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Liftings and covering morphisms of crossed modules in group-groupoids

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Abstract: In this work we introduce lifting and covering of a crossed module in the category of group-groupoids; and then we prove the categorical equivalence of horizontal actions of a double group-groupoid and lifting crossed modules of corresponding crossed module in group-groupoids. These allow us to produce more examples of double group-groupoids.

Key words: Crossed module, group-groupoid, double group-groupoid, action, covering morphism

1. Introduction

The concept of covering groupoid has a significant role in the utilizations of groupoids (see [4, 16]). It is well known that the groupoid actions on sets and the covering morphisms of a certain groupoid G are categorically equivalent (see [6] for topological version). An analogous equivalence was given in [9, Proposition 3.1] for a group-groupoid G which is used under the name 2-group in [2] and G-groupoids or group object in the category of groupoids in [11]. In [1], this result was generalized by assuming G is an internal groupoid in the category of groups with operations appeared in [27, 28]. That result is adapted to Leibniz algebras setting in [29], to categorical groups in [25] and to categorical ring in [22].

Double groupoids which are useful for Seifert-van Kampen Theorem to determine the fundamental groupoids of topological spaces [7] were defined by Ehresmann in [13, 14] to be internal groupoids in the category of groupoids. It was proved in [10] that double groupoids are categorically equivalent to crossed modules in the sense of Whitehead [31, 32]. Due to this equivalence some algebraic structures such as normality and quotient of double groupoids were characterized in [21] (see [23] for similar structures in group-groupoids).

By Loday [18] cat^1-groups and crossed modules in groups; cat^2-groups and crossed squares are categorically equivalent. More generally by Ellis and Steiner [15] cat^n-groups are equivalent to crossed n-cubes. The readers are also referred to [3] for algebraic structures on groupoids and algebraic descriptions of homotopy n-types. Due to [30] crossed modules in group-groupoids are equivalent to double group-groupoids and to crossed squares; and therefore to cat^2-groups.

It was proved in [8, Theorem1.7] that the categories of horizontal actions and horizontal action morphisms for a double Lie groupoid are equivalent. Recently this result is extended to double group-groupoids in [12]; and action and covering notions of double group-groupoids are characterized.

In this paper, by means of the latter equivalence, we aim to introduce the notions of lifting and covering

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of a crossed module in group-groupoids and prove the categorical equivalence of them. For the convenience of the reader in the second section we give preliminaries on groupoids, actions, coverings and group-groupoids. Section 3 contains a brief summary of double groupoids, double group-groupoids together with actions, coverings and crossed module in the category of group-groupoids. In Section 4, we characterize a crossed module in the category of group-groupoids corresponding to a double group-groupoid horizontally acting on a group-groupoid, define the lifting notion of crossed modules in group-groupoids and we give some examples. We prove that the category of horizontal actions of a double group-groupoid on group-groupoids and the category of lifting crossed modules in the category of group-groupoids are equivalent; and obtain a categorical equivalence between liftings and coverings of a crossed module in group-groupoids. These results extend [24, Theorem 4.6] and [24, Theorem 5.1] respectively. The results of the paper enable us to produce more examples of double group-groupoids.

