Turkish Journal of Mathematics

Volume 45 | Number 3 Article 5

1-1-2021

Existence results for a class of boundary value problems for fractional differential equations

ABDÜLKADİR DOĞAN

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 1149 – 1165
© TÜBİTAK
doi:10.3906/mat-2012-33

Existence results for a class of boundary value problems for fractional differential equations

Abdülkadir DOĞAN*
Department of Mathematics, Faculty of Computer Sciences, Abdullah Gül University, Kayseri, Turkey

Received: 08.12.2020 • **Accepted/Published Online:** 17.03.2020 • **Final Version:** 20.05.2021

Abstract: By application of some fixed point theorems, that is, the Banach fixed point theorem, Schaefer's and the Leray-Schauder fixed point theorem, we establish new existence results of solutions to boundary value problems of fractional differential equations. This paper is motivated by Agarwal et al. (Georgian Math. J. 16 (2009) No.3, 401-411).

Key words: Fractional differential equations, boundary value problems, Caputo fractional derivative, uniqueness, existence, fixed point

1. Introduction

In this manuscript, we study the fractional differential equations

$$ {}^cD^{\nu}u(t) = f(t, u(t)), \qquad t \in \mathcal{J} = [0, b], \tag{1.1}$$

subject to boundary conditions

$$u(0) - u'(0) = u_0, \qquad u(b) + u'(b) = u_b, \qquad u''(b) = u_b^*. \tag{1.2}$$

Here $2 < \nu \leq 3$, ${}^cD^{\nu}$ designates the Caputo fractional derivative of order ν, $f : \mathcal{J} \times \mathcal{R} \to \mathcal{R}$ is a continuous and u_0, u_b, u_b^* are real constants. We investigate the existence and uniqueness of solutions for boundary value problems (BVPs) for fractional differential equations.

Fractional differential equations have captivated much concentration since they can be practiced in diverse areas of science and engineering. Many phenomena in viscoelasticity, electrochemistry, control theory, porous media, electromagnetism, etc. can be modelled as fractional differential eqautions. For structures, view the documents of Kilbas et al. [16], Kiryakova [17], Miller and Ross [20], Oldham and Spanier [21], Podlubny [22], and Samko et al.[23]. Some the latest improvements to the hypothesis of fractional differential equations may be observed in [1–14, 18, 19, 24–28].

In [5], Bai and Lü studied the BVP of fractional order

$$D^{\alpha}_{0+}x(t) + f(t, x(t)) = 0, \qquad t \in (0, 1), \qquad \alpha \in (1, 2], \tag{1.3}$$

$$x(0) = x(1) = 0. \tag{1.4}$$

*Correspondence: abdulkadir.dogan@agu.edu.tr
2010 *AMS Mathematics Subject Classification:* 26A33

Here $D^{\alpha}_{0+}x(t)$ is the Caputo fractional derivative, and $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous.They found some existence and multiplicity results of positive solutions by employing some fixed point theorems .

In [26], Zhang investigated the existence and multiplicity of positive solutions for BVPs of nonlinear fractional differential equations

$$D^{\alpha}_{0+}x(t)=f(t,x(t)), \qquad t\in(0,1), \qquad \alpha\in(1,2], \tag{1.5}$$

$$x(0)+x'(0)=0, \qquad x(1)+x'(1)=0. \tag{1.6}$$

Here $D^{\alpha}_{0+}x(t)$ is the Caputo fractional derivative, $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous. He found some existence results of positive solutions by employing fixed point theorem on cones.

In [28], the authors established the existence of multiple positive solutions for the nonlinear fractional differential equation BVP

$$D^{\alpha}_{0+}x(t)+f(t,x(t))=0, \qquad t\in(0,1), \qquad \alpha\in(2,3], \tag{1.7}$$

$$x(0)=x'(0)=x'(1)=0, \tag{1.8}$$

where $D^{\alpha}_{0+}x(t)$ is the Caputo fractional derivative. They established some new existence criteria for singular and nonsingular fractional differential equation BVP by the characteristics of the Green function, the lower and upper solution method and fixed point theorem.

Inspired by the work of the above papers, in this paper we discuss the BVPs (1.1),(1.2). Using the Banach fixed point theorem, for the applications of Schaefer's and the Leray-Schauder fixed point theorem, we give some new existence results for BVPs (1.1),(1.2). Recently, Agarwal et al. examined the BVPs for fractional differential equations (1) and (2) in [1]. As far as we know, BVPs for fractional differential equations (1.1),(1.2) have not been studied. We will complete this opening in the written matter.

