[Turkish Journal of Mathematics](https://journals.tubitak.gov.tr/math)

[Volume 45](https://journals.tubitak.gov.tr/math/vol45) | [Number 2](https://journals.tubitak.gov.tr/math/vol45/iss2) Article 1

1-1-2021

Notes on multivalent Bazilevic functions defined by higher order derivatives

MOHAMED K. AOUF

ADELA O. MOSTAFA

TEODOR BULBOACA

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 624 – 633
© TÜBİTAK
doi:10.3906/mat-1911-6

Notes on multivalent Bazilević functions defined by higher order derivatives

Mohamed K. AOUF[1,*], **Adela O. MOSTAFA**[1], **Teodor BULBOACĂ**[2]
[1]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt
[2]Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Cluj-Napoca, Romania

Received: 02.11.2019 • **Accepted/Published Online:** 09.03.2020 • **Final Version:** 26.03.2021

Abstract: In this paper we consider two subclasses $B(p,q,\alpha,\beta)$ and $B_1(p,q,\alpha,\beta)$ of p-valently Bazilević functions defined by higher order derivatives, and we defined and studied some properties of the images of the functions of these classes by the integral operators $\mathrm{I}_{n,p}$ and $\mathrm{J}_{n,p}$ for multivalent functions, defined by using higher order derivatives.

Key words: p-valent functions, p-valent starlike and convex functions, Bazilević functions, higher order derivatives, integral operator

1. Introduction

Let us denote by $\mathcal{A}(p)$, $p \in \mathbb{N} := \{1, 2, \dots\}$, the class of multivalent analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,\ z \in \mathbb{U} := \{z \in \mathbb{C} : |z| < 1\},$$

and let $\mathcal{A} := \mathcal{A}(1)$.

For $0 \le \gamma < p - q$, $p > q$, $p \in \mathbb{N}$, and $q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we say that the function $f \in \mathcal{A}(p)$ is in the class $\mathbb{S}^*_{p,q}(\gamma)$ if it satisfies the inequality

$$\operatorname{Re} \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} > \gamma,\ z \in \mathbb{U},$$

and is in the class $\mathbb{K}_{p,q}(\gamma)$ if it satisfies

$$\operatorname{Re} \left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right) > \gamma,\ z \in \mathbb{U}.$$

The classes $\mathbb{S}^*_{p,q}(\gamma)$ and $\mathbb{K}_{p,q}(\gamma)$, were introduced and studied by Aouf [5, 7, 8]. Note that $\mathbb{S}^*_{p,0}(\gamma) =: \mathbb{S}^*_p(\gamma)$ and $\mathbb{K}_{p,0}(\gamma) =: \mathbb{K}_p(\gamma)$, which are, respectively, the classes of p-valent starlike and convex functions of order γ, with $0 \le \gamma < p$ (see Owa [17] and Aouf [1, 2, 10]).

*Correspondence: mkaouf127@yahoo.com
2010 *AMS Mathematics Subject Classification:* 30C45

Definition 1.1 *(i) A function* $f \in \mathcal{A}(p)$ *is said to be* p-valently Bazilević functions defined by higher order derivative of type α, $(\alpha > 0)$ and order β $(0 \le \beta < p-q,\ p > q)$, *if there exists a function* $g \in \mathbb{S}^*_{p,q}(0) =: \mathbb{S}^*_{p,q}$ *such that*

$$\operatorname{Re}\left[\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\left(\frac{f^{(q)}(z)}{g^{(q)}(z)}\right)^{\alpha}\right] > \beta,\ z \in \mathbb{U},$$

where the power is the principal one, and we denote by $B(p,q,\alpha,\beta)$ *to the class of such functions.*

(ii) Further, let $B_1(p,q,\alpha,\beta) \subset B(p,q,\alpha,\beta)$ *the subclass of functions for which* $g \in \mathcal{A}(p)$, *such that* $g^{(q)}(z) = \delta(p,q)z^{p-q}$, *and therefore* $g \in \mathbb{S}^*_{p,q}$, *where*

$$\delta(p,q) = \frac{p!}{(p-q)!},\ (p > q).$$

Remark that for special choices of the parameters we obtain the following previously studied subclasses of $B(p,q,\alpha,\beta)$ and $B_1(p,q,\alpha,\beta)$:

