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Second Hankel determinant for Mocanu type bi-starlike functions related to shell-shaped region

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Abstract: In this paper, we investigate the coefficient bound estimates, second Hankel determinant, and Fekete–Szegö inequality for the analytic bi-univalent function class, which we call Mocanu type bi-starlike functions, related to a shell-shaped region in the open unit disk in the complex plane. Some interesting special cases of the results are also discussed.

Key words: Bi-univalent function, bi-starlike and bi-convex functions, second Hankel determinant, Fekete–Szegö inequality

1. Introduction and preliminaries

Let A denote the class of all complex valued functions f given by

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... + a_n z^n + ... = z + \sum_{n=2}^{\infty} a_n z^n, a_n \in \mathbb{C}, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane. Furthermore, let S be the class of all univalent functions in A. Some of the important and well-investigated subclasses of S that include the classes $\mathrm{S}^*(\alpha)$ and $\mathrm{C}(\alpha)$ are given below, as the class of starlike and convex functions of order $\alpha\,(\alpha \in [0,1))$, respectively. By definition (see [6, 9, 22])

$$\mathrm{S}^*(\alpha) = \left\{ f \in \ \mathrm{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in \mathbb{U} \right\}$$

and

$$\mathrm{C}(\alpha) = \left\{ f \in \mathrm{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \mathbb{U} \right\}, \alpha \in [0,1)$$

An analytic function f is subordinate to an analytic function φ, written as $f(z) \prec \varphi(z)$, provided that there is an analytic function (that is, Schwarz function) ω defined on $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying

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$f(z) = \varphi(\omega(z))$. Ma and Minda [14] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1+\frac{zf''(z)}{f'(z)}$ is subordinate to a more general function. For this purpose, they considered φ an analytic function with positive real part in $\mathbb{U}$, satisfying $\varphi(0)=1$, $\varphi'(0)>0$ and $\varphi(\mathbb{U})$ is symmetric with respect to real axis. Such a function has a series expansion of the following form:

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + ... = 1 + \sum_{n=2}^{\infty} b_n z^n, b_1 > 0.$$

It is well-known that (see [6]) every function $f \in \mathrm{S}$ has an inverse f^{-1} defined by

$$\begin{aligned} f^{-1}(f(z)) &= z, z \in \mathbb{U}, f^{-1}(f(w)) = w, w \in \mathbb{U}_{r_0} = \{w \in \mathbb{C} : |w| < r_0(f)\}, \\ r_0(f) &\geq 1/4, \end{aligned}$$

and

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + A_4 w^4 + ..., w \in \mathbb{U}_{r_0}, \tag{1.2}$$

where

$$A_2 = -a_2, A_3 = 2a_2^2 - a_3, A_4 = -5a_2^3 + 5a_2 a_3 - a_4.$$

A function $f \in \mathrm{A}$ is called bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a short history and examples of functions in the class Σ, see [21].

Firstly, Lewin [13] introduced the class of bi-univalent functions, obtaining the estimate $|a_2| \leq 1.51$. Subsequently, Brannan and Clunie [3] developed the result of Lewin to $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Accordingly, Netanyahu [17] showed that $|a_2| \leq \frac{4}{3}$. Earlier, Brannan and Taha [2] introduced certain subclasses of bi-univalent function class Σ, namely bi-starlike function of order α denoted $\mathrm{S}^*_\Sigma(\alpha)$ and bi-convex function of order α denoted $\mathrm{C}_\Sigma(\alpha)$ corresponding to the function classes $\mathrm{S}^*(\alpha)$ and $\mathrm{C}(\alpha)$, respectively. For each of the function classes $\mathrm{S}^*_\Sigma(\alpha)$ and $\mathrm{C}_\Sigma(\alpha)$, nonsharp estimates on the first two Taylor–Maclaurin coefficients were found in [2, 23]. Many researchers (see [21, 24]) have introduced and investigated several interesting subclasses of bi-univalent function class Σ, and they have found nonsharp estimates on the first two Taylor–Maclaurin coefficients. However, even so, the sharp estimates for each of the Taylor–Maclaurin coefficients $|a_n|, n = 2, 3, 4, ...$ is still an open problem (see, for example, [13, 17]). Recently, the upper bounds of $|H_2(2)| = |a_2 a_4 - a_3^2|$ for the classes $\mathrm{S}^*_\Sigma(\alpha)$ and $\mathrm{C}_\Sigma(\alpha)$ were obtained by Deniz et al. [4] . Very soon, Orhan et al. [19] reviewed the study of bound for the second Hankel determinant of the subclass $M^\alpha_\Sigma(\beta)$ of bi-univalent functions and Mustafa et al.[16] improved the results obtained in [4].

Definition 1.1 *[20] Let $f \in \mathrm{A}$ be normalized by $f(0) = f'(0) - 1 = 0$ in the unit disc $\mathbb{U}$. We denote by $\mathrm{S}^*(\varphi)$ the class of analytic functions and satisfying the condition that*

$$\frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2} =: \varphi(z),$$

where the branch of the square root is chosen to be the principal one, that is $\varphi(0) = 1$.

The function $\varphi(z) := z + \sqrt{1+z^2}$ maps the unit disc $\mathbb{U}$ onto a shell-shaped region on the right half plane, and it is analytic and univalent on $\mathbb{U}$. The range $\varphi(\mathbb{U})$ is symmetric respecting the real axis and $\varphi(z)$ is a function with positive real part in $\mathbb{U}$, with $\varphi(0) = \varphi'(0) = 1$. Moreover, it is a starlike domain with respect to the point $\varphi(0) = 1$ (see Figure).