The plan of this manuscript is the following way. In Section 2, we recall several useful preliminaries that will be used to verify main results. Main results are given in Section 3. In Section 4, an illustration is displayed to clarify the principal conclusions.

2. Background

For the usefulness of the reader, one gives several notation, descriptions, preliminary facts, which can be found in [16, 22]. They will be used in the proofs of our results.

By $\mathcal{C}(\mathcal{J},\mathcal{R})$ one represents the Banach space of all continuous functions from $\mathcal{J}$ into $\mathcal{R}$ endowed with

$$||u||_{\infty}:=\sup\{|u(t)|:t\in\mathcal{J}\}.$$

Definition 2.1 The fractional integral of order $\nu\in\mathcal{R}_{+}$ for a function $y\in L^{1}([a,b],\mathcal{R}_{+})$ is given as

$$I^{\nu}_{a}y(t)=\frac{1}{\Gamma(\nu)}\int_{a}^{t}(t-s)^{\nu-1}y(s)ds,$$

here Γ is the gamma function. If $a=0$, then one has $I^{\nu}y(t)=y(t)\star\varphi_{\nu}(t)$, where $\varphi_{\nu}(t)=\frac{t^{\nu-1}}{\Gamma(\nu)}$ for $t>0$, $\varphi_{\nu}(t)=0$ for $t\leq 0$, $\varphi_{\nu}\to\delta(t)$ as $\nu\to 0$, here δ is the delta function.

Definition 2.2 The Riemann–Liouville fractional derivative of order ν for a function y defined on $[a,b]$ can be described by

$$D^{\nu}_{a+}y(t) = \frac{1}{\Gamma(k-\nu)}\left(\frac{d}{dt}\right)^{k}\int_a^t \frac{y(s)}{(t-s)^{\nu-k+1}}ds,$$

where

$$k = \begin{cases} [\nu]+1 & \text{if } \ k \notin \{0,1,2,\ldots\} \\ \nu & \text{if } \ k \in \{0,1,2,\ldots\} \end{cases}$$

and $[\nu]$ denotes the integer part of ν.

Definition 2.3 Caputo's fractional derivative of order ν for a function y given on $[a,b]$ can be prescribed as

$$^{c}D^{\nu}_{a+}y(t) = \frac{1}{\Gamma(k-\nu)}\int_a^t \frac{y^{(k)}(s)}{(t-s)^{\nu-k+1}}ds.$$

3. Main results

We shall begin with the following definition.

Definition 3.1 $u \in \mathcal{C}^2(\mathcal{J},\mathcal{R})$ with its ν-derivative occuring on $\mathcal{J}$ is called to be a solution of BVP (1.1) and (1.2) if u solves the equation $^{c}D^{\nu}u(t) = f(t,u(t))$ on $\mathcal{J}$ and boundary conditions $u(0)-u'(0) = u_0, \quad u(b)+u'(b) = u_b, \quad u''(b) = u_b^{\star}$.
Here the notation $\mathcal{C}^2(\mathcal{J},\mathcal{R})$ means continuously twice differentiable.

In order to establish the existence of solutions for the BVP (1.1) and (1.2), one has to use lemma as follows:

Lemma 3.2 *([26]). For $\nu > 0$, the solution of the differential equation $^{c}D^{\nu}y(t) = 0$ is presented as*

$$y(t) = c_0 + c_1 t + c_2 t^2 + \cdots c_{n-1}t^{n-1}, \quad \textit{here} \ \ c_j \in \mathcal{R}, \quad j = 0,1,2,\ldots,n-1$$

where

$$n = \begin{cases} [\nu]+1 & \textit{if } \ n \notin \{0,1,2,\ldots\} \\ \nu & \textit{if } \ n \in \{0,1,2,\ldots\}. \end{cases}$$

Taking into account Lemma 3.2, one has that

$$I^{\nu c}D^{\nu}y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots c_{n-1}t^{n-1}$$

for $c_j \in \mathcal{R}, \quad j = 0,1,2,\ldots,n-1$, where

$$n = \begin{cases} [\nu]+1 & \text{if } \ n \notin \{0,1,2,\ldots\} \\ \nu & \text{if } \ n \in \{0,1,2,\ldots\}. \end{cases}$$

By means of Lemma 3.2 one has the next result which is valuable.

