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NGUYEN DAT THUC

LE THI PHUONG NGOC

NGUYEN THANH LONG

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Solvability, stability, smoothness and compactness of the set of solutions for a nonlinear functional integral equation

Nguyen Dat THUC[1,2], **Le Thi Phuong NGOC**[3], **Nguyen Thanh LONG**[1,2,*]
[1]Department of Mathematics and Computer Science, University of Science, Ho Chi Minh City, Vietnam
[2]Vietnam National University, Ho Chi Minh City, Vietnam
[3]University of Khanh Hoa, Nha Trang City, Vietnam

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Abstract: This paper is devoted to the study of the following nonlinear functional integral equation

$$f(x)=\sum_{i=1}^{q}\alpha_i(x)f(\tau_i(x))+\int_0^{\sigma_1(x)}\Psi\left(x,t,f(\sigma_2(t)),\int_0^{\sigma_3(t)}f(s)ds\right)dt+g(x),\ \forall x\in[0,1], \tag{E}$$

where $\tau_i,\ \sigma_1,\ \sigma_2,\ \sigma_3:[0,1]\to[0,1];\ \alpha_i,\ g:[0,1]\to\mathbb{R};\ \Psi:[0,1]\times[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are the given continuous functions and $f:[0,1]\to\mathbb{R}$ is an unknown function. First, two sufficient conditions for the existence and some properties of solutions of Eq. (E) are proved. By using Banach's fixed point theorem, we have the first sufficient condition yielding existence, uniqueness and stability of the solution. By applying Schauder's fixed point theorem, we have the second sufficient condition for the existence and compactness of the solution set. An example is also given in order to illustrate the results obtained here. Next, in the case of $\Psi\in C^2([0,1]\times[0,1]\times\mathbb{R}^2;\mathbb{R})$, we investigate the quadratic convergence for the solution of Eq. (E). Finally, the smoothness of the solution depending on data is established.

Key words: Nonlinear functional integral equation, quadratic convergence, stability and smoothness of solutions, compactness of the solution set

1. Introduction

In this paper, we consider the following nonlinear functional integral equation

$$f(x)=\sum_{i=1}^{q}\alpha_i(x)f(\tau_i(x))+\int_0^{\sigma_1(x)}\Psi\left(x,t,f(\sigma_2(t)),\int_0^{\sigma_3(t)}f(s)ds\right)dt+g(x), \tag{1.1}$$

for all $x\in[0,1]$, where $\tau_i,\ \sigma_1,\ \sigma_2,\ \sigma_3:[0,1]\to[0,1];\ \alpha_i,\ g:[0,1]\to\mathbb{R};\ \Psi:[0,1]\times[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are given continuous functions and $f:[0,1]\to\mathbb{R}$ is an unknown function.

It is well known that, integral equations/functional integral equations have attracted great interest in the field of nonlinear analysis for a long time not only because of their mathematical context but also because of a large number of their applications. These equations can be used to model many problems of science and theoretical physics such as engineering, mechanic, electrostatics, population dynamics, economics, and other

*Correspondence: longnt2@gmail.com
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fields of science. They occur in a natural way in the description of many physical phenomena, for example, see the books written by Corduneanu [4], Deimling [6].

The existence, uniqueness, stability, and some other properties of solutions for functional integral equations as above have been proved by using many different methods together with various techniques in functional analysis. In general, existence results of these equations have been obtained via the fundamental methods in which the fixed point theorems are often applied, for example, we refer to the [1–3, 5, 7–23] and references therein.

In [1], Abdou et al. proved the existence of integrable solutions of the following nonlinear integral equation, by using the technique of measure of weak noncompactness and Schauder's fixed point theorem

$$\varphi(x,t) = f(x,t) + \int_0^1 k(x,y)\gamma(y,\varphi(y,t))dy + \int_0^t F(t,\tau)\varphi(x,\tau)d\tau,\ x \in [0,1],\ t \in [0,M],$$

where $M > 0$, the known functions $k(x,y)$, $F(t,\tau)$ are called the kernels of the integral terms with respect to position and time, respectively, $f(x,t)$, $\gamma(y,\varphi(y,t))$ are given functions in the space $L^1[0,1] \times C[0,M]$, and the function $\varphi(x,t)$ is unknown. Also, in [2], Abdou et al. considered a mixed nonlinear integral equation of the second kind in position and time, in which, the existence of a unique solution is proved using Banach's fixed point theorem.

In [3], Avramescu and Vladimirescu have proved the existence of asymptotically stable solutions to the following integral equations

$$u(t) = q(t) + f(t,u(t)) + \int_0^t V(t,s)u(s)ds + \int_0^t G(t,s,u(s))ds,\ t \in \mathbb{R}_+,$$

under suitable hypotheses. In the proofs, a fixed point theorem of Krasnosel'skii type is used. Also applying a fixed point theorem of Krasnosel'skii type and giving the suitable assumptions, Dhage and Ntouyas [7], Purnaras [22] obtained some results on the existence of solutions to the following nonlinear functional integral equation

$$x(t) = q(t) + \int_0^{\mu(t)} k(t,s)f(s,x(\theta(s)))ds + \int_0^{\sigma(t)} v(t,s)g(s,x(\eta(s)))ds,\ t \in [0,1],$$

where $0 \le \mu(t) \le t;\ 0 \le \sigma(t) \le t;\ 0 \le \theta(t) \le t;\ 0 \le \eta(t) \le t$, for all $t \in [0,1]$. Afterwards, Purnaras showed that the technique used in [22] can be applied to yield existence results for the following equation

$$x(t) = q(t) + \int_{\alpha(t)}^{\mu(t)} k(t,s)f(s,x(\theta(s)))ds + \int_{\beta(t)}^{\lambda(t)} \widehat{k}(t,s)F\left(s, x(\nu(s)), \int_0^{\sigma(s)} k_0\left(s,v,x(\eta(v))\right)dv\right)ds,\ t \in [0,1].$$

Using the technique of the measure of noncompactness and Darbo's fixed point theorem, Liu et al. [12] have proved the existence and asymptotic stability of solutions for the equation

$$x(t) = f\left(t,\ x(t),\ \int_0^t u\left(t,s,x(a(s)),x(b(s))\right)ds\right),\ t \in \mathbb{R}_+.$$

In [14, 16], similar types of (1.1) have been studied corresponding the following form

$$f_i(x) = \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ijk}\left(x, f_j(S_{ijk}(x))\right) + g_i(x),\ i = 1,\cdots,n,\ x \in I \subset \mathbb{R},$$

where I is a bounded or an unbounded interval. By using Banach's fixed point theorem, the authors have established the existence, uniqueness and stability of the solution for the equation considered with respect to the functions g_i. Furthermore, the quadratic convergence and an asymptotic expansion of solutions were also investigated. Motivated by the ideas and the techniques used in [14, 16], recently, similar (and more general) results were obtained in [19] for the following system of nonlinear functional equations

$$f_i(x) = \sum_{k=1}^{m}\sum_{j=1}^{n}\left[\varepsilon a_{ijk}\Psi\left(x, f_j(R_{ijk}(x)), \int_0^{X_{ijk}(x)} f_j(t)dt\right) + b_{ijk}f_j(S_{ijk}(x))\right] + g_i(x),$$

$i = 1,\cdots,n,\ x \in \Omega = [-b,b]$, where $a_{ijk},\ b_{ijk}$ are the given real constants; $R_{ijk},\ S_{ijk},\ X_{ijk} : \Omega \to \Omega$, $g_i : \Omega \to \mathbb{R},\ \Psi : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are the given continuous functions and $f_i : \Omega \to \mathbb{R}$ are unknown functions, ε is a small parameter.

In [5, 17, 18, 20, 21], some nonlinear functional integral equations in one variable or two variables, or N variables were also considered. Applying fixed point theorems such as Banach's fixed point theorem, a fixed point theorem of Krasnosel'skii type and using tools of functional analysis, the solvability and the existence of asymptotically stable solutions of these equations were proved. On the other hand, the set of solutions obtained in [5] is compact and obtained in [21] is a continuum.

Inspired and motivated by the idea of the abovementioned studies, we continue to investigate the more general nonlinear functional integral equation of the form (1.1). This paper consists of four sections. In Section 2, two theorems for the existence and some properties of the solutions of Eq. (1.1) are proved (Theorems 2.3 and 2.5). By applying Banach's fixed point theorem, Theorem 2.3 establishes the first sufficient condition for the unique existence and stability of the solution. By applying Schauder's fixed point theorem, Theorem 2.5 establishes the second sufficient condition for the existence and the compactness of the solution set. In order to illustrate the results obtained here, an example is given. Section 3 is devoted to investigating the quadratic convergence for the solution of Eq. (1.1) in the case of $\Psi \in C^2(\Omega \times \Omega \times \mathbb{R}^2;\mathbb{R})$. Finally, in Section 4, the smoothness of the solution of Eq. (1.1) depending on given data is established.