(i) $B(p,0,\alpha,\beta) =: B(p,\alpha,\beta)$, the class of p-valently Bazilević functions of type α $(\alpha > 0)$ and order β $(0 \le \beta < p)$ (see Irmak et al. [14], Goswami and Bansal [13], Aouf [6] and Owa [19]);

(ii) $B_1(p,0,\alpha,\beta) =: B_1(p,\alpha,\beta)$ (see Owa [19] and Aouf [6]);

(iii) $B(1,0,\alpha,\beta) =: B(\alpha,\beta)$ and $B_1(1,0,\alpha,\beta) =: B_1(\alpha,\beta)$ (see Owa and Obradović [20]);

(iv) $B(p,q,1,\beta) =: C_{p,q}(\beta) = \left\{ f \in \mathcal{A}(p) : \operatorname{Re}\dfrac{zf^{(1+q)}(z)}{g^{(q)}(z)} > \beta,\ z \in \mathbb{U},\ g \in \mathbb{S}^*_{p,q} \right\}$ (see Aouf [4]), and $C_{p,0}(\beta) =: C_p(\beta)$ (see Aouf [3, 9]).

2. Integral operator $\mathrm{I}_{n,p}f^{(q)}$

Unless stated otherwise, we assume that $\alpha > 0$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, $p > q$, $0 \le \beta < p-q$, $z = re^{i\theta} \in \mathbb{U}$, and all the powers are the principal ones.

For $f \in \mathcal{A}(p)$, we define the integral operator $\mathrm{I}_{n,p}f^{(q)}$ by

$$\mathrm{I}_{0,p}f^{(q)}(z) := \left(\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right)^{\alpha},$$

and

$$\mathrm{I}_{n,p}f^{(q)}(z) := z^{-1}\int_0^1 \mathrm{I}_{n-1,p}f^{(q)}(t)dt,\ n \in \mathbb{N}.$$

Note that the integral operator $\mathrm{I}_{n,p}f^{(0)} =: \mathrm{I}_{n,p}f$ $(f \in \mathcal{A}(p))$ was studied by Owa [17, 18] and the integral operator $\mathrm{I}_{n,1}f =: \mathrm{I}_n f$ $(f \in \mathcal{A})$ was studied by Halenbeck [12], Thomas [25] and Halim and Thomas [11].

For $f \in \mathcal{A}(p)$, Owa [19] proved the following result:

Theorem A *If* $f \in B_1(p,0,\alpha,\beta) =: B_1(p,\alpha,\beta)$ $(p \in \mathbb{N},\ \alpha > 0,\ 0 \le \beta < p)$, *then*

$$\operatorname{Re}\mathrm{I}_{n,p}f(z) \ge \gamma_n(r) > \gamma_n(1),\ z \in \mathbb{U},\ (n \in \mathbb{N}_0) \tag{2.1}$$

where

$$\frac{\beta}{p} < \gamma_n(r) := \frac{\beta}{p} + \left(1 - \frac{\beta}{p}\right)\left(-1 + 2p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k^n(k-1+p\alpha)}\right).$$

The equality in (2.1) is attained for the function f given by

$$f(z)=\left\{\alpha\int_0^z t^{p\alpha-1}\left[\beta+(p-\beta)\frac{1-t}{1+t}\right]dt\right\}^{\frac{1}{\alpha}}.$$

Also, for $f\in\mathcal{A}(p)$, Owa [18] proved that:

Theorem B *If $f\in\mathcal{A}(p)$ satisfies*

$$\operatorname{Re}\left[\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha}\right]>0,\ z\in\mathbb{U},\ (\alpha>0),$$

then

$$\operatorname{Re}\mathrm{I}_{n,p}f(z)\geq\widetilde{\gamma}_n(r)>\widetilde{\gamma}_n(1),\ z\in\mathbb{U},\ (n\in\mathbb{N}_0) \tag{2.2}$$

and

$$0<\widetilde{\gamma}_n(r):=-1+2p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k^n(k-1+p\alpha)}<1.$$

The equality in (2.2) is attained for the function f given by

$$f(z)=\left(p\alpha\int_0^z t^{p\alpha-1}\left(\frac{1-t}{1+t}\right)dt\right)^{\frac{1}{\alpha}}.$$

The main result regarding this integral operator is the next theorem:

Theorem 2.1 *If $f\in B_1(p,q,\alpha,\beta)$, then*

$$\operatorname{Re}\mathrm{I}_{n,p}f^{(q)}(z)\geq\gamma_{p,q}^n(r)>\gamma_{p,q}^n(1),\ z\in\mathbb{U},\ r=|z|,\ (n\in\mathbb{N}_0) \tag{2.3}$$

and

$$\frac{\beta}{p-q}<\gamma_{p,q}^n(r):=\frac{\beta}{p-q}+\left(1-\frac{\beta}{p-q}\right)\left(-1+2(p-q)\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k^n[k-1+(p-q)\alpha]}\right). \tag{2.4}$$

The equality in (2.3) is attained for the function $f\in\mathcal{A}(p)$ given by

$$f^{(q)}(z)=\delta(p,q)\left\{\alpha\int_0^z t^{(p-q)\alpha-1}\left[\beta+(p-q-\beta)\frac{1-t}{1+t}\right]dt\right\}^{\frac{1}{\alpha}}.$$

Proof Since $f\in B_1(p,q,\alpha,\beta)$, then we have

$$\operatorname{Re}h(z)>\frac{\beta}{p-q},\ z\in\mathbb{U},$$

where the function h is defined by

$$h(z)=\frac{f^{(1+q)}(z)}{\delta(p,q+1)z^{p-q-1}}\left(\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right)^{\alpha-1},\ z\in\mathbb{U},$$

and $h(0)=1$. Thus, it is easy to check that

$$\left(\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right)^{\alpha}=\frac{(p-q)\alpha}{z^{(p-q)\alpha}}\int\limits_0^z t^{(p-q)\alpha-1}h(t)dt,\ z\in\mathbb{U},$$

that is

$$\operatorname{Re}\mathrm{I}_{0,p}f^{(q)}(z)=\operatorname{Re}\left(\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right)^{\alpha}=\operatorname{Re}\left[\frac{(p-q)\alpha}{z^{(p-q)\alpha}}\int\limits_0^z t^{(p-q)\alpha-1}h(t)dt\right],\ z=re^{i\theta}. \tag{2.5}$$

Substituting $t=\rho e^{i\theta}$ in (2.5), we have

$$\operatorname{Re}\mathrm{I}_{0,p}f^{(q)}(z)=\frac{(p-q)\alpha}{r^{(p-q)\alpha}}\int\limits_0^r \rho^{(p-q)\alpha-1}\operatorname{Re}h\left(\rho e^{i\theta}\right)d\rho,\ z=re^{i\theta}. \tag{2.6}$$

It is well-known that for $q\in\mathcal{A}$, with $\operatorname{Re}q(z)>0$ for all $z\in\mathbb{U}$, (see [16, p. 532]) the next inequality holds:

$$\operatorname{Re}q(z)\geq\frac{1-r}{1+r},\ |z|=r<1, \tag{2.7}$$

therefore

$$\operatorname{Re}h(z)\geq\frac{\beta}{p-q}+\left(1-\frac{\beta}{p-q}\right)\frac{1-r}{1+r},\ |z|=r<1. \tag{2.8}$$

From (2.6) and (2.8) we obtain

$$\operatorname{Re}\mathrm{I}_{0,p}f^{(q)}(z)\geq\frac{(p-q)\alpha}{r^{(p-q)\alpha}}\int\limits_0^r \rho^{(p-q)\alpha-1}\left[\frac{\beta}{p-q}+\left(1-\frac{\beta}{p-q}\right)\frac{1-\rho}{1+\rho}\right]d\rho$$

$$=\frac{\beta}{p-q}+\left(1-\frac{\beta}{p-q}\right)\left[-1+\frac{2(p-q)\alpha}{r^{(p-q)\alpha}}\int\limits_0^r\frac{\rho^{(p-q)\alpha-1}}{1+\rho}d\rho\right],\ |z|=r<1. \tag{2.9}$$

Taking $\rho=r\varphi$ in (2.9) we deduce

$$\operatorname{Re}\mathrm{I}_{0,p}f^{(q)}(z)\geq\frac{\beta}{p-q}+\left(1-\frac{\beta}{p-q}\right)\left[-1+2(p-q)\alpha\int\limits_0^1\frac{\varphi^{(p-q)\alpha-1}}{1+r\varphi}d\varphi\right],\ |z|=r<1,$$

and using that

$$\int\limits_0^1\frac{\varphi^{(p-q)\alpha-1}}{1+r\varphi}d\varphi=\int\limits_0^1\left[\varphi^{(p-q)\alpha-1}\sum_{s=0}^{\infty}(-1)^s r^s\varphi^s\right]d\varphi=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{(p-q)\alpha+k-1},$$

we have

$$\operatorname{Re} \mathrm{I}_{0,p} f^{(q)}(z) \geq \frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right) \left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k-1+(p-q)\alpha}\right] = \gamma_{p,q}^{0}(r), \ |z| = r < 1.$$