2. Groupoids and group-groupoids

A groupoid is a small category whose morphisms are isomorphisms (see [4, 19] for more details). Indeed a groupoid G includes a set G of morphisms or arrows and G_0 of objects with source and target point maps $d_0, d_1\colon G\to G_0$ and object inclusion map $\epsilon\colon G_0\to G$ with the property that $d_0\epsilon=d_1\epsilon=1_{G_0}$. An associative partial composition $G_{d_1}\times_{d_0}G\to G, (g,h)\mapsto g\circ h$, where $G_{d_1}\times_{d_0}G$ is the pullback of d_0 and d_1 is defined. Here if $g,h\in G$ and $d_1(g)=d_0(h)$, then the composite $g\circ h$ is well defined such that $d_0(g\circ h)=d_0(g)$ and $d_1(g\circ h)=d_1(h)$. Moreover, for $x\in G_0$ the morphism $\epsilon(x)$ acts as the identity and it is denoted by 1_x. There is a map $G\to G$ called inversion which assigns to every element g its inverse g^{-1} such that $d_0(g^{-1})=d_1(g)$, $d_1(g^{-1})=d_0(g)$, $g\circ g^{-1}=\epsilon(d_0(g))$, $g^{-1}\circ g=\epsilon(d_1((g))$. In a groupoid G, all maps defined above are called structural maps. If $x\in G_0$, the star St_Gx of x is defined by the set $\{g\in G; d_0(g)=x\}$. The fundamental groupoid πX of a topological space X is an example of groupoid whose objects are the points of X and morphisms are the homotopy classes of the paths relative to the end points.

A morphism $f\colon G\to H$ of groupoids includes the maps $f_1\colon G\to H$ and $f_0\colon G_0\to H_0$ satisfying $d_0f_1=f_0d_0$, $d_1f_1=f_0d_1$, $f_1\varepsilon=\varepsilon f_0$ and preserving the composite $f(g\circ h)=f(g)\circ f(h)$, for $g,h\in G$.

A groupoid G whose sets of objects and morphisms are equipped with group structures is called a group-groupoid whenever the group operation written additively $G\times G\to G, (g,h)\mapsto g+h$, the inverse $G\to G, g\mapsto -g$ and the unit map $\{0\}\to G$, where $\{0\}$ is singleton, are groupoid morphisms. Here note that the additive map is a morphism of groupoids if and only if the interchange rule

$$(g+h)\circ(k+l)=(g\circ k)+(h\circ l)$$

is satisfied for $g,h,k,l\in G$ whenever the composites are well defined. A group-groupoid morphism is a group structure preserving morphism of underlying groupoids. We thus obtain a category GpGd of group-groupoids.

A groupoid G whose morphism and object sets have topologies and the structural maps are continuous is called topological groupoid (see [5, 19]). A topological group-groupoid is defined in [17] to be a topological groupoid which has topological group structures on the sets of objects and morphisms.

A covering morphism of groupoids, $p\colon\widetilde{G}\to G$, is a groupoid morphism with the property that for every $\tilde{x}\in\widetilde{G_0}$ the restriction $\mathsf{St}_{\widetilde{G}}\tilde{x}\to\mathsf{St}_G p(\tilde{x})$ is bijective. A covering morphism of topological groupoid is a covering morphism of groupoids in which each restriction to the star is a homeomorphism. A covering morphism of topological group-groupoids is defined in [20, Definition 4.1] as a covering morphism of topological groupoid.

The following construction on action appears in [4, p.373].

An action of a groupoid G on a set X includes a function $\omega\colon X \to G_0$ and a partial function $\varphi\colon X_\omega \times_{d_0} G \to X, (x,g) \mapsto x \bullet g$ where $X_\omega \times_{d_0} G$ is pullback of ω and d_0 with the following properties.

(i) $\omega(x \bullet g) = d_1(g)$ for $(x,g) \in X_\omega \times_{d_0} G$;

(ii) $x \bullet (g \circ h) = (x \bullet g) \bullet h$ for $(g,h) \in G_{d_1} \times_{d_0} G$ and $(x,g) \in X_\omega \times_{d_0} G$;

(iii) $x \bullet \epsilon(\omega(x)) = x$ for $x \in X$.

Such an action is denoted by (X,ω). A morphism of these actions from (X,ω) to (X',ω') is a function $f\colon X \to X'$ with the properties $\omega' f = \omega$ and $f(x \bullet g) = f(x) \bullet g$. So for a given groupoid G, we have a category denoted by $\mathrm{GpdAct(G)}$.