2. Existence, uniqueness, stability and compactness of the set of solutions

Let $\Omega = [0,1]$ and denote by $X = C(\Omega;\mathbb{R})$ the Banach space of continuous functions $f : \Omega \to \mathbb{R}$ with respect to the norm

$$\|f\|_X = \sup_{x\in\Omega}|f(x)|, \quad f \in X.$$

For any positive integer r, we put

$$X_r = C^r(\Omega;\mathbb{R}) = \left\{f \in X : f^{(k)} = \frac{d^k f}{dx^k} \in X,\ 1 \le k \le r\right\},$$

and note that $C^r(\Omega;\mathbb{R})$ is the Banach space with respect to the norm $\|f\|_{X_r} = \sum_{k=0}^{r}\left\|f^{(k)}\right\|_X$ (see [23]).

In order to prove sufficient conditions for the existence and some properties of solutions of Eq. (1.1) by using fixed point theorems, we first write Eq. (1.1) in the form of an operational equation in X as follows:

$$f = Uf + Vf + g, \tag{2.1}$$

where

$$\begin{cases} (Uf)(x) = \sum_{i=1}^{q} \alpha_i(x) f(\tau_i(x)), \\ (Vf)(x) = \int_0^{\sigma_1(x)} \Psi\left(x, t, f(\sigma_2(t)), \int_0^{\sigma_3(t)} f(s)ds\right) dt, \ x \in \Omega = [0,1]. \end{cases} \tag{2.2}$$

We now make the following assumptions to consider the unique existence and stability of the solution:

(H_1) $g \in X$;

(H_2) $\tau_i,\ \alpha_i \in X,\ (i = 1, \cdots, q)$, such that
(i) $\tau_i(x) \in [0,1],\ \forall x \in [0,1],\ (i = 1, \cdots, q)$,
(ii) $\sum_{i=1}^{q} \|\alpha_i\|_X < 1$;

(H_3) $\sigma_j \in C(\Omega; \Omega),\ (j = 1, 2, 3)$;

(H_4) $\Psi \in C([0,1] \times [0,1] \times \mathbb{R}^2; \mathbb{R})$, such that
$\forall M > 0,\ \exists K_M > 0 : |\Psi(x,t,y,z) - \Psi(x,t,\bar{y},\bar{z})| \le K_M \left(|y - \bar{y}| + |z - \bar{z}|\right)$,
for all $(x,t,y,z),\ (x,t,\bar{y},\bar{z}) \in [0,1] \times [0,1] \times [-M,M]^2$.

We state the following lemmas (Lemmas 2.1 and 2.2) which are useful to establish our main results. In Lemma 2.1 below, we denote by $\mathcal{L}(X)$ the Banach space of continuous (or bounded) linear operators $U : X \to X$ equipped with the norm $\|U\|_{\mathcal{L}(X)} = \sup_{0 \ne f \in X} \frac{\|Uf\|_X}{\|f\|_X}$.

Lemma 2.1. *Let (H_2) hold. Then the linear operator $I - U : X \to X$ is invertible and*

$$\left\|(I - U)^{-1}\right\|_{\mathcal{L}(X)} \le \frac{1}{1 - \sum_{i=1}^{q} \|\alpha_i\|_X}.$$

Proof of Lemma 2.1. By (H_2) holds, we have that the operator $U : X \to X$ is well defined and linear. On the other hand, for all $f \in X,\ |(Uf)(x)| \le \sum_{i=1}^{q} \|\alpha_i\|_X \|f\|_X,\ \forall x \in \Omega$, it leads to $\|U\|_{\mathcal{L}(X)} \le \sum_{i=1}^{q} \|\alpha_i\|_X < 1$. Therefore, the linear operator $I - U : X \to X$ is invertible (see [23]) and it is easy to verify that

$$\left\|(I - U)^{-1}\right\|_{\mathcal{L}(X)} \le \frac{1}{1 - \|U\|_{\mathcal{L}(X)}} \le \frac{1}{1 - \sum_{i=1}^{q} \|\alpha_i\|_X}. \ \square$$

In view of Lemma 2.1, we can rewrite Eq. (2.1) in the form

$$f = (I - U)^{-1}(Vf + g) \equiv Tf. \tag{2.3}$$

Lemma 2.2. *Let $(H_1) - (H_4)$ hold. Then, for every $M > 0$, we have*

(i) $\|Vf\|_X \le (2MK_M + \Psi_*) \|\sigma_1\|_X,\ \ \forall f \in B_M$;
(ii) $\left\|Vf - V\bar{f}\right\|_X \le 2K_M \|\sigma_1\|_X \left\|f - \bar{f}\right\|_X,\ \ \forall f,\ \bar{f} \in B_M$,

where $B_M = \{f \in X : \|f\|_X \le M\}$, $\Psi_* = \sup_{(x,t) \in \Delta_{\sigma_1}} |\Psi(x,t,0,0)|$, *and* $\Delta_{\sigma_1} = \{(x,t) : 0 \le t \le \sigma_1(x),\ 0 \le x \le 1\}$.

Proof of Lemma 2.2.

(i) $\forall f \in B_M,$ we have

$$\begin{aligned} |(Vf)(x)| &\leq \left| \int_0^{\sigma_1(x)} \left[\Psi\left(x,t,f(\sigma_2(t)),\int_0^{\sigma_3(t)} f(s)ds\right) - \Psi(x,t,0,0) \right] dt \right| \\ &\quad + \left| \int_0^{\sigma_1(x)} \Psi(x,t,0,0)\, dt \right| \\ &\leq \int_0^{\|\sigma_1\|_X} \left| \Psi\left(x,t,f(\sigma_2(t)),\int_0^{\sigma_3(t)} f(s)ds\right) - \Psi(x,t,0,0) \right| dt \\ &\quad + \int_0^{\|\sigma_1\|_X} |\Psi(x,t,0,0)|\, dt,\ \forall x \in \Omega, \end{aligned}$$

in which

$$\begin{aligned} |f(\sigma_2(t))| &\leq \|f\|_X \leq M, \\ 0 &\leq t \leq \|\sigma_1\|_X \leq 1, \\ \left| \int_0^{\sigma_3(t)} f(s)ds \right| &\leq \int_0^1 |f(s)|\, ds \leq \|f\|_X \leq M, \end{aligned}$$

$$\begin{aligned} \left| \Psi\left(x,t,f(\sigma_2(t)),\int_0^{\sigma_3(t)} f(s)ds\right) - \Psi(x,t,0,0) \right| &\leq K_M \left[|f(\sigma_2(t))| + \left| \int_0^{\sigma_3(t)} f(s)ds \right| \right] \\ &\leq 2MK_M, \\ \int_0^{\|\sigma_1\|_X} \left| \Psi\left(x,t,f(\sigma_2(t)),\int_0^{\sigma_3(t)} f(s)ds\right) - \Psi(x,t,0,0) \right| dt &\leq 2MK_M \|\sigma_1\|_X, \\ \int_0^{\|\sigma_1\|_X} |\Psi(x,t,0,0)|\, dt &\leq \|\sigma_1\|_X \Psi_*, \end{aligned}$$

hence

$$|(Vf)(x)| \leq (2MK_M + \Psi_*) \|\sigma_1\|_X,\ \forall x \in \Omega,$$

therefore (i) is verified.

(ii) $\forall f,\ \bar{f} \in B_M,$ we have

$$\begin{aligned} &\left| (Vf)(x) - (V\bar{f})(x) \right| \\ &\leq \int_0^{\|\sigma_1\|_X} \left| \Psi\left(x,t,f(\sigma_2(t)),\int_0^{\sigma_3(t)} f(s)ds\right) - \Psi\left(x,t,\bar{f}(\sigma_2(t)),\int_0^{\sigma_3(t)} \bar{f}(s)ds\right) \right| dt \\ &\leq K_M \int_0^{\|\sigma_1\|_X} \left[\left| f(\sigma_2(t)) - \bar{f}(\sigma_2(t) \right| + \left| \int_0^{\sigma_3(t)} (f(s) - \bar{f}(s))ds \right| \right] dt \\ &\leq 2K_M \|\sigma_1\|_X \left\| f - \bar{f} \right\|_X,\ \forall x \in \Omega, \end{aligned}$$

hence (ii) is also verified and Lemma 2.2 is proved.

We have the following theorem.

Theorem 2.3. *Let* $(H_1)-(H_4)$ *hold. In addition, suppose that there exists a constant* $M>0$ *such that the following condition is satisfied*

$$\frac{(2MK_M+\Psi_*)\left\|\sigma_1\right\|_X+\left\|g\right\|_X}{1-\sum_{i=1}^q\left\|\alpha_i\right\|_X}<M. \tag{2.4}$$

Then:

(i) *Eq.* (2.3) *has a unique solution* $f\in B_M$;

(ii) *The solution* f *of Eq.* (2.3) *is stable with respect to* g *in* X, *i.e. if* $f,\ \tilde{f}$ *are two solutions of Eq.* (2.3) *corresponding to two functions* $g,\ \tilde{g}$ *in* X, *and if* $\left\|g-\tilde{g}\right\|_X\to 0$ *then* $\left\|f-\tilde{f}\right\|_X\to 0.$

Proof.