Lemma 3.3 *Suppose that $y:\mathcal{J}\to\mathcal{R}$ is continuous,and $2<\nu\leq 3$. Let $d=b+2\neq 0$. Then u is a solution of the integral expression*

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\nu)}\int_0^t (t-w)^{\nu-1}y(w)dw - \frac{1}{2\Gamma(\nu-2)}t^2\int_0^b (b-w)^{\nu-3}y(w)dw \\ &+\frac{b^2}{2d\Gamma(\nu-2)}t\int_0^b (b-w)^{\nu-3}y(w)dw - \frac{1}{d\Gamma(\nu)}t\int_0^b (b-w)^{\nu-1}y(w)dw \\ &+\frac{b}{d\Gamma(\nu-2)}t\int_0^b (b-w)^{\nu-3}y(w)dw - \frac{1}{d\Gamma(\nu-1)}t\int_0^b (b-w)^{\nu-2}y(w)dw \\ &+\frac{b^2}{2d\Gamma(\nu-2)}\int_0^b (b-w)^{\nu-3}y(w)dw - \frac{1}{d\Gamma(\nu)}\int_0^b (b-w)^{\nu-1}y(w)dw \\ &+\frac{b}{d\Gamma(\nu-2)}\int_0^b (b-w)^{\nu-3}y(w)dw - \frac{1}{d\Gamma(\nu-1)}\int_0^b (b-w)^{\nu-2}y(w)dw \\ &+\frac{u_b^*}{2}t^2+\frac{u_b}{d}t-\frac{u_0}{d}t-\frac{u_b^*b^2}{2d}t-\frac{u_b^*b}{d}t+u_0+\frac{u_b}{d}-\frac{u_0}{d}-\frac{u_b^*b^2}{2d}-\frac{u_b^*b}{d} \end{aligned} \tag{3.1}$$

$\Longleftrightarrow$ *u is a solution of the BVP*

$$^cD^{\nu}u(t)=y(t), \qquad t\in\mathcal{J}, \tag{3.2}$$

$$u(0)-u'(0)=u_0, \quad u(b)+u'(b)=u_b, \quad u''(b)=u_b^{\star}. \tag{3.3}$$

Proof Express $u(t)$ using the RL integral as

$$u(t)=c_0+c_1t+c_2t^2+(I_a^{\nu}y)(t), \tag{3.4}$$

and then point out that the Caputo derivative annihilates the polynomial terms in the front and acts as an inverse to the RL integral in this situation (see[[16], Lemma 2.21]).

Applying the boundary conditions for (3.3), we find that

$$\begin{aligned} c_0 &= u_0+\frac{u_b}{d}-\frac{u_0}{d}-\frac{u_b^*}{2d}b^2+\frac{b^2}{2d\Gamma(\nu-2)}\int_0^b (b-w)^{\nu-3}y(w)dw \\ &-\frac{1}{d\Gamma(\nu)}\int_0^b (b-w)^{\nu-1}y(w)dw-\frac{u_b^*}{d}b+\frac{b}{d\Gamma(\nu-2)}\int_0^b (b-w)^{\nu-3}y(w)dw \\ &-\frac{b}{d\Gamma(\nu-1)}\int_0^b (b-w)^{\nu-2}y(w)dw, \end{aligned}$$

$$\begin{aligned} c_1 &= \frac{u_b}{d}-\frac{u_0}{d}-\frac{u_b^*}{2d}b^2+\frac{b^2}{2d\Gamma(\nu-2)}\int_0^b (b-w)^{\nu-3}y(w)dw \\ &-\frac{1}{d\Gamma(\nu)}\int_0^b (b-w)^{\nu-1}y(w)dw-\frac{u_b^*}{d}b+\frac{b}{d\Gamma(\nu-2)}\int_0^b (b-w)^{\nu-3}y(w)dw \\ &-\frac{b}{d\Gamma(\nu-1)}\int_0^b (b-w)^{\nu-2}y(w)dw, \end{aligned}$$

and

$$c_2 \quad = \quad \frac{u_b^*}{2} - \frac{1}{2\Gamma(\nu-2)} \int_0^b (b-w)^{\nu-3} y(w) dw.$$

If we substitute the values $c_0, c_1,\ c_2$ in (3.4), then we obtain the solution given by (3.1). Inversely, it is obvious that if u fulfills Eq. (3.1), then Eq. (3.2) and (3.3) are satisfied. □

By applying the Banach fixed point theorem, we give our first result.

Theorem 3.4 *Let $f : \mathcal{J} \times \mathcal{R} \to \mathcal{R}$ be a continuous function satisfying the Lipschitz conditions:*

(C1) $|f(t,w) - f(t,\overline{w})| \leq N_1 |w - \overline{w}|$, *for* $t \in \mathcal{J}, \quad w, \overline{w} \in \mathcal{R}, \quad N_1 > 0.$

Let

$$N_1 b^\nu \left[\frac{3}{\Gamma(\nu+1)} + \frac{2}{\Gamma(\nu)} + \frac{7}{2\Gamma(\nu-1)} \right] < 1. \tag{3.5}$$

There exists a unique solution for BVP (1.1) and (1.2) on $\mathcal{J}$.