Following [4], for such an action there is a groupoid $G \ltimes X$, called semidirect product groupoid. Here the object set is X. The elements of $(G \ltimes X)(x,y)$ are the pairs (g,x) in which $g \in G(\omega(x), \omega(y))$ and $x \bullet g = y$. The groupoid composition is as follows.

$$(g,x) \circ (h,y) = (g \circ h, x)$$

The projection map $p\colon G \ltimes X \to G$ is a covering morphism of groupoids. This assignment determines a categorical equivalence between actions and coverings of G [6].

An action of a group-groupoid G on a group X by a group morphism $\omega\colon X \to G_0$ (See [9, Section 3] for more details) is a groupoid action of G on the underlying set of X by ω, satisfying the interchange rule

$$(x \bullet g) + (y \bullet h) = (x + y) \bullet (g + h)$$

for $g, h \in G$ and $x, y \in X$.

A morphism from a group-groupoid action (X,ω) to (X,ω') include $f\colon X \to X'$ as a morphism of group and of underlying operations of G. Therefore there is a category $\mathrm{GpGpdAct(G)}$ of group-groupoid actions and morphisms of them.

Besides, the categories of group-groupoid coverings and actions are equivalent for a fixed group-groupoid [9, Proposition 3.1].

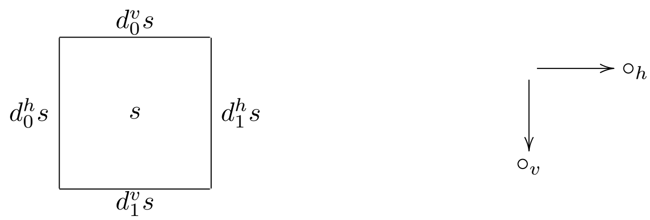
3. Double group-groupoids and crossed modules

A double groupoid is a groupoid object in the category of groupoids. In other words it consists of a quadruple of sets $(S; H, V; P)$ such that there are two groupoid structures on H and V with object set P and two groupoid structures on S which are vertical one based on H denoted by S_V and horizontal one based on V denoted as S_H. Therefore a double groupoid has four related groupoid structures and compatible structural maps.

In a double groupoid we write multiplication for groupoid compositions in H and V; and $1_b^H \in H$ and $1_b^V \in V$ for the identity elements for $b \in P$. The source, target, object inclusion, composition for H are $d_0^H, d_1^H\colon H \to P$, $\varepsilon^H\colon P \to H$ and $m^H\colon H * H \to H$ respectively and similar notations are used for V.

The horizontal groupoid S_H has source and target $d_0^h, d_1^h\colon S \to V$, object inclusion $\varepsilon^h\colon V \to S$ and partial composition $\circ_h\colon S * S \to S$, $(s_1, s_2) \to s_1 \circ_h s_2$. The vertical groupoid S_V has source and target $d_0^v, d_1^v\colon S \to H$, object inclusion $\varepsilon^v\colon H \to S$ and partial composition $\circ_v\colon S * S \to S$, $(s_1, s_2) \to s_1 \circ_v s_2$. For a square s write s^{-h} and s^{-v} for the inverses of s in S_H and S_V, respectively.

Elements of S are squares with boundaries as follows:



A double groupoid has the following interchange rule for $s_1, s_2, s_3, s_4 \in S$

$$(s_1 \circ_h s_2) \circ_v (s_3 \circ_h s_4) = (s_1 \circ_v s_3) \circ_h (s_2 \circ_v s_4)$$

A morphism $\varphi = (\varphi_s, \varphi_h, \varphi_v, \varphi_p)\colon (S'; H', V'; P') \to (S; H, V; P)$ of double groupoids consists of four maps that commute with structural maps. These form a category DGpd of double groupoids.