It is clear that $T:X\to X$. Consider $f,\ \bar{f}\in B_M$, by Lemmas 2.1 and 2.2, and Eq. (2.4), we easily verify that

$$\begin{aligned}\left\|Tf\right\|_X &= \left\|(I-U)^{-1}(Vf+g)\right\|_X\leq\left\|(I-U)^{-1}\right\|_{\mathcal{L}(X)}\left\|Vf+g\right\|_X \\ &\leq \frac{1}{1-\sum_{i=1}^q\left\|\alpha_i\right\|_X}\left(\left\|Vf\right\|_X+\left\|g\right\|_X\right) \\ &\leq \frac{(2MK_M+\Psi_*)\left\|\sigma_1\right\|_X+\left\|g\right\|_X}{1-\sum_{i=1}^q\left\|\alpha_i\right\|_X}<M;\end{aligned} \tag{2.5}$$

$$\left\|Tf-T\bar{f}\right\|_X=\left\|(I-U)^{-1}\left(Vf-V\bar{f}\right)\right\|_X\leq\frac{2K_M\left\|\sigma_1\right\|_X}{1-\sum_{i=1}^q\left\|\alpha_i\right\|_X}\left\|f-\bar{f}\right\|_X. \tag{2.6}$$

We note more that, by (2.4), we have $\sigma\equiv\dfrac{2K_M\left\|\sigma_1\right\|_X}{1-\sum_{i=1}^q\left\|\alpha_i\right\|_X}<1$. Therefore, by (2.5)–(2.6), we obtain that $T:B_M\to B_M$ is a contraction mapping. Applying Banach's fixed point theorem, we conclude that there exists a unique function $f\in B_M$ such that $f=Tf$. It means that Eq. (2.3) has a unique solution $f\in B_M$.

It remains to prove that this solution is stable with respect to g in X. Let $f,\ \tilde{f}$ be two solutions of Eq. (2.3) corresponding to two functions $g,\ \tilde{g}$ in X. Then

$$\begin{aligned}\left\|f-\tilde{f}\right\|_X &= \left\|(I-U)^{-1}\left(Vf-V\tilde{f}+g-\tilde{g}\right)\right\|_X \\ &\leq \frac{2K_M\left\|\sigma_1\right\|_X}{1-\sum_{i=1}^q\left\|\alpha_i\right\|_X}\left\|f-\tilde{f}\right\|_X+\frac{\left\|g-\tilde{g}\right\|_X}{1-\sum_{i=1}^q\left\|\alpha_i\right\|_X}.\end{aligned}$$

As this gives

$$\left\|f-\tilde{f}\right\|_X\leq\frac{1}{(1-\sigma)\left(1-\sum_{i=1}^q\left\|\alpha_i\right\|_X\right)}\left\|g-\tilde{g}\right\|_X,$$

obviously, if $\left\|g-\tilde{g}\right\|_X\to 0$ then $\left\|f-\tilde{f}\right\|_X\to 0$. Theorem 2.3 is proved.

Corollary 2.4. *Let $(H_1)-(H_4)$ hold. Then, there exist constants $M>0$ and $\rho_*>0$ such that, if $\|\sigma_1\|_X<\rho_*$, then Eq. (2.3) has a unique solution $f\in B_M$. Furthermore, f is stable with respect to g in X.*

Proof of Corollary 2.4. First, we have

$$\begin{aligned} 0 &\leq \Psi_* \|\sigma_1\|_X \leq \left(\sup_{0\leq x\leq 1,\ 0\leq t\leq \|\sigma_1\|_X} |\Psi(x,t,0,0)| \right) \|\sigma_1\|_X \\ &\leq \left(\sup_{0\leq x\leq 1,\ 0\leq t\leq 1} |\Psi(x,t,0,0)| \right) \|\sigma_1\|_X \to 0, \text{ as } \|\sigma_1\|_X \to 0. \end{aligned}$$

Now, by choosing $M>\dfrac{\|g\|_X}{1-\sum_{i=1}^q \|\alpha_i\|_X}$ to get

$$\lim_{\|\sigma_1\|_X\to 0} \frac{(2MK_M+\Psi_*)\|\sigma_1\|_X+\|g\|_X}{1-\sum_{i=1}^q \|\alpha_i\|_X} = \frac{\|g\|_X}{1-\sum_{i=1}^q \|\alpha_i\|_X} < M,$$

we imply that there exists a constant $\rho_*\in(0,1)$ such that $\forall \sigma_1\in X,\ \|\sigma_1\|_X\leq \rho_*$, we have

$$\frac{(2MK_M+\Psi_*)\|\sigma_1\|_X+\|g\|_X}{1-\sum_{i=1}^q \|\alpha_i\|_X} < M,$$

which means that the condition (2.4) is satisfied. Applying Theorem 2.3, Corollary 2.4 follows.

Remark 2.1. Based on the proof of Theorem 2.3 with using Banach's fixed point theorem, we have a consecutive approximate algorithm

$$f^{(\nu)} = Tf^{(\nu-1)},\ \nu=1,2,\cdots, \tag{2.7}$$

where $f^{(0)}\in X$ is given, to derive the sequence $\{f^{(\nu)}\}$ such that $\{f^{(\nu)}\}$ converges in X to the solution f of Eq. (2.3), with the error estimation obtained as follows:

$$\left\| f^{(\nu)}-f \right\|_X \leq \frac{\sigma^\nu}{1-\sigma} \left\| Tf^{(0)}-f^{(0)} \right\|_X \text{ for all } \nu\in\mathbb{N}. \tag{2.8}$$

We next change the assumptions to get the existence and the compactness of the solution set. We shall replace the assumption (H_4) concerned to local Lipschitz property of Ψ by the assumption $(\bar{H}_4)$ as follows:

$(\bar{H}_4)$ $\Psi\in C([0,1]\times[0,1]\times\mathbb{R}^2;\mathbb{R})$ such that, there exists
a nonnegative function $k_0:[0,1]\times[0,1]\to\mathbb{R}$ and a constant $\alpha>1$ satisfying
(i) $\sup_{0\leq x\leq 1} \int_0^1 k_0^\alpha(x,t)dt<+\infty$,
(ii) $|\Psi(x,t,y,z)|\leq k_0(x,t)(1+|y|+|z|)$, for all $(x,t,y,z)\in[0,1]\times[0,1]\times\mathbb{R}^2$.

We have the following theorem.

Theorem 2.5. *Let $(H_1)-(H_3)$, $(\bar{H}_4)$ hold. In addition, suppose that there exists a constant $M>0$ such that the following condition is satisfied*

$$\frac{(1+2M)\left(\|\sigma_1\|_X\right)^{\frac{\alpha-1}{\alpha}} \left(\sup_{0\leq x\leq 1} \int_0^1 k_0^\alpha(x,t)dt \right)^{1/\alpha} + \|g\|_X}{1-\sum_{i=1}^q \|\alpha_i\|_X} < M. \tag{2.9}$$

Then, Eq. (2.3) *has a solution* $f \in B_M$. *Furthermore, the set of solutions is compact in* X.

Proof. We proceed in two steps.

Step 1. *The existence.* We first consider the operators U, V as in (2.2) and T as in (2.3).

Obviously, $Uf,\ Vf,\ Tf \in X,\ \forall f \in X$. We also note that, for all $f \in B_M = \{f \in X : \|f\|_X \leq M\}$, we have

$$\begin{aligned}
|(Vf)(x)| &\leq \left|\int_0^{\sigma_1(x)} \left[\Psi\left(x, t, f(\sigma_2(t)), \int_0^{\sigma_3(t)} f(s)ds\right)\right] dt\right| \\
&\leq \int_0^{\|\sigma_1\|_X} k_0(x,t)\left(1 + |f(\sigma_2(t))| + \left|\int_0^{\sigma_3(t)} f(s)ds\right|\right) dt \\
&\leq \int_0^{\|\sigma_1\|_X} k_0(x,t)\left(1 + 2\|f\|_X\right) dt \\
&\leq (1+2M)\left(\|\sigma_1\|_X\right)^{\frac{\alpha-1}{\alpha}}\left(\int_0^1 k_0^{\alpha}(x,t)dt\right)^{1/\alpha} \\
&\leq (1+2M)\left(\|\sigma_1\|_X\right)^{\frac{\alpha-1}{\alpha}}\left(\sup_{0\leq x\leq 1}\int_0^1 k_0^{\alpha}(x,t)dt\right)^{1/\alpha},\ \forall x \in \Omega.
\end{aligned}$$

It implies that the following estimation for $\|Vf\|_X$ is fulfilled

$$\|Vf\|_X \leq (1+2M)\left(\|\sigma_1\|_X\right)^{\frac{\alpha-1}{\alpha}}\left(\sup_{0\leq x\leq 1}\int_0^1 k_0^{\alpha}(x,t)dt\right)^{1/\alpha}. \tag{2.10}$$

Combining Lemma 2.1 with Eqs. (2.9) and (2.10), we verify that

$$\begin{aligned}
\|Tf\|_X &= \left\|(I-U)^{-1}(Vf+g)\right\|_X \leq \left\|(I-U)^{-1}\right\|_{\mathcal{L}(X)}\|Vf+g\|_X \qquad (2.11)\\
&\leq \frac{1}{1-\sum_{i=1}^q\|\alpha_i\|_X}\left(\|Vf\|_X + \|g\|_X\right) \\
&\leq \frac{(1+2M)\left(\|\sigma_1\|_X\right)^{\frac{\alpha-1}{\alpha}}\left(\sup\limits_{0\leq x\leq 1}\int_0^1 k_0^{\alpha}(x,t)dt\right)^{1/\alpha} + \|g\|_X}{1-\sum_{i=1}^q\|\alpha_i\|_X} < M,
\end{aligned}$$

for all $f \in B_M$. Thus $\overline{T(B_M)} \subset B_M$. Next, we show that the conditions (a), (b) below are satisfied.