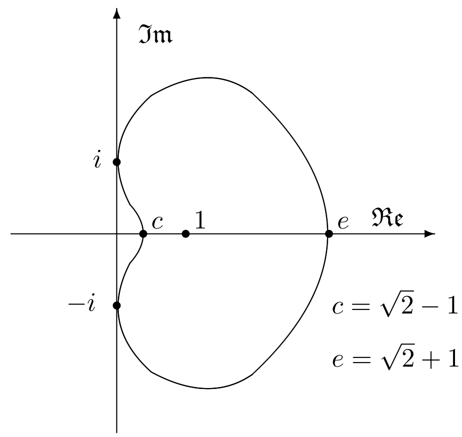

Figure. The boundary of the set $\varphi(\mathbb{U})$.

Inspired by the aforementioned works, we define a subclass of bi-univalent functions namely Σ as follows.

Definition 1.2 *A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathrm{M}_{\Sigma}(\varphi, \beta)$, $\beta \geq 0$, if the following conditions are satisfied*

$$(1-\beta)\frac{zf'(z)}{f(z)} + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) = z + \sqrt{1+z^2}, z \in \mathbb{U}$$

and

$$(1-\beta)\frac{wg'(w)}{g(w)} + \beta\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \varphi(w) = w + \sqrt{1+w^2}, w \in \mathbb{U}_{r_0},$$

where $g = f^{-1}$.

Remark 1.3 *Taking $\beta = 0$ in the above definition, we have bi-starlike function class $S^*_{\Sigma}(\varphi)$, which satisfied the following conditions*

$$f \in \mathrm{S}^*_{\Sigma}(\varphi) \Longleftrightarrow \frac{zf'(z)}{f(z)} \prec \varphi(z) = z + \sqrt{1+z^2}, z \in \mathbb{U}$$

and

$$\frac{wg'(w)}{g(w)} \prec \varphi(w) = w + \sqrt{1+w^2}, w \in \mathbb{U}_{r_0}$$

where $g = f^{-1}$.

Remark 1.4 *Taking $\beta = 1$ in the above defination, we have bi-convex function class $C_{\Sigma}(\varphi)$, which satisfied the following conditions.*

$$f \in \mathrm{C}_{\Sigma}(\varphi) \Longleftrightarrow 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) = z + \sqrt{1+z^2}, z \in \mathbb{U}$$

and

$$1 + \frac{wg''(w)}{g'(w)} \prec \varphi(w) = w + \sqrt{1+w^2}, w \in \mathbb{U}_{r_0},$$

where $g = f^{-1}$.

In this paper, we give coefficient bound estimates, determine the upper bound estimate for the second Hankel determinant, and solve the Fekete–Szegö problem for the functions belonging to the class $\mathrm{M}_{\Sigma}(\varphi, \beta)$ and in the special cases.

In order to prove our main results, we shall need the following lemma.

Lemma 1.5 *[6] Let P be the class of all analytic functions p of the form*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.3}$$

satisfying $\Re(p(z)) > 0, z \in \mathbb{U}$ and $p(0) = 1$. Then,

$$|p_n| \leq 2, \;\; n = 1, 2, 3, ...$$

This ineuqality is sharp for each n. In particular, equality holds for all n for the function

$$p(z) = \frac{1+z}{1-z}.$$

Lemma 1.6 *[12] Let P be the class of all analytic functions p of the form*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} p_n z^n$$

satisfying $\Re(p(z)) > 0, z \in \mathbb{U}$ and $p(0) = 1$. Then,

$$\begin{aligned} 2p_2 &= p_1^2 + \left(4 - p_1^2\right) x, \\ 4p_3 &= p_1^3 + 2\left(4 - p_1^2\right) p_1 x - \left(4 - p_1^2\right) p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right) z, \end{aligned}$$

for some x, z with $|x| \leq 1, \;\; |z| \leq 1$.

Lemma 1.7 *[8] The power series given in (1.3) converges in* $\mathbb{U}$ *to the function* p *in* P *if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_n \\ p_{-1} & 2 & p_1 & \cdots & p_{n-1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ p_{-n} & p_{-n+1} & p_{-n+2} & \cdots & 2 \end{vmatrix}, n = 1, 2, 3, ...$$

and $p_{-n} = \bar{p_n}$, *are all nonnegative. They are strictly positive except for*

$$p(z) = \sum_{n=1}^{n} \rho_n p_0 \left(e^{it_n z}\right), \ \rho_n > 0, t_n \text{ real}$$

and $t_n \neq t_k$ *for* $n \neq k$ *in this case* $D_n > 0$ *for* $n < m-1$ *and* $D_n = 0$ *for* $n \geq m$.

Notation 1.8 *Since* $p \in$ P, *according to Lemma 1.7* $D_1 \geq 0$ *and* $p_{-1} = \bar{p_1} \geq 0$. *According to this* $D_1 = \begin{vmatrix} 2 & p_1 \\ p_1 & 2 \end{vmatrix} \geq 0$ *and* $p_1 = \bar{p_1} = p_{-1} \geq 0$. *So* $4 - p_1^2 \geq 0$ *and* $p_1 \geq 0$, *which is equivalet to* $p_1 \in [0, 2]$. *For these reasons, for* p_1, *which is first coefficient in (1.3), we will assume that* $\left|4 - p_1^2\right| = \left|4 - |p_1|^2\right| = 4 - |p_1|^2$ *throughout our study.*

2. Coefficients bound estimates

In this section, we prove the following theorem on upper bound estimates for the few initial coefficients of the functions belonging to the class $\mathrm{M}_\Sigma(\varphi, \beta)$.

Theorem 2.1 *Let the function* f *given by (1.1) be in the class* $\mathrm{M}_\Sigma(\varphi, \beta)$. *Then,*

$$\begin{aligned} |a_2| &\leq \frac{1}{1+\beta}, \\ |a_3| &\leq \begin{cases} \frac{1}{(1+\beta)^2} & if \quad \beta \in \left[0, 1+\sqrt{2}\right] \\ \frac{1}{2(1+2\beta)} & if \quad \beta \geq 1+\sqrt{2}, \end{cases} \\ |a_4| &\leq \frac{1}{3(1+3\beta)} \begin{cases} \frac{2(1+4\beta)}{(1+\beta)^3} & if \quad \beta \in [0, \beta_0], \\ 1 & if \quad \beta \geq \beta_0, \end{cases} \end{aligned}$$

where $\beta_0 = 1.3289$ *is the numerical solution of the equation* $\beta^3 + 3\beta^2 - 5\beta - 1 = 0$.