A double group-groupoid is defined in [30] to be an internal groupoid in the category GpGd. Hence it consists of four related group-groupoids S_H, S_V, H and V provided with the following interchange rules

$$(s_1 \circ_h s_2) + (s_3 \circ_h s_4) = (s_1 + s_3) \circ_h (s_2 + s_4)$$

,

$$(s_1 \circ_v s_2) + (s_3 \circ_v s_4) = (s_1 + s_3) \circ_v (s_2 + s_4)$$

.

By the interchange rule in double group-groupoid, horizontal and vertical groupoid compositions can be written in terms of group operation for $d_1^h(s_1) = d_0^h(s)$ and $d_1^v(\alpha_1) = d_0^v(\alpha)$ as follows:

$$s_1 \circ_h s = s_1 - \varepsilon^h (d_1)^h (s_1) + s = s - \varepsilon^h (d_1)^h (s) + s_1 \tag{3.1}$$

$$\alpha_1 \circ_v \alpha = \alpha_1 - \varepsilon^v (d_1)^v (\alpha_1) + \alpha = \alpha - \varepsilon^v (d_1)^v (\alpha) + \alpha_1 \tag{3.2}$$

whenever the necessary operations are defined; and for the squares $s, s_1 \in \operatorname{Ker} d_0^h$ and $\alpha, \alpha_1 \in \operatorname{Ker} d_0^v$, we have

$$s + s_1 - s = \varepsilon^h d_1^h(s) + s_1 - \varepsilon^h d_1^h(s)$$

and

$$\alpha + \alpha_1 - \alpha = \varepsilon^v d_1^v(\alpha) + \alpha_1 - \varepsilon^v d_1^v(\alpha).$$

There is a category of double group-groupoids denoted by DGpGpd.

Action of a double group-groupoid on a group-groupoid below comes from [12].

A horizontal action of double group-groupoid $\mathcal{S} = (S; H, V; P)$ on a group-groupoid G via a morphism $\omega\colon G \to V$ of group-groupoids includes an action of horizontal group-groupoid S_H on V via $\omega\colon G \to V$ and an action of H on P via $\omega_0\colon G_0 \to P$ with the following properties.

(i) $d_1^G(g \bullet s) = d_1^G(g) \bullet d_1^v(s)$ and $d_0^G(g \bullet s) = d_0^G(g) \bullet d_0^v(s)$ for each $s \in S, g \in G$ with $d_0^h(s) = \omega(g)$.

(ii) For $s_1, s_2 \in S$ and $g_1, g_2 \in G$ we have

$$(g_1 g_2) \bullet (s_1 \circ_v s_2) = (g_1 \bullet s_1) \circ_v (g_2 \bullet s_2) \tag{3.3}$$

(iii) For all $a \in H$ and $x \in G_0$ with $d_0^H(a) = \omega_0(x)$ we have $1_x^G \bullet \varepsilon^v(a) = 1_{ax}^G$ and for $x, x_1 \in G_0$ and $a, a_1 \in H$ we have

$$(x_1 + x) \bullet (a_1 + a) = (x_1 \bullet a_1) + (x \bullet a)$$

We write (G, ω) for such an action. Due to structure of group-groupoid we have an interchange rule

$$(g_1 + g_2) \bullet (s_1 + s_2) = (g_1 \bullet s_1) + (g_2 \bullet s_2). \tag{3.4}$$

Similarly vertical action of double group-groupoids can be restated. See [8] for the study about horizontal action of Lie double groupoid and related examples.

A morphism $f\colon (G, \omega) \to (G', \omega')$ of such actions consists of group homomorphisms $f\colon G \to G'$ and $f_0\colon G_0 \to G'_0$ provided that $f(g \bullet s) = f(g) \bullet s$ and $f_0(x \bullet h) = f_0(x) \bullet h$ such that $\omega' f = \omega$ and $\omega'_0 f_0 = \omega_0$. Thus for a fixed double group-groupoid S we have a category $\mathsf{DGpGpdAct_H(S)}$ of horizontal actions of double group-groupoids.