(a) $T : B_M \to B_M$ is continuous.

(b) $\mathcal{F} = T(B_M)$ is relatively compact in X.

To prove (a), for $\{f_p\} \subset B_M$, $\|f_p - f\|_X \to 0$, as $p \to \infty$, it suffices to show that $\|Tf_p - Tf\|_X \to 0$, as $p \to \infty$.

By

$$\begin{aligned}
\|Tf_p - Tf\|_X &= \left\|(I-U)^{-1}(Vf_p - Vf)\right\|_X \leq \left\|(I-U)^{-1}\right\|_{\mathcal{L}(X)}\|Vf_p - Vf\|_X \\
&\leq \frac{1}{1-\sum_{i=1}^q\|\alpha_i\|_X}\|Vf_p - Vf\|_X,
\end{aligned}$$

all we have to prove is that $\|Vf_p - Vf\|_X \to 0$, as $p \to \infty$.

The function Ψ is uniformly continuous on $D_M = [0,1] \times [0,1] \times [-M, M]^2$, so for all $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $\forall (x,t,y,z)$, $(x,t,\bar{y},\bar{z}) \in D_M$,

$$|y - \bar{y}| + |z - \bar{z}| < \delta_1 \Longrightarrow |\Psi(x,t,y,z) - \Psi(x,t,\bar{y},\bar{z})| < \frac{\varepsilon}{1 + \|\sigma_1\|_X}. \tag{2.12}$$

Because of $\|f_p - f\|_X \to 0$, as $p \to \infty$, there exists $p_0 \in \mathbb{N}$, such that

$$\|f_p - f\|_X < \frac{\delta_1}{2}, \ \forall p > p_0. \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$\left| \Psi\left(x, t, f_p(\sigma_2(t)), \int_0^{\sigma_3(t)} f_p(s)ds\right) - \Psi\left(x, t, f(\sigma_2(t)), \int_0^{\sigma_3(t)} f(s)ds\right) \right| < \frac{\varepsilon}{1 + \|\sigma_1\|_X},$$

$\forall p > p_0$, $\forall (x,t) \in [0,1] \times [0,1]$. Therefore, we get

$$\begin{aligned} & |(Vf_p)(x) - (Vf)(x)| \\ \leq \ & \int_0^{\sigma_1(x)} \left| \Psi\left(x, t, f_p(\sigma_2(t)), \int_0^{\sigma_3(t)} f_p(s)ds\right) - \Psi\left(x, t, f(\sigma_2(t)), \int_0^{\sigma_3(t)} f(s)ds\right) \right| dt \\ \leq \ & \|\sigma_1\|_X \frac{\varepsilon}{1 + \|\sigma_1\|_X} < \varepsilon, \end{aligned}$$

which means that $\|Vf_p - Vf\|_X < \varepsilon$, $\forall p > p_0$. Condition (a) is satisfied.

To prove (b), note that $Tf = (I-U)^{-1}(Vf + g) = (I-U)^{-1}Vf + (I-U)^{-1}g$, so we need to show that $V : B_M \to X$ is compact. We shall prove that $V(B_M)$ is bounded and equicontinuous in X.

It is easy to see that $V(B_M)$ is bounded, since (2.9) and (2.10) lead to

$$\begin{aligned} \|Vf\|_X \ & \leq \ (1 + 2M)(\|\sigma_1\|_X)^{\frac{\alpha-1}{\alpha}} \left(\sup_{0 \leq x \leq 1} \int_0^1 k_0^\alpha(x,t)dt \right)^{1/\alpha} \\ & < \ M\left(1 - \sum\nolimits_{i=1}^q \|\alpha_i\|_X\right) - \|g\|_X \equiv \tilde{M}. \end{aligned} \tag{2.14}$$

On the other hand, $\forall f \in B_M$, $\forall x$, $\bar{x} \in [0,1]$, we have

$$\begin{aligned} & |(Vf)(x) - (Vf)(\bar{x})| \\ \leq \ & \int_0^{\sigma_1(x)} \left| \Psi\left(x, t, f(\sigma_2(t)), \int_0^{\sigma_3(t)} f_p(s)ds\right) - \Psi\left(\bar{x}, t, f(\sigma_2(t)), \int_0^{\sigma_3(t)} f(s)ds\right) \right| dt \\ & + \left| \int_{\sigma_1(\bar{x})}^{\sigma_1(x)} \Psi\left(\bar{x}, t, f(\sigma_2(t)), \int_0^{\sigma_3(t)} f(s)ds\right) \right| dt \\ \leq \ & \int_0^{\sigma_1(x)} \left| \Psi\left(x, t, f(\sigma_2(t)), \int_0^{\sigma_3(t)} f_p(s)ds\right) - \Psi\left(\bar{x}, t, f(\sigma_2(t)), \int_0^{\sigma_3(t)} f(s)ds\right) \right| dt \\ & + \Psi_M^* \ |\sigma_1(x) - \sigma_1(\bar{x})|, \end{aligned} \tag{2.15}$$

in which $\Psi_M^* = \sup_{(x,t,y,z)\in D_M} |\Psi(x,t,y,z)|$.

Similarly to (2.12), Ψ is uniformly continuous on $D_M = [0,1]\times[0,1]\times[-M,M]^2$, so for all $\varepsilon > 0$, there also exists $\delta_{*1} > 0$ such that $\forall (x,t,y,z),\ (\bar{x},t,y,z) \in D_M$,

$$|x-\bar{x}| < \delta_{*1} \Longrightarrow |\Psi(x,t,y,z) - \Psi(\bar{x},t,y,z)| < \frac{\varepsilon}{2(1+\|\sigma_1\|_X)}. \tag{2.16}$$

It follows that if $|x-\bar{x}| < \delta_{*1}$ then

$$\begin{aligned} &\int_0^{\sigma_1(x)} \left| \Psi\left(x,t,f(\sigma_2(t)),\int_0^{\sigma_3(t)} f_p(s)ds\right) - \Psi\left(\bar{x},t,f(\sigma_2(t)),\int_0^{\sigma_3(t)} f(s)ds\right)\right| dt \qquad (2.17)\\ \leq\ & \int_0^{\sigma_1(x)} \frac{\varepsilon}{2(1+\|\sigma_1\|_X)} dt \\ <\ & \|\sigma_1\|_X \frac{\varepsilon}{2(1+\|\sigma_1\|_X)} < \frac{\varepsilon}{2},\ \forall f \in B_M. \end{aligned}$$

As σ_1 is uniformly continuous on $[0,1]$, there exists $\delta_{*2} > 0 : \forall x,\bar{x} \in [0,1]$,

$$|x-\bar{x}| < \delta_{*2} \Longrightarrow \Psi_M^* |\sigma_1(x) - \sigma_1(\bar{x})| < \frac{\varepsilon}{2}. \tag{2.18}$$

Consequently, for all $x,\ \bar{x} \in [0,1]$, if $|x-\bar{x}| < \min\{\delta_{*1},\delta_{*2}\}$, then from (2.15), (2.17), and (2.18), we have that

$$|(Vf)(x) - (Vf)(\bar{x})| < \varepsilon,\ \forall f \in B_M.$$

This means that $V(B_M)$ is equicontinuous in X, (b) follows. Therefore, the operator $T : B_M \to B_M$ is compact. Applying Schauder's fixed point theorem for the operator T, the existence of a solution of Eq. (2.3) is proved.

Step 2. *The compactness of the solution set.*

Consider the set of solutions obtained in step 1, that is, $S = \{f \in B_M : f = Tf\} \subset X$. By the compactness of the operator $T : B_M \to B_M$ and $S = T(S)$, in order to prove the compactness of S, we only check that T is closed.

Let $\{f_p\} \subset S$, $\|f_p - f\|_X \to 0$. By the continuity of T, it gives $\|Tf_p - Tf\|_X \to 0$, this implies that

$$\|f - Tf\|_X \leq \|f - f_p\|_X + \|f_p - Tf\|_X = \|f - f_p\|_X + \|Tf_p - Tf\|_X \to 0.$$

Thus, $f = Tf \in S$. Theorem 2.5 is proved.