Proof Let $f \in \mathrm{M}_\Sigma(\varphi, \beta), \beta \geq 0$ and $g = f^{-1}$. Then, there are analytic functions $\omega : \mathbb{U} \to \mathbb{U}, \varpi : \mathbb{U}_{r_0} \to \mathbb{U}_{r_0}$ with $\omega(0) = 0 = \varpi(0), |\omega(z)| \leq 1, |\varpi(w)| \leq 1$ satisfying the following conditions

$$(1-\beta)\frac{zf'(z)}{f(z)} + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) = \varphi(\omega(z)) = \omega(z) + \sqrt{1+\omega^2(z)}, z \in \mathbb{U} \tag{2.1}$$

and

$$(1-\beta)\frac{wg'(w)}{g(w)}+\beta\left(1+\frac{wg''(w)}{g'(w)}\right)\prec\varphi(\varpi(w))=\varpi(w)+\sqrt{1+\varpi^2(w)}, w\in\mathbb{U}_{r_0}\ . \tag{2.2}$$

We define the functions $p, q \in \mathrm{P}$ as follows:

$$p(z):=\frac{1+\omega(z)}{1-\omega(z)}=1+p_1z+p_2z^2+p_3z^3+...=1+\sum_{n=1}^{\infty}p_nz^n$$

and

$$q(w):=\frac{1+\varpi(w)}{1-\varpi(w)}=1+q_1w+q_2w^2+q_3w^3+...=1+\sum_{n=1}^{\infty}q_nw^n.$$

It follows that

$$\omega(z):=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left[p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)z^3+...\right] \tag{2.3}$$

and

$$\varpi(z):=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}\left[q_1w+\left(q_2-\frac{q_1^2}{2}\right)w^2+\left(q_3-q_1q_2+\frac{q_1^3}{4}\right)w^3+...\right]. \tag{2.4}$$

Thus, from (2.1), (2.3) and (2.2), (2.4), we write

$$(1-\beta)\frac{zf'(z)}{f(z)}+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)=\frac{p(z)-1}{p(z)+1}+\sqrt{1+\left(\frac{p(z)-1}{p(z)+1}\right)^2}, z\in\mathbb{U} \tag{2.5}$$

and

$$(1-\beta)\frac{wg'(w)}{g(w)}+\beta\left(1+\frac{wg''(w)}{g'(w)}\right)=\frac{q(w)-1}{q(w)+1}+\sqrt{1+\left(\frac{q(w)-1}{q(w)+1}\right)^2}, w\in\mathbb{U}_{r_0}\ . \tag{2.6}$$

Also, from (2.1), (2.3) and (2.2), (2.4) for the functions $\varphi(\omega(z))$ and $\varphi(\varpi(w))$, we can write the following expressions, respectively.

$$\varphi(\omega(z))=1+\frac{p_1}{2}z+\left(\frac{p_2}{2}-\frac{p_1^2}{8}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{4}\right)z^3+... \tag{2.7}$$

and

$$\varphi(\varpi(w))=1+\frac{q_1}{2}w+\left(\frac{q_2}{2}-\frac{q_1^2}{8}\right)w^2+\left(\frac{q_3}{2}-\frac{q_1q_2}{4}\right)w^3+...\ . \tag{2.8}$$

From (2.1), (2.7) and (2.2), (2.8), we easily write

$$(1-\beta)\frac{zf'(z)}{f(z)}+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)=1+\frac{p_1}{2}z+\left(\frac{p_2}{2}-\frac{p_1^2}{8}\right)z^2+\left(\frac{p_3}{2}-\frac{p_1p_2}{4}\right)z^3+... \tag{2.9}$$

and

$$(1-\beta)\frac{wg'(w)}{g(w)}+\beta\left(1+\frac{wg''(w)}{g'(w)}\right)=1+\frac{q_1}{2}w+\left(\frac{q_2}{2}-\frac{q_1^2}{8}\right)w^2+\left(\frac{q_3}{2}-\frac{q_1q_2}{4}\right)w^3+\ldots \tag{2.10}$$

From the equalities (2.9) and (2.10) obtained

$$(1+\beta)\,a_2 = \frac{p_1}{2}, \tag{2.11}$$

$$2(1+2\beta)\,a_3-(1+3\beta)\,a_2^2 = \frac{p_2}{2}-\frac{p_1^2}{8}, \tag{2.12}$$

$$3(1+3\beta)\,a_4-3(1+5\beta)\,a_2a_3+(1+7\beta)\,a_2^3 = \frac{p_3}{2}-\frac{p_1p_2}{4} \tag{2.13}$$

and

$$(1+\beta)\,A_2 = \frac{q_1}{2}, \tag{2.14}$$

$$2(1+2\beta)\,A_3-(1+3\beta)\,A_2^2 = \frac{q_2}{2}-\frac{q_1^2}{8}, \tag{2.15}$$

$$3(1+3\beta)\,A_4-3(1+5\beta)\,A_2A_3+(1+7\beta)\,A_2^3 = \frac{q_3}{2}-\frac{q_1q_2}{4}\;. \tag{2.16}$$

Substituting the expressions of A_2, A_3, and A_4 in the (2.14), (2.15), and (2.16), we obtain

$$-(1+\beta)\,a_2 = \frac{q_1}{2}, \tag{2.17}$$

$$-2(1+2\beta)\,a_3+(3+5\beta)\,a_2^2 = \frac{q_2}{2}-\frac{q_1^2}{8}, \tag{2.18}$$

$$-3(1+3\beta)\,a_4+(12+30\beta)\,a_2a_3-(10+22\beta)\,a_2^3 = \frac{q_3}{2}-\frac{q_1q_2}{4} \tag{2.19}$$

From (2.11) and (2.17), we write

$$\frac{p_1}{2(1+\beta)}=a_2=-\frac{q_1}{2(1+\beta)} \text{ and } p_1=-q_1. \tag{2.20}$$

From this, the first result of the theorem is clear.