It easy to see that

$$\operatorname{Re} \mathrm{I}_{1,p} f^{(q)}(z) = \operatorname{Re}\left[\frac{1}{z}\int_0^z \mathrm{I}_{0,p} f^{(q)}(t)dt\right] = \frac{1}{r}\int_0^r \operatorname{Re} \mathrm{I}_{0,p} f^{(q)}\left(\rho e^{i\theta}\right) d\rho$$

$$\geq \frac{1}{r}\int_0^r \left\{\frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right)\left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\rho^{k-1}}{k-1+(p-q)\alpha}\right]\right\} d\rho$$

$$= \frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right)\left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k(k-1+(p-q)\alpha)}\right] = \gamma_{p,q}^{1}(r), \ |z| = r < 1,$$

and by mathematical induction, we conclude that

$$\operatorname{Re} \mathrm{I}_{n+1,p} f^{(q)}(z) = \operatorname{Re}\left[\frac{1}{z}\int_0^z \mathrm{I}_{n,p} f^{(q)}(t)dt\right] = \frac{1}{r}\int_0^r \operatorname{Re} \mathrm{I}_{n,p} f^{(q)}\left(\rho e^{i\theta}\right) d\rho$$

$$\geq \frac{1}{r}\int_0^r \left\{\frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right)\left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^n[k-1+(p-q)\alpha]}\right]\right\} d\rho$$

$$= \frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right)\left[-1 + 2(p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^n(k-1+(p-q)\alpha)}\right] = \gamma_{p,q}^{n+1}(r), \ |z| = r < 1.$$

If we define the function $\Phi_{p,q}^{n,\alpha}$ by

$$\Phi_{p,q}^{n,\alpha}(r) = (p-q)\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^n[k-1+(p-q)\alpha]}, \ 0 < r < 1,$$

according to the result of Thomas [25, page 20] we get $\frac{1}{2} < \Phi_{p,q}^{n,\alpha}(r) < 1$, and this inequality implies our conclusion (2.4). Moreover,

$$r\Phi_{p,q}^{n,\alpha}(r) = \int_0^r \Phi_{p,q}^{n-1,\alpha}(\rho)d\rho, \ n \in \mathbb{N},$$

thus $\left(\Phi_{p,q}^{n,\alpha}(r)\right)' < 0$ and $\gamma_{p,q}^{n}(r)$ decreases with r as $r \to 1$ for fixed n, and increases to 1 when $n \to \infty$ for fixed r, which completes our proof. □

Remark 2.2 *(i) Taking $q = 0$ in Theorem 2.1 we obtain Theorem A of Owa [19];*

(ii) Putting $\beta = q = 0$ in Theorem 2.1 we obtain Theorem B due to Owa [18];

(iii) Taking $\beta = q = 0$ and $p = 1$, in Theorem 2.1 we obtain the result of Thomas [25] and Halim and Thomas [11];

(iv) For $\beta = q = 0$ and $p = \alpha = 1$, Theorem 2.1 reduces to the result of Hallenbeck [12];

(v) Our result of Theorem 2.1 with (i) $q = 0$, (ii) $q = \beta = 0$, (iii) $q = \beta = 0$ and $\alpha = p^{-1}$ $(p \in \mathbb{N})$ improve the results of Owa [19, Lemma 4, Corollaries 3 and 4, respectively].