Proof Convert BVP (1.1) and (1.2) into a fixed point problem. Define an operator $Q : \mathcal{C}(\mathcal{J}, \mathcal{R}) \to \mathcal{C}(\mathcal{J}, \mathcal{R})$ as

$$\begin{aligned} Qu(t) \quad = \quad & \frac{1}{\Gamma(\nu)} \int_0^t (t-\sigma)^{\nu-1} f(\sigma, u(\sigma)) d\sigma - \frac{1}{2\Gamma(\nu-2)} t^2 \int_0^b (b-\sigma)^{\nu-3} f(\sigma, u(\sigma)) d\sigma \\ & + \frac{b^2}{2d\Gamma(\nu-2)} t \int_0^b (b-\sigma)^{\nu-3} f(\sigma, u(\sigma)) d\sigma - \frac{1}{d\Gamma(\nu)} t \\ & \times \int_0^b (b-\sigma)^{\nu-1} f(\sigma, u(\sigma)) d\sigma + \frac{b}{d\Gamma(\nu-2)} t \int_0^b (b-\sigma)^{\nu-3} f(\sigma, u(\sigma)) d\sigma \\ & - \frac{1}{d\Gamma(\nu-1)} t \int_0^b (b-\sigma)^{\nu-2} f(\sigma, u(\sigma)) d\sigma + \frac{b^2}{2d\Gamma(\nu-2)} \\ & \times \int_0^b (b-\sigma)^{\nu-3} f(\sigma, u(\sigma)) d\sigma - \frac{1}{d\Gamma(\nu)} \int_0^b (b-\sigma)^{\nu-1} f(\sigma, u(\sigma)) d\sigma \\ & + \frac{b}{d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} f(\sigma, u(\sigma)) d\sigma - \frac{1}{d\Gamma(\nu-1)} \\ & \times \int_0^b (b-\sigma)^{\nu-2} f(\sigma, u(\sigma)) d\sigma + \frac{u_b^*}{2} t^2 + \frac{u_b}{d} t - \frac{u_0}{d} t - \frac{u_b^* b^2}{2d} t - \frac{u_b^* b}{d} t \\ & + u_0 + \frac{u_b}{d} - \frac{u_0}{d} - \frac{u_b^* b^2}{2d} - \frac{u_b^* b}{d}. \end{aligned}$$

Undoubtedly, the fixed points of the operator Q are solutions of the BVP (1.1) and (1.2). We will apply the Banach contraction principle to show that Q has a fixed point. We shall verify that Q is a contraction.

If $u_1, u_2 \in \mathcal{C}(\mathcal{J}, \mathcal{R})$, for $t \in \mathcal{J}$ we have