Corollary 2.6. *Let $(H_1)-(H_3)$, $(\bar{H}_4)$ hold. Then, there exist the constants $M > 0$ and $\rho_* > 0$ such that, if $\|S_1\|_X < \rho_*$, then Eq. (2.3) has a solution $f \in B_M$. Furthermore, the set of solutions is compact in X.*

Proof of Corollary 2.6. By choosing $M > \dfrac{\|g\|_X}{1-\sum_{i=1}^q \|\alpha_i\|_X}$, we obtain that

$$\lim_{\|\sigma_1\|_X \to 0} \frac{(1+2M)(\|\sigma_1\|_X)^{\frac{\alpha-1}{\alpha}} \left(\sup_{0\leq x\leq 1} \int_0^1 k_0^\alpha(x,t)dt\right)^{1/\alpha} + \|g\|_X}{1-\sum_{i=1}^q \|\alpha_i\|_X} = \frac{\|g\|_X}{1-\sum_{i=1}^q \|\alpha_i\|_X} < M,$$

so there exists a constant $\rho_* \in (0,1)$ such that $\forall \sigma_1 \in X,\ \|\sigma_1\|_X \le \rho_*,$ we get

$$\frac{(1+2M)\left(\|\sigma_1\|_X\right)^{\frac{\alpha-1}{\alpha}} \left(\sup\limits_{0\le x\le 1} \int_0^1 k_0^{\alpha}(x,t)dt\right)^{1/\alpha} + \|g\|_X}{1-\sum_{i=1}^{q}\|\alpha_i\|_X} < M.$$

Therefore, the condition (2.9) is satisfied. Applying Theorem 2.5, Corollary 2.6 is proved.

An example. To illustrate the results obtained in Theorems 2.3 and 2.5, we give an example as follows: Considering the following nonlinear functional integral equation

$$\begin{aligned} f(x) &= \sum_{i=1}^{q} \alpha_i(x) f(\tau_i(x)) + \int_0^{\sigma_1(x)} k_1(x,t) \left|f(\sigma_2(t))\right|^{\gamma_1} dt \qquad (2.19) \\ &\quad + \int_0^{\sigma_1(x)} k_2(x,t) \left| \int_0^{\sigma_3(t)} f(s)ds \right|^{\gamma_2} dt + g(x),\ x \in [0,1], \end{aligned}$$

where the functions $\alpha_i,\ \tau_i,\ \sigma_1,\ \sigma_2,\ \sigma_3,\ g$ are given and satisfy $(H_1)-(H_3)$; the functions $k_1,\ k_2$, and constants $\gamma_1 > 0,\ \gamma_2 > 0$ are also given and satisfy some suitable condition which will be defined later. Eq. (2.19) has the form (1.1) with the function Ψ defined by

$$\Psi(x,t,y,z) = k_1(x,t)\,|y|^{\gamma_1} + k_2(x,t)\,|z|^{\gamma_2}. \qquad (2.20)$$

We shall consider two cases of the function Ψ:

(i) **Case 1**. $k_1,\ k_2 \in C([0,1]\times[0,1])$ and $\gamma_1,\ \gamma_2 \ge 1$.

In this case, $\Psi \in C([0,1]\times[0,1]\times\mathbb{R}^2;\mathbb{R})$ satisfies (H_4). Indeed, by using the inequality

$$\left||y|^{\gamma} - |\bar{y}|^{\gamma}\right| \le \gamma M^{\gamma-1}\,|y-\bar{y}|,\ \forall y,\bar{y}\in[-M,M],\ \forall M>0,\ \forall \gamma>1,$$

we have that, for each $M>0$, for all $(x,t,y,z),\ (x,t,\bar{y},\bar{z}) \in [0,1]\times[0,1]\times[-M,M]^2$:

$$\begin{aligned} |\Psi(x,t,y,z)-\Psi(x,t,\bar{y},\bar{z})| &\le |k_1(x,t)|\,\left||y|^{\gamma_1}-|\bar{y}|^{\gamma_1}\right| + |k_2(x,t)|\,\left||z|^{\gamma_2}-|\bar{z}|^{\gamma_2}\right| \qquad (2.21) \\ &\le \gamma_1 M^{\gamma_1-1}\,|k_1(x,t)|\,|y-\bar{y}| + \gamma_2 M^{\gamma_2-1}\,|k_2(x,t)|\,|z-\bar{z}| \\ &\le K_M\left(|y-\bar{y}|+|z-\bar{z}|\right), \end{aligned}$$

in which $K_M = \gamma_1 M^{\gamma_1-1} \sup\limits_{0\le x,t\le 1} |k_1(x,t)| + \gamma_2 M^{\gamma_2-1} \sup\limits_{0\le x,t\le 1} |k_2(x,t)|.$

Therefore, (H_4) holds. It leads to Theorem 2.3 is true for Eq. (2.19).

(ii) **Case 2**. $k_1,\ k_2 \in C([0,1]\times[0,1])$ and $0<\gamma_1,\ \gamma_2<1$.

In this case, $\Psi \in C([0,1]\times[0,1]\times\mathbb{R}^2;\mathbb{R})$ satisfies $(\bar{H}_4)$. We check it as follows.

Using the inequality $|y|^{\gamma} \le 1+|y|,\ \forall y\in\mathbb{R},\ \forall\gamma\in(0,1)$, we get

$$\begin{aligned} |\Psi(x,t,y,z)| &\le |k_1(x,t)|\,|y|^{\gamma_1} + |k_2(x,t)|\,|z|^{\gamma_2} \qquad (2.22) \\ &\le |k_1(x,t)|\,(1+|y|) + |k_2(x,t)|\,(1+|z|) \\ &\le k_0(x,t)\,(1+|y|+|z|), \end{aligned}$$

with $k_0(x,t) = |k_1(x,t)| + |k_2(x,t)|$. It is clear that

$$\sup_{0\le x\le 1}\int_0^1 k_0^2(x,t)dt \le \left(\sup_{0\le x,t\le 1} k_0(x,t)\right)^2 < +\infty.$$

Thus, there exists a nonnegative function $k_0 : [0,1]\times[0,1] \to \mathbb{R}$ defined as above and a constant $\alpha = 2 > 1$ such that $(\bar{H}_4)$ holds. Theorem 2.5 is also true for Eq. (2.19).

3. Second order algorithm

In this section, we begin with making the hypothesis $\Psi \in C^1([0,1]^2 \times \mathbb{R}^2;\mathbb{R})$ and use the approximation of $\Psi(x,t,u^{(\nu)},v^{(\nu)})$ in the form

$$\begin{aligned}\Psi(x,t,u^{(\nu)},v^{(\nu)}) \quad &\cong \quad \Psi(x,t,u^{(\nu-1)},v^{(\nu-1)}) + D_3\Psi(x,t,u^{(\nu-1)},v^{(\nu-1)})\left(u^{(\nu)} - u^{(\nu-1)}\right)\\ &\quad +D_4\Psi(x,t,u^{(\nu-1)},v^{(\nu-1)})\left(v^{(\nu)} - v^{(\nu-1)}\right),\end{aligned}$$

where $u^{(\nu)} = f^{(\nu)}(\sigma_2(t))$, $v^{(\nu)} = \int_0^{\sigma_3(t)} f^{(\nu)}(s)ds$, to obtain the following algorithm for Eq. (1.1)

$$\begin{aligned} f^{(\nu)}(x) \quad &= \quad \sum_{i=1}^q \alpha_i(x) f^{(\nu)}(\tau_i(x)) + \int_0^{\sigma_1(x)} \Psi\left(W^{(\nu)}(x,t)\right)dt \\ &\quad + \int_0^{\sigma_1(x)} D_3\Psi\left(W^{(\nu)}(x,t)\right)\left(f^{(\nu)}(\sigma_2(t)) - f^{(\nu-1)}(\sigma_2(t))\right)dt \\ &\quad + \int_0^{\sigma_1(x)} D_4\Psi\left(W^{(\nu)}(x,t)\right)\left(\int_0^{\sigma_3(t)} f^{(\nu)}(s)ds - \int_0^{\sigma_3(t)} f^{(\nu-1)}(s)ds\right)dt + g(x), \end{aligned} \tag{3.1}$$

for all $x \in \Omega$ and $\nu = 1,2,\cdots$, where $f^{(0)} \in K_M$ is given and $W^{(\nu)}(x,t) \in [0,1]^2 \times \mathbb{R}^2$ is

$$W^{(\nu)}(x,t) = \left(x,t,f^{(\nu-1)}(\sigma_2(t)), \int_0^{\sigma_3(t)} f^{(\nu-1)}(s)ds\right). \tag{3.2}$$

Rewrite (3.1) in the form

$$\begin{aligned} f^{(\nu)}(x) \quad &= \quad (Uf^{(\nu)})(x) + \int_0^{\sigma_1(x)}\left[\Psi_3^{(\nu)}(x,t)f^{(\nu)}(\sigma_2(t)) + \Psi_4^{(\nu)}(x,t)\int_0^{\sigma_3(t)} f^{(\nu)}(s)ds\right]dt \\ &\quad +g^{(\nu)}(x), \end{aligned} \tag{3.3}$$

for $x \in \Omega$ and $\nu = 1,2,\cdots$, with $\Psi_3^{(\nu)}(x,t)$, $\Psi_4^{(\nu)}(x,t)$ and $g^{(\nu)}(x)$ depending on $f^{(\nu-1)}$ as follows:

$$\Psi_i^{(\nu)}(x,t) = D_i\Psi\left(W^{(\nu)}(x,t)\right),\ i = 3,4, \tag{3.4}$$

and

$$g^{(\nu)}(x) = g(x)+\int_0^{\sigma_1(x)} \Psi_0^{(\nu)}(x,t)dt \tag{3.5}$$
$$- \int_0^{\sigma_1(x)} \left[\Psi_3^{(\nu)}(x,t)f^{(\nu-1)}(\sigma_2(t)) + \Psi_4^{(\nu)}(x,t)\int_0^{\sigma_3(t)} f^{(\nu-1)}(s)ds \right] dt,$$
$$\Psi_0^{(\nu)}(x,t) = \Psi\left(W^{(\nu)}(x,t)\right).$$

Then, we have the following.