Subtracting (2.18) from (2.12) and considering second equality of (2.20), we get

$$a_3=a_2^2+\frac{p_2-q_2}{8(1+2\beta)};$$

that is,

$$a_3=\frac{p_1^2}{4(1+\beta)^2}+\frac{p_2-q_2}{8(1+2\beta)}. \tag{2.21}$$

Also, subtracting (2.19) from (2.13) and considering (2.20) and (2.21), we get

$$a_4=\frac{(1+4\beta)\,p_1^3}{12(1+3\beta)(1+\beta)^3}+\frac{5(p_2-q_2)\,p_1}{32(1+\beta)(1+2\beta)}+\frac{p_3-q_3}{12(1+3\beta)}-\frac{(p_2+q_2)\,p_1}{24(1+3\beta)}. \tag{2.22}$$

If we take into account that $p_1 = -q_1$, according to Lemma 1.6, we have

$$p_2 - q_2 = \frac{4-p_1^2}{2}(x-y), p_2 + q_2 = p_1^2 + \frac{4-p_1^2}{2}(x+y) \tag{2.23}$$

and

$$\begin{aligned} p_3 - q_3 &= \frac{p_1^3}{2} + \frac{\left(4-p_1^2\right)p_1}{2}(x+y) - \frac{\left(4-p_1^2\right)p_1}{4}\left(x^2+y^2\right) \\ &+ \frac{4-p_1^2}{2}\left[\left(1-|x|^2\right)z - \left(1-|y|^2\right)w\right] \end{aligned} \tag{2.24}$$

for some x, y, z, w with $|x| \le 1, |y| \le 1, |z| \le 1, |w| \le 1$.
Substituting the first expression (2.23) in (2.21) for the coefficient a_3, we write the following equality

$$a_3 = \frac{p_1^2}{4(1+\beta)^2} + \frac{4-p_1^2}{16(1+2\beta)}(x-y).$$

Note that, if we take $|p_1| = t$, we can write $\left|4-p_1^2\right| = \left|4-|p_1|^2\right| = \left|4-t^2\right| = 4-t^2$ (see, Note 1.8 at the end of the first section). That is, we may assume without restriction that $t \in [0,2]$. In that case, setting $|x| = \xi$ and $|y| = \eta$ and using triangle inequality for $|a_3|$, we can write the following inequality

$$|a_3| \le \frac{t^2}{4(1+\beta)^2} + \frac{4-t^2}{16(1+2\beta)}(\xi+\eta) := F(\xi,\eta), \ \xi,\eta \in [0,1].$$

Since $F(\xi,\eta) \le F(1,1)$, from the last inequality, we can write

$$|a_3| \le \frac{-\beta^2+2\beta+1}{8(1+2\beta)(1+\beta)^2}t^2 + \frac{1}{2(1+2\beta)} := \psi(t), t \in [0,2].$$

Since $1+2\beta-\beta^2 \ge 0$ for $\beta \in \left[0, 1+\sqrt{2}\right]$ and $1+2\beta-\beta^2 \le 0$ for $\beta \ge 1+\sqrt{2}$, the function ψ is increasing function for $\beta \in \left[0,1+\sqrt{2}\right]$ and decreasing for $\beta \ge 1+\sqrt{2}$. Therefore,

$$|a_3| \le \max\{\psi(t), t \in [0,2]\} = \begin{cases} \frac{1}{(1+\beta)^2} & if \quad \beta \in \left[0,1+\sqrt{2}\right], \\ \frac{1}{2(1+2\beta)} & if \quad \beta \ge 1+\sqrt{2}. \end{cases}$$

From (2.22), using (2.23), (2.24) and triangle inequality, we obtain the following inequality for $|a_4|$

$$|a_4| \le c_1(t) + c_2(t)(\xi+\eta) + c_3(t)\left(\xi^2+\eta^2\right) := G(\xi,\eta),$$

where

$$c_1(t) = \frac{1+4\beta}{12(1+3\beta)(1+\beta)^2}t^3 + \frac{4-t^2}{12(1+3\beta)},$$

$$c_2(t) = \frac{\left(8\beta^2+57\beta+19\right)\left(4-t^2\right)t}{192(1+\beta)(1+2\beta)(1+3\beta)},$$

$$c_3(t) = \frac{\left(4-t^2\right)(t-2)}{48(1+3\beta)}.$$

Now we need to maximize the function G in the square $\Omega = \{(\xi,\eta) : \xi,\eta \in [0,1]\}$ for $t \in [0,2]$. We must investigate the maximum of G in the cases $t = 0, t = 2$, and $t \in (0,2)$.

For $t = 0$, we write

$$\begin{aligned} G(\xi,\eta) &= \frac{1}{3(1+3\beta)} - \frac{1}{6(1+3\beta)}\left(\xi^2+\eta^2\right) \le \max\{G(\xi,\eta) : \xi,\eta \in [0,1]\} \\ &= \frac{1}{3(1+3\beta)}. \end{aligned}$$

Let $t = 2$, then the function G is constant as follows:

$$G(\xi,\eta) = \frac{2(1+4\beta)}{3(1+3\beta)(1+\beta)^3}.$$

Now, let $t \in (0,2)$. In this case, we must investigate the maximum of the function G according to $t \in (0,2)$ by taking into account the sign of $\Delta(G) = G_{\xi\xi}(\xi,\eta)\, G_{\eta\eta}(\xi,\eta) - (G_{\xi\eta}(\xi,\eta))^2$. It is clear that the point (ξ_0,η_0), where $\xi_0 = \eta_0 = \frac{-c_2(t)}{2c_3(t)} > 0$, is a critical point according to extremum for the function G. Let say that $-c_2(t) \ge 2c_3(t)$, some values of $t \in (0,2)$ and $\beta \ge 0$, so $(\xi_0,\eta_0) \in \Omega$ (in the case $-c_2(t) \le 2c_3(t)$ $(\xi_0,\eta_0) \notin \Omega$,).