A morphism $\varphi = (\varphi_s, \varphi_h, \varphi_v, \varphi_p)\colon (S'; H', V'; P') \to (S; H, V; P)$ of double group-groupoids is called covering morphism associated with the horizontal action if (φ_s, φ_v) and (φ_h, φ_p) are covering morphisms of ordinary group-groupoids [12, Definition 2.2]. Then we have a category $\mathsf{Cov_H DGpGpd/S}$ of coverings of S.

A crossed module which is due to Whitehead in [31, 32] is defined to be group homomorphism $\partial\colon A \to B$ with a right action $(a, b) \mapsto a.b$ of B on A with the following rules.

[CM1] $\partial(a.b) = -b + \partial(a) + b$, and

[CM2] $a_1.\partial(a) = -a + a_1 + a$.

We know by [30, Proposition 3.9] that (G, H, ∂) is a crossed module in group-groupoids if (G, H, ∂_1) is a crossed module in groups. A morphism (f, g) from (G', H', ∂) to (G, H, ∂) is defined to be two group-groupoid morphisms $f\colon G' \to G$ and $g\colon H' \to H$ with the property that $(f, g)\colon (G', H', \partial) \to (G, H, \partial)$ is a morphism of crossed module in groups. Therefore there is a category $\mathsf{XModGpGd}$ of crossed modules in group-groupoids.

We need some techniques of the proof for the following result in later parts and hence we only state the main ideas.

Theorem 3.1 [30, Theorem 4.7] Crossed modules in group-groupoids and double group-groupoids are categorically equivalent.

$$\mathsf{XModGpGd} \simeq \mathsf{DGpGpd}$$

Proof For a crossed module in group-groupoid (G, H, ∂), one has a corresponding double group-groupoid $(H \ltimes G, H_0 \ltimes G_0, H, H_0)$ in which the compositions are defined as follows:

$$(h', g') \circ_h (h, g) = (h' \circ h, g' \circ g)$$

,

$$(h', g') \circ_v (h, g) = (h, g' + g')$$

. By [30, Lemma 3.4], group operation of $G \ltimes H$ is

$$(h_1, g_1) + (h, g) = (h_1 + h, g_1 + h_1.g). \tag{3.5}$$

Conversely for a double group-groupoid $(S; H, V; P)$ one has a crossed module in group-groupoid (K, V, ∂) where

$$K = (\operatorname{Ker} d_0^h, \operatorname{Ker} d_0^H, d_0^v, d_1^v, \varepsilon^v, n^v, m^v)$$

and

$$V = (V, P, d_0^V, d_1^V, \varepsilon^V, n^V, m^V)$$

the boundary map is $\partial = (\partial_1 = d_1^h, \partial_0 = d_1^H)$ and the action of V on $\operatorname{Ker} d_0^h$ is given by

$$a.b = -\varepsilon^h(b) + a + \varepsilon^h(b) \tag{3.6}$$

for $a \in \operatorname{Ker} d_0^h$ and $b \in V$. We refer to the cited reference for more details.

□

We now state how to construct a double group-groupoid from a topological group-groupoid and obtain the corresponding crossed module in group-groupoids.

Example 3.2 We know that for a topological group X, the fundamental groupoid πX is a group-groupoid. Hence if G is a topological group-groupoid, then G and G_0 are topological groups and then we have related four group-groupoids $(\pi G, \pi G_0)$, $(\pi G, G)$, $(\pi G_0, G_0)$ and (G, G_0). So we have a quadruple $(\pi G; \pi G_0, G; G_0)$ which becomes a double group-groupoid. Therefore by Theorem 3.1 we have a corresponding crossed module $d_1 \colon St_{\pi G}0 \to G$ in group-groupoids.

4. Liftings and coverings of crossed modules in group-groupoids

In this section evaluating the equivalence of the categories in Theorem 3.1 we obtain the lifting notion for a crossed module in the category of group-groupoids associated with a horizontal action of a double group-groupoid on a group-groupoid. We first have the following preparation.