Theorem 3.1. *Let* $(H_1)-(H_3)$ *hold and let* $\Psi \in C^1([0,1]^2\times\mathbb{R}^2;\mathbb{R})$. *If* $f^{(\nu-1)} \in X$ *satisfies*

$$\gamma_\nu = \sum\nolimits_{i=1}^q \|\alpha_i\|_X + \sup_{(x,t,y,z)\in D^{(\nu)}} \left(|D_3\Psi(x,t,y,z)| + |D_4\Psi(x,t,y,z)|\right)\|\sigma_1\|_X < 1,$$

where $D^{(\nu)} = [0,1]^2 \times \left[-\left\|f^{(\nu-1)}\right\|_X, \left\|f^{(\nu-1)}\right\|_X\right]^2$, *then Eq. (3.3) has a unique solution* $f^{(\nu)} \in X$.

Proof. We write Eq. (3.3) in the form of an operational equation in X

$$f^{(\nu)} = T_\nu f^{(\nu)}, \tag{3.6}$$

where

$$(T_\nu f)(x) = (Uf)(x)+\int_0^{\sigma_1(x)} \left[\Psi_3^{(\nu)}(x,t)f(\sigma_2(t)) + \Psi_4^{(\nu)}(x,t)\int_0^{\sigma_3(t)} f(s)ds \right] dt \tag{3.7}$$
$$+g^{(\nu)}(x),$$

for $x \in \Omega$ and $f \in X$.

It is not difficult to check that $T_\nu : X \to X$ and

$$\left\|T_\nu f - T_\nu \bar{f}\right\|_X \le \gamma_\nu \left\|f-\bar{f}\right\|_X \text{ for all } f,\ \bar{f}\in X.$$

Using Banach's fixed point theorem for $T_\nu : X \to X$, it follows that Eq. (3.6) has a unique solution $f^{(\nu)} \in X$. Theorem 3.1 is proved.

Next, we strengthen the following hypotheses:

(H_5) $\Psi \in C^2([0,1]^2\times\mathbb{R}^2;\mathbb{R})$,

(H_6) $\|g\|_X + \left(M_0 + M_{1*} + 2M_{2*}M^2\right)M\|\sigma_1\|_X \le M\left(1-\sum_{i=1}^q \|\alpha_i\|_X\right)$,

where

$$\begin{cases} M_0 = \sup\limits_{0\le x\le 1,\ 0\le t\le 1} |\Psi(x,t,0,0)| \\ M_{0*} = \sup\left\{|\Psi(x,t,y,z)| : (x,t,y,z)\in D_M\right\}, \\ M_{1*} = \sup\left\{\left(|D_3\Psi| + |D_4\Psi|\right)(x,t,y,z) : (x,t,y,z)\in D_M\right\}, \\ M_{2*} = \sup\left\{\left(\left|D_3^2\Psi\right| + |D_3D_4\Psi| + \left|D_4^2\Psi\right|\right)(x,t,y,z) : (x,t,y,z)\in D_M\right\}, \end{cases}$$

with $D_M = \{(x,t,y,z) : x,t\in[0,1],\ y,z\in[-M,M]\}$.

Theorem 3.2. *Let* $(H_1)-(H_3)$, (H_5), (H_6) *hold. Assume that* f *is the solution of Eq.* (1.1) *and the sequence* $\{f^{(\nu)}\}$ *is defined by algorithm* (3.3)–(3.5).

(i) If $\|f^{(0)}\|_X \le M$, *then*

$$\left\|f^{(\nu)}-f\right\|_X \le \beta_M \left\|f^{(\nu-1)}-f\right\|_X^2, \quad \forall \nu = 1,2,\cdots, \tag{3.8}$$

where

$$\beta_M = \frac{2\|\sigma_1\|_X M_{2*}}{1-\sum_{i=1}^q \|\alpha_i\|_X - \|\sigma_1\|_X M_{1*}} > 0. \tag{3.9}$$

(ii) If the first term $f^{(0)}$ *is given sufficiently near* f *such that* $\beta_M \|f^{(0)}-f\|_X < 1$, *then the sequence* $\{f^{(\nu)}\}$ *converges quadratically to* f *and furthermore the estimate is obtained as follows:*

$$\left\|f^{(\nu)}-f\right\|_X \le \frac{1}{\beta_M}\left(\beta_M \left\|f^{(0)}-f\right\|_X\right)^{2^\nu}, \quad \forall \nu = 1,2,\cdots. \tag{3.10}$$

Proof. First, we verify that if $\|f^{(0)}\|_X \le M$, then

$$\left\|f^{(\nu)}\right\|_X \le M, \quad \forall \nu = 1,2,\cdots. \tag{3.11}$$

Indeed, by induction we assume that

$$\left\|f^{(\nu-1)}\right\|_X \le M, \tag{3.12}$$

we deduce from (3.3) that

$$\left\|f^{(\nu)}\right\|_X \le \left(\sum_{i=1}^q \|\alpha_i\|_X + M_{1*}\|\sigma_1\|_X\right)\left\|f^{(\nu)}\right\|_X + \left\|g^{(\nu)}\right\|_X. \tag{3.13}$$

Note that (H_6) leads to $\sum_{i=1}^q \|\alpha_i\|_X + M_{1*}\|\sigma_1\|_X < 1$, therefore (3.13) gives

$$\left\|f^{(\nu)}\right\|_X \le \frac{\|g^{(\nu)}\|_X}{1-\sum_{i=1}^q \|\alpha_i\|_X - M_{1*}\|\sigma_1\|_X}. \tag{3.14}$$

Estimating the term $\|g^{(\nu)}\|_X$: From (3.4) and (3.5), we obtain

$$\begin{aligned} g^{(\nu)}(x) &= g(x) \\ &\quad + \int_0^{\sigma_1(x)} \left[\Psi\left(W^{(\nu)}(x,t)\right) - D_3\Psi\left(W^{(\nu)}(x,t)\right) f^{(\nu-1)}(\sigma_2(t))\right. \\ &\quad \left. - D_4\Psi\left(W^{(\nu)}(x,t)\right)\int_0^{\sigma_3(t)} f^{(\nu-1)}(s)ds\right]dt. \end{aligned} \tag{3.15}$$

On the other hand, by using the Taylor's expansion of the function $\Psi(x,t,0,0)$ up to order 2 at $W^{(\nu)}(x,t)$

(given in (3.2)), we get

$$\begin{aligned}\Psi\left(x,t,0,0\right) &= \Psi(W^{(\nu)}(x,t)) - D_3\Psi\left(W^{(\nu)}(x,t)\right) f^{(\nu-1)}(\sigma_2(t)) \\ &-D_4\Psi\left(W^{(\nu)}(x,t)\right)\int_0^{\sigma_3(t)} f^{(\nu-1)}(s)ds + \frac{1}{2}D_3^2\Psi\left(\bar{W}^{(\nu)}(x,t)\right)\left|f^{(\nu-1)}(\sigma_2(t))\right|^2 \\ &+D_3D_4\Psi\left(\bar{W}^{(\nu)}(x,t)\right) f^{(\nu-1)}(\sigma_2(t))\int_0^{\sigma_3(t)} f^{(\nu-1)}(s)ds \\ &+\frac{1}{2}D_4^2\Psi\left(\bar{W}^{(\nu)}(x,t)\right)\left|\int_0^{\sigma_3(t)} f^{(\nu-1)}(s)ds\right|^2,\end{aligned} \tag{3.16}$$

where

$$\bar{W}^{(\nu)}(x,t) = \left(x,t,\theta f^{(\nu-1)}(\sigma_2(t)),\ \theta\int_0^{\sigma_3(t)} f^{(\nu-1)}(s)ds\right),\quad 0<\theta<1.$$