Putting $q = 0$ and $\alpha = 1$ in Theorem 2.1 we get the following special case:

Corollary 2.3 *If $f \in \mathcal{A}(p)$ satisfies*

$$\operatorname{Re} \frac{f'(z)}{z^{p-1}} > \beta, \ z \in \mathbb{U}, \ (0 \le \beta < p)$$

then

$$\operatorname{Re} \mathrm{I}_{n,p} f(z) \ge \gamma_p^n(r) > \gamma_p^n(1), \ z \in \mathbb{U}, \ r = |z|, \ (n \in \mathbb{N}_0) \tag{2.10}$$

and

$$\frac{\beta}{p} < \gamma_p^n(r) = \frac{\beta}{p} + \left(1 - \frac{\beta}{p}\right)\left(-1 + 2p \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^n (k-1+p)}\right).$$

The equality in (2.10) is attained for the function

$$f(z) = z^p + 2(p-\beta) \sum_{k=1}^{\infty} (-1)^k \frac{z^{p+k}}{p+k}.$$

Remark 2.4 *Our result of Corollary 2.3 is an improvement of the result of Saitoh [24, Theorem 1, with $j = 1$ and Corollary 2], and of Aouf [6, Theorem 2, with $\alpha = n = 1$]*

For the special case $\alpha = \dfrac{1}{p-q}$, $(p > q)$ Theorem 2.1 reduces to the next special case:

Corollary 2.5 *If $f \in \mathcal{A}(p)$ satisfies*

$$\operatorname{Re} \left[\frac{f^{(1+q)}(z)}{f^{(q)}(z)} \left(\frac{f^{(q)}(z)}{\delta(p,q)} \right)^{\frac{1}{p-q}} \right] > \beta, \ z \in \mathbb{U}, \ (0 \le \beta < p-q)$$

then

$$\operatorname{Re} \mathrm{I}_{n,p} f^{(q)}(z) \ge \gamma_{p,q}^n(r) > \gamma_{p,q}^n(1), \ z \in \mathbb{U}, \ r = |z|, \ (n \in \mathbb{N}_0) \tag{2.11}$$

and

$$\frac{\beta}{p-q} < \gamma_{p,q}^n(r) = \frac{\beta}{p-q} + \left(1 - \frac{\beta}{p-q}\right)\left(-1 + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^{n+1}}\right) < 1.$$

The equality in (2.11) is attained for the function $f \in \mathcal{A}(p)$ given by

$$f^{(q)}(z) = \delta(p,q) \left\{ \left(\frac{2\beta}{p-q} - 1 \right) z + 2\left(1 - \frac{\beta}{p-q}\right) \log(1+z) \right\}^{p-q}.$$

Remark 2.6 *For the special case $q=0$, the result of Corollary 2.5 is an improvement of the result due to Owa [19, Corollary 7].*

Putting $p=1$ and $q=0$ in Corollary 2.5 we get:

Corollary 2.7 *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} f'(z) > \beta,\ z \in \mathbb{U}, (0 \le \beta < 1)$$

then

$$\operatorname{Re} \mathrm{I}_n f(z) \ge \gamma_n(r) > \gamma_n(1),\ z \in \mathbb{U},\ r=|z|,\ (n \in \mathbb{N}_0) \tag{2.12}$$

and

$$\beta < \gamma_n(r) = \beta + (1-\beta)\left(-1 + 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1} r^{k-1}}{k^{n+1}}\right).$$

The equality in (2.12) is attained for the function

$$f(z) = (2\beta-1)z + 2(1-\beta)\log(1+z).$$

Remark 2.8 *(i) The result of Corollary 2.7 was also obtained by Owa [19, Corollary 8], Hallenbeck [12, with $n=\beta=0$], Ling et al. [15, Corollary 3], and Patel and Rout [21, Corollary 3];*

(ii) The above corollary improve the results of Owa and Obradović [20, Theorem 4 with $\alpha=1$ and Corollary 4], Saitoh [23, Corollary 3], Saitoh [24, Corollary 8 with $\lambda=1$], and Ponnusamy and Karunakran [22, with $k=m=1$].

3. Integral operator $\mathbf{J}_n f^{(q)}$

For $f \in \mathcal{A}(p)$, we define the integral operator

$$\mathrm{J}_0 f^{(q)}(z) := \frac{f^{(q)}(z)}{\delta(p,q) z^{p-q}},$$

and

$$\mathrm{J}_n f^{(q)}(z) := \frac{a+1}{z^{a+1}} \int_0^z t^a \mathrm{J}_{n-1} f^{(q)}(t)dt,\ (a > -1,\ n \in \mathbb{N}).$$

For the operator $\mathrm{J}_n f^{(q)}$ we obtained the next result:

Theorem 3.1 *If $f \in \mathcal{A}(p)$ satisfies*

$$\operatorname{Re} \frac{f^{(q)}(z)}{\delta(p,q) z^{p-q}} > \alpha,\ z \in \mathbb{U},\ (\alpha < 1),$$

then

$$\operatorname{Re} \mathrm{J}_n f^{(q)}(z) \ge \gamma_n(r) > \gamma_n(1),\ z \in \mathbb{U},\ r=|z|,\ (n \in \mathbb{N}_0) \tag{3.1}$$

and

$$0 < \gamma_n(r) := 1 + 2(a+1)^n(1-\alpha)\sum_{k=1}^{\infty}\frac{(-r)^k}{(k+a+1)^n} < 1.$$

The equality in (3.1) is attained for the function $f \in \mathcal{A}(p)$ given by

$$f^{(q)}(z) = \delta(p,q)z^{p-q}\left[\alpha + (1-\alpha)\frac{1-z}{1+z}\right].$$

Proof For $n=0$ the implication is trivial. For $n=1$, if we denote

$$g(z) = \frac{1}{\alpha}\left[\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} - \alpha\right],\ z \in \mathbb{U},$$

then, from our assumption we have $\operatorname{Re} g(z) > 0$, $z \in \mathbb{U}$, and $g(0) = 1$. Using the inequality (2.7) for the function g and letting $z = re^{i\theta}$ and $t = \rho e^{i\theta}$, for $a > -1$ we get

$$\operatorname{Re} \mathrm{J}_1 f^{(q)}(z) = \operatorname{Re}\left(\frac{a+1}{z^{a+1}}\int_0^z t^a \mathrm{J}_0 f^{(q)}(t)dt\right) \geq \frac{a+1}{r^{a+1}}\int_0^r \rho^a\left[\alpha + (1-\alpha)\frac{1-\rho}{1+\rho}\right]d\rho$$

$$= \frac{a+1}{r^{a+1}}\int_0^r \rho^a\left[1 + 2(1-\alpha)\sum_{k=1}^{\infty}(-\rho)^k\right]d\rho = 1 + \frac{2(a+1)(1-\alpha)}{r^{a+1}}\int_0^r\sum_{k=1}^{\infty}(-1)^k\rho^{k+a}d\rho$$

$$= 1 + 2(a+1)(1-\alpha)\sum_{k=1}^{\infty}\frac{(-r)^k}{k+a+1},\ |z| = r,$$

thus (3.1) holds for $n=1$. Further, assuming that (3.1) holds for a fixed $n \in \mathbb{N}$, we have

$$\operatorname{Re} \mathrm{J}_{n+1} f^{(q)}(z) = \operatorname{Re}\left(\frac{a+1}{z^{a+1}}\int_0^z t^a \mathrm{J}_n f^{(q)}(t)dt\right) = \frac{a+1}{r^{a+1}}\int_0^r \rho^a \operatorname{Re} \mathrm{J}_n f^{(q)}\left(\rho e^{i\theta}\right)d\rho$$

$$\geq \frac{a+1}{r^{a+1}}\int_0^r\left(\rho^a + 2(a+1)^n(1-\alpha)\sum_{k=1}^{\infty}\frac{(-1)^k\rho^{k+a}}{(k+a+1)^n}\right)d\rho$$

$$= 1 + 2(a+1)^{n+1}(1-\alpha)\sum_{k=1}^{\infty}\frac{(-r)^k}{(k+a+1)^{n+1}} = \gamma_{n+1}(r),\ |z| = r.$$

Moreover, it is easy to see that $0 < \gamma_n < 1$, which completes our proof. □

Taking $\alpha = \dfrac{p-q}{p-q+\beta}$, $(p > q,\ \beta > 0)$ in the above theorem we get the next special case:

Corollary 3.2 *If $f \in \mathcal{A}(p)$ satisfies*

$$\operatorname{Re}\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} > \frac{p-q}{p-q+\beta},\ z \in \mathbb{U},\ (\beta > 0),$$

then

$$\operatorname{Re} \mathrm{J}_n f^{(q)}(z) \geq \gamma_n(r) > \gamma_n(1), \; z \in \mathbb{U}, \; r = |z|, \; (n \in \mathbb{N}_0) \tag{3.2}$$

and

$$0 < \gamma_n(r) := 1 + \frac{2\beta(a+1)^n}{p-q+\beta} \sum_{k=1}^{\infty} \frac{(-r)^k}{(k+a+1)^n} < 1.$$

The equality in (3.2) is attained for the function $f \in \mathcal{A}(p)$ given by

$$f^{(q)}(z) = \frac{\delta(p,q) z^{p-q}}{p-q+\beta} \frac{1+\beta+(1-\beta)z}{1+z}.$$

Remark 3.3 *Putting $q = 0$ in Theorem 3.1 and in Corollary 3.2 we obtain the results of Owa [18, Theorem 2 and Corollary 4] and Owa [19, Theorem 2 and Corollary 10].*

References

[1] Aouf MK. On a class of p-valent starlike functions of order α. Panamerican Mathematical Journal 1987; 10 (4): 733-744.