For given a double group-groupoid $\mathcal{S} = (S; H, V; P)$ horizontally acting on a group-groupoid G via a morphism $\omega \colon G \to V$ of group-groupoids suppose that (A, B, ∂) is the crossed module of group-groupoids associated with $\mathcal{S}$ by Theorem 3.1. We then actually have the following.

(i) a morphism of group-groupoids $\omega \colon G \to B$

(ii) an action of G on $A = \operatorname{Ker} d_0^h$ via ω defined by

$$A \times G \to A; \quad (a, g) \mapsto a.g = -\varepsilon^h(\omega(g)) + a + \varepsilon^h(\omega(g)) \tag{4.1}$$

(iii) an action of G_0 on $A_0 = \operatorname{Ker} d_0^H$ via ω_0 defined by

$$A_0 \times G_0 \to A_0; \quad (x, y) \mapsto x.y = -\varepsilon^H(\omega_0(y)) + x + \varepsilon^H(\omega_0(y))$$

(iv) a group-groupoid morphism

$$\varphi \colon A \to G \qquad \varphi_1(a) = 0_G \bullet a, \qquad \varphi_0(x) = 0_{G_0} \bullet x \tag{4.2}$$

such that $\omega\varphi = \partial$.

We now state the following theorem.

Theorem 4.1 (A, G, φ) is a crossed module in group-groupoids.

Proof By [30, Proposition 3.9], we need to prove that (A,G,φ_1) satisfies the axioms of a crossed module of groups.

[CM1]

$$\begin{aligned}
\varphi_1(a.g) &= \varphi_1(-\varepsilon^h(\omega(g))+a+\varepsilon^h(\omega(g))) && \text{(by Eq.4.1)}\\
&= 0_G \bullet (-\varepsilon^h(\omega(g))+a+\varepsilon^h(\omega(g))) && \text{(by Eq.4.2)}\\
&= (-g+g)\bullet(-\varepsilon^h(\omega(g))+a+\varepsilon^h(\omega(g))) &&\\
&= ((-g)\bullet\varepsilon^h(\omega(-g)))+g\bullet(a+\varepsilon^h(\omega(g))) && \text{(by Eq. 3.4)}\\
&= (-g)+(0_G+g)\bullet(a+\varepsilon^h(\omega(g))) && (\text{by } g\bullet\varepsilon^h(\omega(g))=g)\\
&= (-g)+0_G\bullet a+g\bullet\varepsilon^h(\omega(g)) && \text{(by Eq. 3.4)}\\
&= -g+\varphi_1(a)+g && \text{(by Eq.4.2)}
\end{aligned}$$

[CM2]

$$\begin{aligned}
a_1\varphi_1(a) &= a_1(0_G\bullet a) && \text{(by Eq.4.2)}\\
&= -\varepsilon^h(\omega(0_G\bullet a))+a_1+\varepsilon^h(\omega(0_G\bullet a)) && \text{(by Eq. 4.1)}\\
&= -\varepsilon^h(d_1^h(a))+a_1+\varepsilon^h(d_1^h(a)) && (\text{by } \omega(0_G\bullet a)=d_1^h(a))\\
&= a_1 d_1^h(a) && \text{(by Eq.3.6)}\\
&= -a+a_1+a
\end{aligned}$$

Therefore (A,G,φ) becomes a crossed module of group-groupoids as required. □

Therefore we can state definition below.

Definition 4.2 Suppose that (A,B,∂) is a crossed module in group-groupoids and $\omega\colon G\to B$ is a morphism of group-groupoids. A crossed module (A,G,φ) in which G acts on A via ω is called a lifting of (A,B,∂) if the following diagram commutes, i.e. $\omega\varphi=\partial$

$$\begin{tikzcd}
 & G \arrow[d, "\omega"] \\
A \arrow[ur, dashed, "\varphi"] \arrow[r, "\partial"'] & B
\end{tikzcd}$$

We will denote such a lifting by (φ,G,ω).