Since

$$G_{\xi\xi}(\xi,\eta) = G_{\eta\eta}(\xi,\eta) = 2c_3(t) = \frac{\left(4-t^2\right)(t-2)}{24(1+3\beta)} < 0$$

and

$$\Delta(G) = G_{\xi\xi}(\xi,\eta)\, G_{\eta\eta}(\xi,\eta) - (G_{\xi\eta}(\xi,\eta))^2 = \left[\frac{\left(4-t^2\right)(t-2)}{24(1+3\beta)}\right]^2 > 0,$$

from the elementary calculus the function G has a local maximum and the maximum occurs at (ξ_0,η_0). Hence,

$$G(\xi,\eta) \le G(\xi_0,\eta_0) = c_1(t) - \frac{c_2^2(t)}{2c_3(t)} := H(t), t \in (0,2)$$

in the square Ω .

We can easily see that

$$H(t) = l_1(\beta)\, t^3 + l_2(\beta)\, t^2 + l_3(\beta),$$

$$l_1(\beta) = \frac{1}{12(1+3\beta)(1+\beta)^2}\left[\frac{\left(8\beta^2+57\beta+19\right)^2}{128(1+2\beta)^2} - \frac{1+4\beta}{1+\beta}\right],$$

$$l_2(\beta) = \frac{1}{12(1+3\beta)}\left[\frac{\left(8\beta^2+57\beta+19\right)^2}{64(1+\beta)^2(1+2\beta)^2} - 1\right], l_3(\beta) = \frac{1}{(1+3\beta)}.$$

Since $l_1(\beta) \geq 0$, $l_2(\beta) \geq 0$ (it can be easily seen that $l_1(\beta) \geq 0$ and $l_2(\beta) \geq 0$) for $\beta \geq 0$, the function H is an increasing function in the interval $(0,2)$. Therefore, the function H cannot have a maximum in the interval $(0,2)$.

Thus, the function G cannot have a local maximum for $(0,2)$ and $\xi, \eta \in [0,1]$.

Now we investigate the maximum of the function G on the boundary of the square Ω.

For $\xi = 0$ and $\eta \in [0,1]$ ($\eta = 0$ and $\xi \in [0,1]$ investigated similarly), we write

$$G(0,\eta) = c_1(t) + c_2(t)\eta + c_3(t)\eta^2.$$

By simple computation, we easily see that $G'(0,\eta) \leq 0$ if $\eta \leq (-c_2(t))/(2c_3(t))$ and $G'(0,\eta) \geq 0$ if $\eta \geq (-c_2(t))/(2c_3(t))$. Therefore, the critical point $\eta_0 = (-c_2(t))/(2c_3(t))$ is a minimum point for the function $G(0,\eta)$. Thus, in this case, the function $G(0,\eta)\,(G(\xi,0))$ cannot have a maximum point in the interval $(0,1)$. However, since $G(0,0) = c_1(t)$ and $G(0,1) = c_1(t) + c_2(t) + c_3(t)$ in the extremes of the interval $[0,1]$, we write

$$\max\{G(0,\eta) : \eta \in [0,1]\} = \max\{c_1(t), c_1(t) + c_2(t) + c_3(t)\} .$$

Thus, in the case $c_2(t) + c_3(t) \leq 0$

$$\max\{G(0,\eta) : \eta \in [0,1]\} = c_1(t)$$

and

$$\max\{G(0,\eta) : \eta \in [0,1]\} = c_1(t) + c_2(t) + c_3(t)$$

in the case $c_2(t) + c_3(t) \geq 0$.

For $\xi = 1$ and $\eta \in [0,1]$ ($\eta = 1$ and $\xi \in [0,1]$ investigated similarly), we write

$$G(1,\eta) = c_1(t) + c_2(t) + c_3(t) + c_2(t)\eta + c_3(t)\eta^2.$$

Similarly, we easily see that $G'(1,\eta) \leq 0$ if $\eta \leq (-c_2(t))/(2c_3(t))$ and $G'(1,\eta) \geq 0$ if $\eta \geq (-c_2(t))/(2c_3(t))$. Therefore, the critical point $\eta_0 = (-c_2(t))/(2c_3(t))$ is a minimum point for the function $G(1,\eta)$. Thus, in this case, the function $G(1,\eta)\,(G(\xi,1))$ cannot have a maximum point in the interval $(0,1)$. However, since $G(1,\eta) = c_1(t) + c_2(t) + c_3(t)$ and $G(1,1) = c_1(t) + 2[c_2(t) + c_3(t)]$ in the extremes of the interval $[0,1]$, we write

$$\max\{G(1,\eta) : \eta \in [0,1]\} = \max\{c_1(t), c_1(t) + 2[c_2(t) + c_3(t)]\} .$$

Thus,

$$\max\{G(1,\eta) : \eta \in [0,1]\} = c_1(t)$$

if $c_2(t) + c_3(t) \leq 0$,

$$\max\{G(1,\eta) : \eta \in [0,1]\} = c_1(t) + 2[c_2(t) + c_3(t)]$$

if $c_2(t) + c_3(t) \geq 0$,

$$\max\{G(0,\eta) : \eta \in [0,1]\} \leq \max\{G(1,\eta) : \eta \in [0,1]\}$$

if $c_2(t)+c_3(t)\geq 0$.