[2] Aouf MK. A generalization of multivalent functions with negative coefficients.Journal of the Korean Mathematical Society 1988; 25 (1): 53-66.

[3] Aouf MK. On a class of p-valent close-to-convex functions of order α and type β.International Journal of Mathematical Sciences 1988; 11 (2): 259-266.

[4] Aouf MK. Certain subclasses of p-valent starlike functions defined by using a differential operator. Applied Mathematics and Computation 2008; 206 (2): 867-875.

[5] Aouf MK. Some families of p-valent functions with negative coefficients. Acta Mathematica Universitatis Comenianae 2009: 78 (1): 121-135.

[6] Aouf MK. On certain subclass of p-valently Bazilević functions. Studia Universitatis Babeş-Bolyai Mathematica 2009; 54 (1): 21-28.

[7] Aouf MK. On certain multivalent functions with negative coefficients defined by using a differential operator. Matematički Vesnik 2010; 62 (1): 23-35.

[8] Aouf MK. Bounded p-valent Robertson functions defined by using a differential operator. Journal of The Franklin Institute 2010; 347 (10): 1927-1941.

[9] Aouf MK. Some inclusion relationships associated with Dziok-Srivastava operator. Applied Mathematics and Computation 2010; 216 (2): 431-437.

[10] Aouf MK, Hossen HM, Srivastava HM. Some families of multivalent functions. Applied Mathematics and Computation 2000; 39 (7-8): 39-48.

[11] Halim SA, Thomas DK. A note on Bazilević functions. Panamerican Mathematical Journal 1991; 14 (4): 821-823.

[12] Hallenbeck DJ. Convex hulls and extreme points of some families of univalent functions. Transactions of the American Mathematical Society 1974; 192: 285-292.

[13] Goswami P, Bansal S. On sufficient conditions and angular properties of starlikeness and convexity for the class of mulivalent Bazilević functions. International Journal of Open Problems in Computer Science and Mathematics 2009; 2 (1): 98-107.

[14] Irmak H, Piejko K, Stankiewicz J. A note involving p-valently Bazilević functions. International Journal of Mathematical Sciences 2005; 7: 1149-1153.

[15] Ling Yi, Gejun B, Shengzbeng L. Some properties of certain analytic functions. Nihonkai Mathematical Journal 1992; 3 (1): 31-37.

[16] MacGregor TH. Functions whose derivative has a positive real part. Transactions of the American Mathematical Society 1962; 104 (3): 532-537.

[17] Owa S. On certain classes of p-valent functions with negative coefficients. Bulletin of the Belgian Mathematical Society Simon Stevin 1985; 59: 385-402.

[18] Owa S. Certain integral operator. Proceedings of the Japan Academy, Series A, Mathematical Sciences 1991; 67 (3): 88-93.

[19] Owa S. Notes on certain p-valent functions. Panamerican Mathematical Journal 1993; 3 (3): 79-93.

[20] Owa S, Obradović M. Certain subclasses of Bazilević functions of type α. International Journal of Mathematical Sciences 1986; 9 (2): 347-359.

[21] Patel J, Rout S. An application of differential subordinations. Rendiconti di Matematica, Serie VII 1994; 14: 367-384.

[22] Ponnusamy S, Karunakaran V. Differential subordination and conformal mappings. Complex Variables, Theory and Application: An International Journal 1989; 11 (2): 79-86.

[23] Saitoh H. Properties of certain analytic functions. Proceedings of the Japan Academy, Series A, Mathematical Sciences 1989; 65 (5): 131-134.

[24] Saitoh H. On certain multivalent functions. Proceedings of the Japan Academy, Series A, Mathematical Sciences 1991; 67 (8): 287-292.

[25] Thomas DK. A note on Bazilević functions. Research Institute for Mathematical Sciences Kyoto University 1990; 714: 18-21.