A morphism $\rho\colon(\varphi,G,\omega)\to(\varphi',G',\omega')$ of such liftings is a morphism $\rho\colon G\to G'$ of group-groupoids satisfying $\rho\varphi=\varphi'$ and $\omega'\rho=\omega$. Therefore we have a category $\mathrm{LXModGpGd}/(A,B,\partial)$ of liftings and morphisms of them.

Example 4.3 For every crossed module (A,B,∂) of group-groupoids, $(\partial,B,1_B)$ becomes a lifting of (A,B,∂).

Example 4.4 If N is a normal subgroup-groupoid of G as defined in [23, Definition 2.10], there exists a morphism $\partial\colon G \to H$ of group-groupoids with $\operatorname{Ker}\partial = N$ by [23, Theorem 2.19]. Hence for a crossed module in group-groupoid $\partial\colon G \to H$ with $\operatorname{Ker}\partial = N$, there is a unique morphism $\tilde{\partial}\colon G/N \to H$ with the commutative diagram below.

$$\begin{tikzcd} & G/N \arrow[d, "\tilde{\partial}"] \\ G \arrow[ur, dashed, "\eta"] \arrow[r, "\partial"'] & H \end{tikzcd}$$

This means that $(G, G/N, \eta)$ is a lifting of (G, H, ∂) by group-groupoid morphism $\tilde{\partial}$.

The following theorem extends [24, Example 4.8].

Theorem 4.5 If $p\colon \widetilde{G} \to G$ is a covering morphism of topological group-groupoids, then there exists a crossed module in group-groupoids $d_1\colon \mathsf{St}_{\pi G}0 \to G$ with a lifting $\tilde{d_1}\colon \mathsf{St}_{\pi G}0 \to \widetilde{G}$.

Proof We observe from Example 3.2 that for a topological group-groupoid G, $d_1\colon St_{\pi G}0 \to G$ is a crossed module in group-groupoids. Since $p\colon \widetilde{G} \to G$ is a covering morphism of groupoids, for a path α in G with initial point 0, identity, there exists a path $\tilde{\alpha}$ in $\widetilde{G}$ with initial point $\tilde{0}$ such that $p(\tilde{\alpha}) = \alpha$. Thus we obtain a function $\tilde{d_1}\colon St_{\pi G}0 \to \widetilde{G}$ which assigns the homotopy class $[\alpha]$ of α to the final point of $\tilde{\alpha}$. So we have the commutative diagram below.

$$\begin{tikzcd} & \widetilde{G} \arrow[d, "p"] \\ St_{\pi G}0 \arrow[ur, dashed, "\tilde{d_1}"] \arrow[r, "d_1"'] & G \end{tikzcd}$$

Hence $(\tilde{d_1}, \widetilde{G}, p)$ becomes a lifting of $(St_{\pi G}0, G, d_1)$. □

As a result we can give the following categorical equivalence.

Theorem 4.6 Let S be a double group-groupoid and (A, B, ∂) be the crossed module in group-groupoid which corresponds to S. Then the following categories are equivalent.

$$\mathsf{DGpGpdAct_H(S)} \simeq \mathrm{LXModGpGd}/(A, B, \partial)$$

Proof Let us begin with defining a functor $\theta\colon \mathsf{DGpGpdAct_H(S)} \to \mathrm{LXModGpGd}/(A, B, \partial)$ which assigns each horizontal action (G, ω) of the double group-groupoid S to a lifting (φ, G, ω) of (A, B, ∂) in which φ is defined by

$$\varphi_1\colon A \to G, \varphi_1(a) = 0_G \bullet a$$

$$\varphi_0\colon A_0 \to G_0, \varphi_0(b) = 0_{G_0} \bullet b$$

such that $\omega\varphi = \partial$.