Define the function $h:\to\mathbb{R}$ as follows

$$h(t)=c_1(t)+2\left[c_2(t)+c_3(t)\right].$$

Since the functions

$$c_1(t)=\frac{1}{3(1+3\beta)}\left[\frac{1+4\beta}{4(1+\beta)^3}t^3-\frac{1}{4}t^2+1\right]$$

and

$$h(t)=\left[L_1(\beta)t^2+L_2(\beta)\right]t,$$

where

$$L_1(\beta)=\frac{1}{12(1+\beta)(1+3\beta)}\left[\frac{1+4\beta}{(1+\beta)^2}-\frac{8\beta^2+57\beta+19}{8(1+2\beta)}\right]\leq 0$$

$$L_2(\beta)=\frac{1}{6(1+3\beta)}\left[1+\frac{8\beta^2+57\beta+19^2}{4(1+\beta)(1+2\beta)}\right]\geq 0,$$

cannot have a maximum in the interval $(0,2)$; we can write

$$\max\left\{G(\xi,\eta):\xi,\eta\in[0,1]\right\}=\frac{1}{3}\max\left\{\frac{2(1+4\beta)}{(1+3\beta)(1+\beta)^3},\frac{1}{1+3\beta}\right\};$$

that is,

$$|a_4|\leq\frac{1}{3(1+3\beta)}\max\left\{\frac{2(1+4\beta)}{(1+\beta)^3},1\right\}$$

Since

$$\max\left\{\frac{2(1+4\beta)}{(1+\beta)^3},1\right\}=\begin{cases}\frac{2(1+4\beta)}{(1+\beta)^3} & if\quad \beta\in[0,\beta_0],\\ 1 & if\quad \beta\geq\beta_0,\end{cases}$$

where $\beta_0=1.3289$ is the numerical solution of the equation $\beta^3+3\beta^2-5\beta-1=0$, from the last inequality we obtain the third inequality of theorem.

Thus, the proof of Theorem2.1 is completed. □

3. The second Hankel determinant and Fekete–Szegö inequality

In this section, we give an upper bound estimate for the second Hankel determinant and to solve Fekete–Szegö problem for the function belonging to the class $M_\Sigma(\varphi,\beta)$.

Firstly, we prove the following theorem on the upper bound estimate of the second Hankel determinant.

Theorem 3.1 *Let the function f given by (1.1) be in the class $M_\Sigma(\varphi,\beta)$. Then,*

$$\left|a_2a_4-a_3^2\right|\leq\begin{cases}\frac{1}{3(1+3\beta)(1+\beta)^3} & if\ \beta\in[0,\beta_1],\\ \frac{1}{4(1+2\beta)^2} & if\ \beta\geq\beta_1\end{cases}\tag{3.1}$$

where $\beta_1 = 0.16357$ is numerical solution of the equation $9\beta^4 + 30\beta^3 + 20\beta^2 + 2\beta - 1 = 0$.

Proof Let $f \in M_\Sigma(\varphi, \beta)$, $\beta \in [0,1]$. Then, from (2.20), (2.21), and (2.22), we write

$$\begin{aligned} a_2a_4 - a_3^2 &= \frac{-1}{48(1+3\beta)(1+\beta)^3}p_1^4 + \frac{(p_2-q_2)p_1^2}{64(1+2\beta)(1+\beta)^2} + \\ & \frac{(p_3-q_3)p_2}{24(1+\beta)(1+3\beta)} - \frac{(p_2+q_2)p_1^2}{48(1+\beta)(1+3\beta)} - \frac{(p_2-q_2)^2}{64(1+2\beta)^2}. \end{aligned}$$

Using (2.23) and (2.24), then triangle inequality and letting $|p_1| = t, |x| = \xi$, $|y| = \eta$ from the last equality, we obtain

$$\left|a_2a_4 - a_3^2\right| \le C_1(t) + C_2(t)(\xi+\eta) + C_3(t)\left(\xi^2+\eta^2\right) + C_4(t)(\xi+\eta)^2 := \Phi(\xi,\eta), \tag{3.2}$$

where

$$\begin{aligned} C_1(t) &= \frac{1}{48(1+3\beta)(1+\beta)^3}t^4 + \frac{\left(4-t^2\right)t}{24(1+\beta)(1+3\beta)} \ge 0, \\ C_2(t) &= \frac{\left(8\beta^2+21\beta+7\right)\left(4-t^2\right)t^2}{384(1+2\beta)(1+3\beta)(1+\beta)^2} \ge 0, \\ C_3(t) &= \frac{\left(4-t^2\right)(t-2)t}{96(1+\beta)(1+3\beta)} \le 0, \\ C_4(t) &= \frac{\left(4-t^2\right)^2}{256(1+2\beta)^2} \ge 0. \end{aligned}$$

Now we need to maximize the function Φ in the closed square $\Omega = \{(\xi,\eta) : \xi, \eta \in [0,1]\}$ for $t \in [0,2]$. We must investigate the maximum of the function Φ in the cases , $t = 0, t = 2$ and $t \in (0,2)$.

Let $t = 0$, then

$$\Phi(\xi,\eta) = \frac{1}{16(1+2\beta)^2}(\xi+\eta)^2 \le \max\{\Phi(\xi,\eta) : \xi,\eta \in \Omega\} = \frac{1}{4(1+2\beta)^2}.$$

For $t = 2$, the function $\Phi(\xi,\eta)$ is constant as follows

$$\Phi(\xi,\eta) = \frac{1}{3(1+3\beta)(1+\beta)^3}.$$

Now, let $t \in (0,2)$. In this case, we must investigate the maximum of the function Φ according to $t \in (0,2)$ taking into account the sign of $\Delta(\Phi) = \Phi_{\xi\xi}(\xi,\eta)\Phi_{\eta\eta}(\xi,\eta) - (\Phi_{\xi\eta}(\xi,\eta))^2$.

Since $\Delta(\Phi) = 4C_3(t)[C_3(t) + 2C_4(t)]$ and $C_3(t) + 2C_4(t) > 0$ for every $t \in (0,2)$ and each $\beta \in [0,1]$, $\Delta(\Phi) < 0$; that is, the function $\Phi(\xi,\eta)$ cannot have a local maximum in the interior of the square Ω.