Therefore

$$\begin{aligned}&\left|\Psi(W^{(\nu)}(x,t)) - D_3\Psi\left(W^{(\nu)}(x,t)\right) f^{(\nu-1)}(\sigma_2(t)) - D_4\Psi\left(W^{(\nu)}(x,t)\right)\int_0^{\sigma_3(t)} f^{(\nu-1)}(s)ds\right| \\ &= \left|\Psi(x,t,0,0) - \frac{1}{2}D_3^2\Psi\left(\bar{W}^{(\nu)}(x,t)\right)\left|f^{(\nu-1)}(\sigma_2(t))\right|^2 - D_3D_4\Psi\left(\bar{W}^{(\nu)}(x,t)\right) f^{(\nu-1)}(\sigma_2(t))\int_0^{\sigma_3(t)} f^{(\nu-1)}(s)ds - \frac{1}{2}D_4^2\Psi\left(\bar{W}^{(\nu)}(x,t)\right)\left|\int_0^{\sigma_3(t)} f^{(\nu-1)}(s)ds\right|^2\right| \\ &\le M_0 + \frac{1}{2}M_{2*}\left\|f^{(\nu-1)}\right\|_X^2 + M_{2*}\left\|f^{(\nu-1)}\right\|_X^2 + \frac{1}{2}M_{2*}\left\|f^{(\nu-1)}\right\|_X^2 \\ &= M_0 + 2M_{2*}\left\|f^{(\nu-1)}\right\|_X^2 \le M_0 + 2M_{2*}M^2.\end{aligned} \tag{3.17}$$

It follows from (3.15) and (3.17) that

$$\left\|g^{(\nu)}\right\|_X \le \|g\|_X + \|\sigma_1\|_X\left(M_0 + 2M_{2*}M^2\right). \tag{3.18}$$

Hence, from (3.14), (3.18), and (H_6), we obtain

$$\left\|f^{(\nu)}\right\|_X \le \frac{\|g\|_X + \|\sigma_1\|_X\left(M_0 + 2M_{2*}M^2\right)}{1 - \sum_{i=1}^q \|\alpha_i\|_X - M_{1*}\|\sigma_1\|_X} \le M, \tag{3.19}$$

which implies (3.11).

Estimating the term $\left\|f - f^{(\nu)}\right\|_X$: Put $e^{(\nu)} = f - f^{(\nu)}$, we obtain from (1.1) and (3.1) that

$$\begin{aligned} e^{(\nu)}(x) = (Ue^{(\nu)})(x) + \textstyle\int_0^{\sigma_1(x)} \Big[\Psi \left(x, t, f(\sigma_2(t)), \textstyle\int_0^{\sigma_3(t)} f(s)ds \right) \\ -D_3\Psi \left(W^{(\nu)}(x,t) \right) f^{(\nu)}(\sigma_2(t)) - D_4\Psi \left(W^{(\nu)}(x,t) \right) \textstyle\int_0^{\sigma_3(t)} f^{(\nu)}(s)ds \Big] dt \\ - \textstyle\int_0^{\sigma_1(x)} \left[\Psi \left(W^{(\nu)}(x,t) \right) - D_3\Psi \left(W^{(\nu)}(x,t) \right) f^{(\nu-1)}(\sigma_2(t)) \right. \\ \left. -D_4\Psi \left(W^{(\nu)}(x,t) \right) \textstyle\int_0^{\sigma_3(t)} f^{(\nu-1)}(s)ds \right] dt. \end{aligned} \tag{3.20}$$

Also using Taylor's expansion of the function $\Psi \left(x, t, f(Y), \int_0^Z f(s)ds \right)$ at the point

$$\left(x, t, f^{(\nu-1)}(Y), \int_0^Z f^{(\nu-1)}(s)ds \right),$$

up to order 2, we obtain

$$\begin{aligned} \Psi \left(x, t, f(Y), \int_0^Z f(s)ds \right) &= \Psi \left(x, t, f^{(\nu-1)}(Y), \int_0^Z f^{(\nu-1)}(s)ds \right) \\ &+ D_3\Psi \left(x, t, f^{(\nu-1)}(Y), \int_0^Z f^{(\nu-1)}(s)ds \right) e^{(\nu-1)}(Y) \\ &+ D_4\Psi \left(x, t, f^{(\nu-1)}(Y), \int_0^Z f^{(\nu-1)}(s)ds \right) \int_0^Z e^{(\nu-1)}(s)ds \\ &+ \frac{1}{2} D_3^2\Psi \left(\omega^{(\nu)}(x,t,Y,Z) \right) \left| e^{(\nu-1)}(Y) \right|^2 \\ &+ D_3D_4\Psi \left(\omega^{(\nu)}(x,t,Y,Z) \right) e^{(\nu-1)}(Y) \int_0^Z e^{(\nu-1)}(s)ds \\ &+ \frac{1}{2} D_4^2\Psi \left(\omega^{(\nu)}(x,t,Y,Z) \right) \left(\int_0^Z e^{(\nu-1)}(s)ds \right)^2, \end{aligned} \tag{3.21}$$

where

$$\omega^{(\nu)}(x,t,Y,Z) = \left(x, t, \ f^{(\nu-1)}(Y) + \theta e^{(\nu-1)}(Y), \ \int_0^Z \left[f^{(\nu-1)}(s) + \theta e^{(\nu-1)}(s) \right] ds \right),$$

$0 < \theta < 1$.

Substituting (3.21) into (3.20), with replacing $Y = \sigma_2(t)$, $Z = \sigma_3(t)$ in the arguments of f, $f^{(\nu-1)}$,

$e^{(\nu-1)},\ \omega^{(\nu)}$ appeared in (3.21), we get

$$\begin{aligned} e^{(\nu)}(x) &= (Ue^{(\nu)})(x) \\ &+\int_0^{\sigma_1(x)}\left[D_3\Psi\left(W^{(\nu)}(x,t)\right)e^{(\nu)}(\sigma_2(t))-D_4\Psi\left(W^{(\nu)}(x,t)\right)\int_0^{\sigma_3(t)}e^{(\nu)}(s)ds\right]dt \\ &+\frac{1}{2}\int_0^{\sigma_1(x)}D_3^2\Psi\left(\bar{\omega}^{(\nu)}(x,t)\right)\left|e^{(\nu-1)}(\sigma_2(t))\right|^2dt \\ &+\int_0^{\sigma_1(x)}\left[D_3D_4\Psi\left(\bar{\omega}^{(\nu)}(x,t)\right)e^{(\nu-1)}(\sigma_2(t))\int_0^{\sigma_3(t)}e^{(\nu-1)}(s)ds\right]dt \\ &+\frac{1}{2}\int_0^{\sigma_1(x)}D_4^2\Psi\left(\bar{\omega}^{(\nu)}(x,t)\right)\left(\int_0^{\sigma_3(t)}e^{(\nu-1)}(s)ds\right)^2dt, \end{aligned} \tag{3.22}$$

where $\bar{\omega}^{(\nu)}(x,t)=\omega^{(\nu)}(x,t,\sigma_2(t),\sigma_3(t))$.

Combining (3.12), (3.22), the result is

$$\begin{aligned} \left\|e^{(\nu)}\right\|_X &\leq \left\|Ue^{(\nu)}\right\|_X+\|\sigma_1\|_X M_{1*}\left\|e^{(\nu)}\right\|_X \\ &\quad +\frac{1}{2}\|\sigma_1\|_X M_{2*}\left\|e^{(\nu-1)}\right\|_X^2+\|\sigma_1\|_X M_{2*}\left\|e^{(\nu-1)}\right\|_X^2+\frac{1}{2}\|\sigma_1\|_X M_{2*}\left\|e^{(\nu-1)}\right\|_X^2 \\ &\leq \left(\sum\nolimits_{i=1}^q\|\alpha_i\|_X+\|\sigma_1\|_X M_{1*}\right)\left\|e^{(\nu)}\right\|_X+2\|\sigma_1\|_X M_{2*}\left\|e^{(\nu-1)}\right\|_X^2. \end{aligned} \tag{3.23}$$

Consequently

$$\left\|e^{(\nu)}\right\|_X \leq \frac{2\|\sigma_1\|_X M_{2*}}{1-\sum_{i=1}^q\|\alpha_i\|_X-\|\sigma_1\|_X M_{1*}}\left\|e^{(\nu-1)}\right\|_X^2 \equiv \beta_M\left\|e^{(\nu-1)}\right\|_X^2. \tag{3.24}$$

Hence, we obtain (3.8) via (3.9) and (3.24). Finally, from (3.8), (3.10) follows. Theorem 3.2 is proved.

Remark 3.1. If μ_0 is chosen large enough such that

$$\beta_M\left\|g^{(\mu_0)}-f\right\|_X \leq \beta_M\left\|Tg^{(0)}-g^{(0)}\right\|_X\frac{\sigma^{\mu_0}}{1-\sigma}<1,$$

by choosing $f^{(0)}=g^{(\mu_0)}$, then the first term $f^{(0)}$ obtained is sufficiently near f such that

$$\beta_M\left\|f^{(0)}-f\right\|_X<1.$$

4. Smoothness of solutions

In this section, we shall prove that the smoothness of the solution of Eq. (1.1) depends on given data. More precisely, by assuming that the functions $\tau_i,\ \sigma_1:[0,1]\to[0,1],\ \alpha_i,\ g:[0,1]\to\mathbb{R},\ \Psi:[0,1]\times[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are of class C^r while the functions $\sigma_2,\ \sigma_3:[0,1]\to[0,1]$ are of class C^{r-1} satisfying some auxiliary conditions, we shall prove that the solution f of Eq. (1.1) is of class C^r.