Let us consider the functor $\delta\colon \mathrm{LXModGpGd}\,/(A,B,\partial)\to \mathsf{DGpGpdAct_H(S)}$ which assigns every lifting (φ,G,ω) to a horizontal action of double group-groupoid (G,ω) of S on G with action

$$S\times G\to G,(s,g)\mapsto g\bullet s=\varphi_1(s-\varepsilon^h(d_0^h(s)))+g$$

$$H\times G_0\to G_0,(h,x)\mapsto x\bullet h=\varphi_0(h-\varepsilon^H(d_0^H(h))+x$$

We now proceed to show that $\theta\circ\delta$ and $\delta\circ\theta$ are naturally isomorphic to $1_{\mathrm{LXModGpGd}\,/(A,B,\partial)}$ and $1_{\mathsf{DGpGpdAct_H(S)}}$, respectively. If $(\varphi,G,\omega)\in \mathrm{LXModGpGd}\,/(A,B,\partial)$ then $(\theta\circ\delta)(\varphi,G,\omega)=(\varphi',G,\omega,)$ where

$$\varphi_1'(a)=0_G\bullet a=\varphi_1(a-\varepsilon^h(d_0^h(a)))+0_G=\varphi_1(a)$$

and

$$\varphi_0'(b)=x\bullet b=\varphi_0(b-\varepsilon^H(d_0^H(b))+x=\varphi_0(b)$$

. Therefore $\theta\circ\delta=1$.

Conversely if $(G,\omega)\in\mathsf{DGpGpdAct_H(S)}$ with an action of S on G by

$$S\times G\to G,(s,g)\mapsto g\bullet s;\qquad H\times G_0\to G_0,(h,x)\mapsto x\bullet h$$

then we have an induced action defined by

$$\begin{aligned}
g\bullet' s&=\varphi_1(s-\varepsilon^h(d_0^h(s)))+g\\
&=0_G\bullet(s-\varepsilon^h(d_0^h(s)))+(g\bullet\varepsilon^h(\omega(g))\\
&=(0_G+g)\bullet(s-\varepsilon^h(d_0^h(s))+\varepsilon^h(\omega(g))) && (\text{by } d_0^h(s)=\omega(g))\\
&=g\bullet s.
\end{aligned}$$

The action of H on G_0 can be checked in a similar way. Hence $\delta\circ\theta=1$ which completes the proof.

□

The definition of covering of double group-groupoid was given in [12]. Evaluating the detailed proof of Theorem 3.1, we can characterize the morphism of crossed modules in group-groupoids corresponding to covering of double group-groupoid as follows.

Definition 4.7 A morphism (f,g) of crossed modules in group-groupoids from (A',B',∂') to (A,B,∂) is called covering morphism if $f\colon A'\to A$ and and $g\colon B'\to B$ are isomorphisms.

Therefore we obtain a category of $\mathsf{XModCov}/(A,B,\partial)$ of coverings for a given crossed module (A,B,∂) in group-groupoids.

Example 4.8 The morphism $(1_A,1_B)\colon(A,B,\partial)\to(A,B,\partial)$ of crossed module in group-groupoids is a covering morphism.

By using the categorical equivalence given in Theorem 3.1, we introduce the below corollary.

Corollary 4.9 The category $\mathsf{Cov_H DGpGpd/S}$ of coverings of double group-groupoid S and the category $\mathsf{XModCov}/(A,B,\partial)$ of coverings of corresponding crossed module in group-groupoids are equivalent.

The proof of the following corollary follows from [24, Theorem 5.2].

Corollary 4.10 Let (A,B,∂) be a crossed module in group-groupoids. Then the category LXModGpGd $/(A,B,\partial)$ of liftings and the category $\mathsf{XModCov}/(A,B,\partial)$ of coverings are equivalent.

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