Now, we investigate the maximum of Φ on the boundary of the square Ω.

For $\xi = 0$ and $\eta \in [0,1]$ (the case $\eta = 0$ and $\xi \in [0,1]$ investigated similarly), we write

$$\Phi(0,\eta) = C_1(t) + C_2(t)\eta + [C_3(t) + C_4(t)]\eta^2 := \varphi_1(\eta).$$

It is clear that $C_3(t)+C_4(t)\leq 0$ and $C_3(t)+C_4(t)\geq 0$ for some values of $t\in(0,2)$.

In the case $C_3(t)+C_4(t)\leq 0$, the function $\varphi_1(\eta)$ cannot have a local maximum in the interval $(0,1)$, but $\varphi_1(0)=c_1(t)$ and $\varphi_1(1)=C_1(t)+C_2(t)+C_3(t)+C_4(t)$ in the extremes of the interval $[0,1]$.

Let $C_3(t)+C_4(t)\geq 0$ for some values of $t\in(0,2)$. Then, the function $\varphi_1(\eta)$ is an increasing function and the maximum occurs at $\eta=1$.

Therefore,

$$\max\{\varphi_1(\eta):\eta\in[0,1]\}=\varphi_1(1)=C_1(t)+C_2(t)+C_3(t)+C_4(t).$$

For $\xi=1$ and $\eta\in[0,1]$ (the case $\eta=1$ and $\xi\in[0,1]$ investigated similarly), we write

$$\begin{aligned}\Phi(1,\eta) &= [C_3(t)+C_4(t)]\eta^2+[C_2(t)+2C_4(t)]\eta \\ &+ C_1(t)+C_2(t)+C_3(t)+C_4(t)=\varphi_2(\eta)\ .\end{aligned}$$

Similar to the above, we write

$$\max\{\varphi_2(\eta):\eta\in[0,1]\}=\varphi_2(1)=C_1(t)+2C_2(t)+2C_3(t)+4C_4(t).$$

Thus, since $\varphi_1(1)\leq\varphi_2(1)$, the maximum of the function $\Phi(\xi,\eta)$ occurs at the point $(1,1)$ and

$$\max\{\Phi(\xi,\eta):\xi,\eta\in\Omega\}=\Phi(1,1)=\varphi_2(1)$$

on the boundary of the square Ω.

Define the function $M:(0,2)\to\mathbb{R}$ as follows

$$M(t)=C_1(t)+2C_2(t)+2C_3(t)+4C_4(t)=\Phi(1,1).$$

Substituting the values $C_1,\ C_2(t),\ C_3(t)$ and $C_4(t)$ in the expression of the function M, we obtain

$$M(t)=-\theta_1(\beta)t^4+\theta_2(\beta)t^2+\theta_3(\beta),$$

where

$$\begin{aligned}\theta_1(\beta) &= \frac{16\beta^3+53\beta^2+36\beta+7}{192(1+2\beta)(1+3\beta)(1+\beta)^3}>0,\\ \theta_2(\beta) &= \frac{14\beta^3+40\beta^2+25\beta+5}{48(1+3\beta)(1+\beta)^2(1+2\beta)^2}>0,\\ \theta_3(\beta) &= \frac{1}{4(1+2\beta)^2}>0.\end{aligned}$$

Setting $t^2=\tau,$ we write

$$N(\tau)=-\theta_1(\beta)\tau^2+\theta_2(\beta)\tau+\theta_3(\beta),\tau\in[0,4].$$

From this, using standard result of solving quadratic equation, we obtain

$$N(\tau)\leq-16\theta_1(\beta)+4\theta_2(\beta)+\theta_3(\beta)=\frac{1}{3(1+3\beta)(1+\beta)^3};$$

that is,

$$\Phi(\xi,\eta) \leq \frac{1}{3(1+3\beta)(1+\beta)^3}.$$

Thus, we have

$$\max\{\Phi(\xi,\eta) : \xi,\eta \in \Omega\} = \max\left\{\frac{1}{4(1+2\beta)^2}, \frac{1}{3(1+3\beta)(1+\beta)^3}\right\}.$$

Also, since

$$\max\left\{\frac{1}{4(1+2\beta)^2}, \frac{1}{3(1+3\beta)(1+\beta)^3}\right\} = \begin{cases} \frac{1}{3(1+3\beta)(1+\beta)^3} & if\ \beta \in [0,\beta_1], \\ \frac{1}{4(1+2\beta)^2} & if\ \beta \geq \beta_1 \end{cases}$$

where $\beta_1 = 0.16357$ is numerical solution of the equation $9\beta^4 + 30\beta^3 + 20\beta^2 + 2\beta - 1 = 0$, from (3.2) and last equality, we obtain the desired result of the theorem.

Thus, the proof of Theorem 3.1 is completed. □

From the Theorem 3.1, we obtain the following results.

Corollary 3.2 *Let the function f given by (1.1) be in the class $S_{\Sigma}^{*}(\varphi)$. Then,*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{3}.$$

Corollary 3.3 *Let the function f given by (1.1) be in the class $C_{\Sigma}(\varphi)$. Then,*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{36}.$$

Now, we give the following theorems on the Fekete–Szegö inequality.

Theorem 3.4 *Let the function f given by (1.1) be in the class $M_{\Sigma}(\varphi,\beta)$ and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{1}{2(1+2\beta)} & if\ |1-\mu| \leq \frac{(1+\beta)^2}{2(1+2\beta)}, \\ \frac{|1-\mu|}{(1+\beta)^2} & if\ |1-\mu| \geq \frac{(1+\beta)^2}{2(1+2\beta)}. \end{cases} \tag{3.3}$$

Proof Let $f \in M_{\Sigma}(\varphi,\beta), \beta \geq 0$ and $\mu \in \mathbb{C}$. Then, from the equalities (2.20), (2.21) and (2.23) the expression $a_3 - \mu a_2^2$ is written as

$$a_3 - \mu a_2^2 = (1-\mu)\frac{p_1^2}{4(1+\beta)^2} + \frac{4-p_1^2}{16(1+2\beta)}(x-y) \tag{3.4}$$

for some x, y with $|x| \leq 1, |y| \leq 1$.

