4] and Owa [19, Theorem 2 and Corollary 10].

**References**