First, it is convenient to use the following lemma.

Lemma 4.1 *Let $\tau_i,\ \alpha_i \in X,\ (i=1,\cdots,q)$, satisfy (H_2). Then, for all $g \in X$, there exists a unique function $f \in X$ such that f is a solution of the following equation*

$$f(x)=\sum_{i=1}^{q}\alpha_i(x)f(\tau_i(x))+g(x),\ \forall x\in[0,1]. \tag{4.1}$$

Proof of Lemma 4.1. It is clear that a unique solution of Eq. (4.1) is $f=(I-U)^{-1}g$. □

We now make the assumptions for the functions $g,\ \tau_i,\ \alpha_i,\ \sigma_j,\ \Psi$ as follows, in which the functions g, $\Psi,\ \tau_i,\ \alpha_i,\ \sigma_1$ are of class C^1:

$(H_1^{(1)})$ $g\in C^1(\Omega;\mathbb{R})$;

$(H_2^{(1)})$ $\tau_i,\ \alpha_i\in C^1(\Omega;\mathbb{R})$, $(i=1,\cdots,q)$, such that
(i) $\tau_i(x)\in[0,1]$, $\forall x\in[0,1]$, $(i=1,\cdots,q)$,
(ii) $\sum_{i=1}^{q}\|\alpha_i\tau_i'\|_X<1$;

$(H_3^{(1)})$ $\sigma_1\in C^1(\Omega;\mathbb{R})$ and $\sigma_2,\ \sigma_3\in X$ such that $\sigma_j(x)\in[0,1]$, $\forall x\in[0,1]$ $(j=1,2,3)$;

$(H_4^{(1)})$ $\Psi\in C^1([0,1]\times[0,1]\times\mathbb{R}^2;\mathbb{R})$.

Let $f\in C^1(\Omega;\mathbb{R})$ be the solution of (1.1). Differentiating the members of (1.1), we obtain that f' satisfies the equation below

$$f'(x)=\sum_{i=1}^{q}\alpha_i^{[1]}(x)f'(\tau_i(x))+g^{[1]}(x),\ \forall x\in[0,1], \tag{4.2}$$

in which

$$\begin{aligned}
\alpha_i^{[1]}(x) &= \alpha_i(x)\tau_i'(x),\ i=1,\cdots,q, \\
g^{[1]}(x) &= \hat{g}^{[1]}(x)+\int_0^{\sigma_1(x)} D_1\Psi\left(\vec{\omega}(x,t)\right)dt, \\
\hat{g}^{[1]}(x) &= g'(x)+\sum_{i=1}^{q}\alpha_i'(x)f(\tau_i(x))+\Psi\left(\vec{W}(x)\right)\sigma_1'(x), \\
\vec{\omega}(x,t) &= \left(x,t,f(\sigma_2(t)),\int_0^{\sigma_3(t)}f(s)ds\right), \\
\vec{W}(x) &= \vec{\omega}(x,\sigma_1(x))=\left(x,\sigma_1(x),(f\circ\sigma_2\circ\sigma_1)(x)),\int_0^{(\sigma_3\circ\sigma_1)(x)}f(s)ds\right).
\end{aligned} \tag{4.3}$$

In (4.1), the derivatives $f'(0),\ f'(1)$ at $x=0,\ x=1$, respectively, mean that $f'(0)=f_+'(0),\ f'(1)=f_-'(1)$.

By the fact that $f\in C^1(\Omega;\mathbb{R})$ and the functions $g,\ \Psi,\ \tau_i,\ \alpha_i,\ \sigma_1$ are of class C^1 while $\sigma_2,\ \sigma_3\in X$, we obtain $g^{[1]},\ \alpha_i^{[1]}\in X$ in (4.3).

Considering the following equation

$$F^{[1]}(x) = \sum_{i=1}^{q} \alpha_i^{[1]}(x) F^{[1]}(\tau_i(x)) + g^{[1]}(x), \ \forall x \in [0,1]. \tag{4.4}$$

Because of $\alpha_i^{[1]} \in X$ satisfying the following condition given in $(H_2^{(1)})$ (ii)

$$\sum_{i=1}^{q} \left\| \alpha_i^{[1]} \right\|_X = \sum_{i=1}^{q} \| \alpha_i \tau_i' \|_X < 1, \tag{4.5}$$

applying Lemma 4.1, there exists a unique function $F^{[1]} \in X$ such that $F^{[1]}$ is the solution of (4.4). On the other hand, f' satisfies (4.2), it leads to $f' \in X$ and f' is also a solution of (4.4). By the uniqueness of (4.4), $F^{[1]} \in X$ is also the derivative f' of f.

We continue to strengthen the assumptions for the functions g, τ_i, α_i, σ_j, Ψ as follows, in which the functions g, Ψ, τ_i, α_i, σ_1 are of class C^r $(r \geq 2)$:

$(H_1^{(r)})$ $g \in C^r(\Omega; \mathbb{R})$;

$(H_2^{(r)})$ $\tau_i, \alpha_i \in C^r(\Omega; \mathbb{R})$, $(i = 1, \cdots, q)$, such that
(i) $\tau_i(x) \in [0,1]$, $\forall x \in [0,1]$, $(i = 1, \cdots, q)$,
(ii) $\sum_{i=1}^{q} \left\| \alpha_i \left(\tau_i'\right)^r \right\|_X < 1$;

$(H_3^{(r)})$ $\sigma_1 \in C^r(\Omega; \mathbb{R})$ and $\sigma_2, \sigma_3 \in C^{r-1}(\Omega; \mathbb{R})$ such that $\sigma_j(x) \in [0,1]$, $\forall x \in [0,1]$ $(j = 1,2,3)$;

$(H_4^{(r)})$ $\Psi \in C^r([0,1] \times [0,1] \times \mathbb{R}^2; \mathbb{R})$.

With the similar arguments as above, let $f \in C^r(\Omega; \mathbb{R})$ be the solution of Eq. (1.1). By differentiating the members of Eq. (1.1) with r times, which implies that $f^{(r)}$ satisfies the equation

$$f^{(r)}(x) = \sum_{i=1}^{q} \alpha_i^{[r]}(x) f^{(r)}(\tau_i(x)) + g^{[r]}(x), \ \forall x \in [0,1], \tag{4.6}$$

in which

$$\begin{aligned} \alpha_i^{[r]}(x) &= \alpha_i^{[r-1]}(x) \tau_i'(x) = \alpha_i(x) \left(\tau_i'(x)\right)^r, \ i = 1, \cdots, q, \\ g^{[r]}(x) &= \hat{g}^{[r]}(x) + \int_0^{\sigma_1(x)} D_1^r \Psi\left(\vec{\omega}(x,t)\right) dt, \\ \hat{g}^{[r]}(x) &= \left(\hat{g}^{[r-1]}\right)'(x) + \sum_{i=1}^{q} \left(\alpha_i^{[r-1]}\right)'(x) f^{(r-1)}(\tau_i(x)) + D_1^{r-1} \Psi\left(\vec{W}(x)\right) \sigma_1'(x). \end{aligned} \tag{4.7}$$

By the fact that $f \in C^r(\Omega; \mathbb{R})$ and the functions g, Ψ, τ_i, α_i, σ_1 are of class C^r, while σ_2, σ_3 are of class C^{r-1}, we obtain $g^{[r]}$, $\alpha_i^{[r]} \in X$ in (4.7).

Applying Lemma 4.1 again for the following equation

$$F^{[r]}(x) = \sum_{i=1}^{q} \alpha_i^{[r]}(x) F^{[r]}(\tau_i(x)) + g^{[r]}(x), \ \forall x \in [0,1], \tag{4.8}$$

with $\alpha_i^{[r]} \in X$ satisfies

$$\sum_{i=1}^{q} \left\| \alpha_i^{[r]} \right\|_X = \sum_{i=1}^{q} \left\| \alpha_i \left(\tau_i'\right)^r \right\|_X < 1, \tag{4.9}$$

as in $(H_2^{(r)})$ (ii), there exists a unique function $F^{[r]} \in X$ which is the solution of (4.8). On the other hand, $f^{(r)}$ satisfies (4.6), so $f^{(r)} \in X$ and $f^{(r)}$ is also a solution of (4.8). By the uniqueness of (4.8), it is clear that $F^{[r]} \in X$ is the r-order derivative $f^{(r)}$ of f.

Finally, we conclude that the following theorem is true and therefore, we have finished the main contents in this section.

Theorem 4.2 *Let the functions* g, τ_i, α_i, σ_j, Ψ *satisfy* $(H_1^{(r)}) - (H_4^{(r)})$. *Then, there exist the functions* $f \in C^r(\Omega; \mathbb{R})$ *and* $F^{[r]} \in X$ *such that* f *is the unique solution of Eq. (1.1) and* $F^{[r]}$ *is the unique solution of Eq. (4.8). Furthermore,* $F^{[r]}$ *is the* r*-order derivative of* f, *i.e.* $F^{[r]} = f^{(r)}$.

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