From the equality (3.4), we obtain

$$\left|a_3 - \mu a_2^2\right| \leq |1-\mu|\frac{t^2}{4(1+\beta)^2} + \frac{4-t^2}{16(1+2\beta)}(\xi+\eta) := \psi(\xi,\eta) \tag{3.5}$$

where $t = |p_1|, \xi = |x|$ and $\eta = |y|$.

It is clear that the maximum $\psi(\xi, \eta)$ occurs at the point (1.1) and

$$\max\{(\xi, \eta) : \psi(\xi, \eta)\} = \psi(1, 1) = \frac{1}{4(1+\beta)^2}\left[|1-\mu| - \frac{(1+\beta)^2}{2(1+2\beta)}\right]t^2 + \frac{1}{2(1+2\beta)}. \tag{3.6}$$

Since the function $\upsilon : (0, 2) \to \mathbb{R}$ defined by

$$\upsilon(t) = \frac{1}{4(1+\beta)^2}\left[|1-\mu| - \frac{(1+\beta)^2}{2(1+2\beta)}\right]t^2 + \frac{1}{2(1+2\beta)}$$

is an increasing function for $|1-\mu| \geq (1+\beta)^2 / 2(1+2\beta)$ and a decreasing function for $|1-\mu| \leq (1+\beta)^2 / 2(1+2\beta)$, from (3.4), (3.5) and (3.9) we obtain the result of theorem.

Thus, the proof of Theorem 3.4 is completed. □

From the Theorem 3.4, we obtain the following results.

Corollary 3.5 *Let the function f given by (1.1) be in the class $\mathrm{S}^*_\Sigma(\varphi)$ and $\mu \in \mathbb{C}$. Then,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2} & if \quad |1-\mu| \leq \frac{1}{2}, \\ |1-\mu| & if \quad |1-\mu| \geq \frac{1}{2}. \end{cases}$$

Corollary 3.6 *Let the function f given by (1.1) be in the class $\mathrm{C}_\Sigma(\varphi)$ and $\mu \in \mathbb{C}$. Then,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{6} & if \quad |1-\mu| \leq \frac{2}{3}, \\ \frac{|1-\mu|}{4} & if \quad |1-\mu| \geq \frac{2}{3}. \end{cases}$$

In the case $\mu \in \mathbb{R}$, the Theorem 3.4 is given as follows.

Theorem 3.7 *Let the function f given by (1.1) be in the class $\mathrm{M}_\Sigma(\varphi, \beta)$ and $\mu \in \mathbb{R}$. Then,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1-\mu}{(1+\beta)^2} & if \quad \mu \leq 1 - \frac{(1+\beta)^2}{2(1+2\beta)}, \\ \frac{1}{2(1+2\beta)} & if \quad 1 - \frac{(1+\beta)^2}{2(1+2\beta)} \leq \mu \leq 1 + \frac{(1+\beta)^2}{2(1+2\beta)}, \\ \frac{\mu-1}{(1+\beta)^2} & if \quad 1 + \frac{(1+\beta)^2}{2(1+2\beta)} \leq \mu. \end{cases} \tag{3.7}$$

Proof Let $f \in M_\Sigma(\varphi, \beta)$, $\beta \geq 0$ and $\mu \in \mathbb{R}$. Since in the case $\mu \in \mathbb{R}$ inequalities

$$|1-\mu| \geq \frac{(1+\beta)^2}{2(1+2\beta)} \text{ and } |1-\mu| \leq \frac{(1+\beta)^2}{2(1+2\beta)}$$

are equivalent to the inequalities

$$\mu \leq 1 - \frac{(1+\beta)^2}{2(1+2\beta)} \text{ or } \quad \mu \geq 1 + \frac{(1+\beta)^2}{2(1+2\beta)} \text{ and } 1 - \frac{(1+\beta)^2}{2(1+2\beta)} \leq \mu \leq 1 + \frac{(1+\beta)^2}{2(1+2\beta)},$$

respectively; from the Theorem 3.4 we obtain the result of the theorem.

This completes the proof. □

From the Theorem 3.7, we obtain the following results.

Corollary 3.8 *Let the function* f *given by (1.1) be in the class* $S_{\Sigma}^{*}(\varphi)$ *and* $\mu \in \mathbb{R}$. *Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 1-\mu & \text{if} \quad \mu \leq \frac{1}{2}, \\ \frac{1}{2} & \text{if} \quad \frac{1}{2} \leq \mu \leq \frac{3}{2}, \\ \mu - 1 & \text{if} \quad \frac{3}{2} \leq \mu. \end{cases}$$

Corollary 3.9 *Let the function* f *given by (1.1) be in the class* $C_{\Sigma}(\varphi)$ *and* $\mu \in \mathbb{R}$. *Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{1-\mu}{4} & \text{if} \quad \mu \leq \frac{1}{3}, \\ \frac{1}{6} & \text{if} \quad \frac{1}{3} \leq \mu \leq \frac{5}{3}, \\ \frac{\mu-1}{4} & \text{if} \quad \frac{5}{3} \leq \mu. \end{cases}$$

Corollary 3.10 *Let the function* f *given by (1.1) be in the class* $\mathrm{M}_{\Sigma}(\varphi, \beta)$. *Then,*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{2(1+2\beta)}.$$

Corollary 3.11 *Let the function* f *given by (1.1) be in the class* $\mathrm{M}_{\Sigma}(\varphi, \beta)$. *Then,*

$$|a_3| \leq \begin{cases} \frac{1}{(1+\beta)^2} & \text{if} \quad \beta \in \left[0, 1+\sqrt{2}\right] \\ \frac{1}{2(1+2\beta)} & \text{if} \quad \beta \geq 1+\sqrt{2}\ . \end{cases}$$

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