$$
\begin{aligned}
|(Qu_1)(t)-(Qu_2)(t)| \leq & \frac{1}{\Gamma(\nu)}\int_0^t (t-\varrho)^{\nu-1}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{b^2}{2\Gamma(\nu-2)}\int_0^b (b-\varrho)^{\nu-3}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{b^3}{2d\Gamma(\nu-2)}\int_0^b (b-\varrho)^{\nu-3}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{b}{d\Gamma(\nu)}\int_0^b (b-\varrho)^{\nu-1}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{b^2}{\Gamma(\nu-2)}\int_0^b (b-\varrho)^{\nu-3}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{b}{d\Gamma(\nu-1)}\int_0^b (b-\varrho)^{\nu-2}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{b^2}{2d\Gamma(\nu-2)}\int_0^b (b-\varrho)^{\nu-3}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{1}{d\Gamma(\nu)}\int_0^b (b-\varrho)^{\nu-1}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{b}{d\Gamma(\nu-2)}\int_0^b (b-\varrho)^{\nu-3}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
& + \frac{1}{d\Gamma(\nu-1)}\int_0^b (b-\varrho)^{\nu-2}|f(\varrho,u_1(\varrho))-f(\varrho,u_2(\varrho))|d\varrho \\
\leq & \frac{N_1||u_1-u_2||_\infty}{\Gamma(\nu)}\int_0^t (t-\rho)^{\nu-1}d\rho + \frac{b^2N_1||u_1-u_2||_\infty}{2\Gamma(\nu-2)} \\
& \times \int_0^b (b-\rho)^{\nu-3}d\rho + \frac{b^3N_1||u_1-u_2||_\infty}{2d\Gamma(\nu-2)}\int_0^b (b-\rho)^{\nu-3}d\rho \\
& + \frac{bN_1||u_1-u_2||_\infty}{d\Gamma(\nu)}\int_0^b (b-\rho)^{\nu-1}d\rho + \frac{b^2N_1||u_1-u_2||_\infty}{d\Gamma(\nu-2)} \\
& \times \int_0^b (b-\rho)^{\nu-3}d\rho + \frac{bN_1||u_1-u_2||_\infty}{d\Gamma(\nu-1)}\int_0^b (b-\rho)^{\nu-2}d\rho \\
& + \frac{b^2N_1||u_1-u_2||_\infty}{2d\Gamma(\nu-2)}\int_0^b (b-\rho)^{\nu-3}d\rho + \frac{N_1||u_1-u_2||_\infty}{(b+2)\Gamma(\nu)} \\
& \times \int_0^b (b-\rho)^{\nu-1}d\rho + \frac{bN_1||u_1-u_2||_\infty}{d\Gamma(\nu-2)}\int_0^b (b-\rho)^{\nu-3}d\rho \\
& + \frac{N_1||u_1-u_2||_\infty}{d\Gamma(\nu-1)}\int_0^b (b-\rho)^{\nu-2}d\rho \\
\leq & N_1 b^\nu\left[\frac{1}{\Gamma(\nu+1)} + \frac{1}{2\Gamma(\nu-1)} + \frac{1}{2\Gamma(\nu-1)} + \frac{1}{\Gamma(\nu+1)} + \frac{1}{\Gamma(\nu-1)}\right. \\
& \left. + \frac{1}{\Gamma(\nu)} + \frac{1}{2\Gamma(\nu-1)} + \frac{1}{\Gamma(\nu+1)} + \frac{1}{\Gamma(\nu-1)} + \frac{1}{\Gamma(\nu)}\right]||u_1-u_2||_\infty.
\end{aligned}
$$

Consequently we obtain

$$
||Q(u_1)-Q(u_2)|| \leq N_1 b^\nu\left[\frac{3}{\Gamma(\nu+1)} + \frac{2}{\Gamma(\nu)} + \frac{7}{2\Gamma(\nu-1)}\right]||u_1-u_2||_\infty,
$$

which, by (3.5), implies that Q is a contraction. We conclude that Q has a fixed point which is a solution of the BVP (1.1) and (1.2) by applying Banach fixed point theorem. □

Next, one will apply Schaefer's fixed point theorem to obtain the second result.

Theorem 3.5 *Suppose that*

(C2) $f : \mathcal{J} \times \mathcal{R} \to \mathcal{R}$ *is continuous;*

(C3) there exists a fixed $N_2 > 0$ *satisfying*

$$|f(t,z)| \leq N_2, \qquad t \in \mathcal{J}, \qquad z \in \mathcal{R}.$$

BVP (1.1) *and* (1.2) *has at least one solution on* $\mathcal{J}$.

Proof We need to prove that the operator Q has a fixed point. The proof can be broken down into steps as follows.

(1) Q **is continuous.**

If $\{u_n\}$ is a sequence such that $u_n \to u$ in $\mathcal{C}(\mathcal{J}, \mathcal{R})$, then we have

$$\begin{aligned}
|(Qu_n)(t) - (Qu)(t)| \leq & \frac{1}{\Gamma(\nu)} \int_0^t (t-\vartheta)^{\nu-1} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{b^2}{2\Gamma(\nu-2)} \int_0^b (b-\vartheta)^{\nu-3} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{b^3}{2d\Gamma(\nu-2)} \int_0^b (b-\vartheta)^{\nu-3} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{b}{d\Gamma(\nu)} \int_0^b (b-\vartheta)^{\nu-1} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{b^2}{d\Gamma(\nu-2)} \int_0^b (b-\vartheta)^{\nu-3} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{b}{d\Gamma(\nu-1)} \int_0^b (b-\vartheta)^{\nu-2} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{b^2}{2d\Gamma(\nu-2)} \int_0^b (b-\vartheta)^{\nu-3} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{1}{d\Gamma(\nu)} \int_0^b (b-\vartheta)^{\nu-1} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{b}{d\Gamma(\nu-2)} \int_0^b (b-\vartheta)^{\nu-3} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta \\
& + \frac{1}{d\Gamma(\nu-1)} \int_0^b (b-\vartheta)^{\nu-2} |f(\vartheta, u_n(\vartheta)) - f(\vartheta, u(\vartheta))| d\vartheta.
\end{aligned}$$

Because f is a continuous function, we have

$$||Q(u_n) - Q(u)||_\infty \to 0 \ \text{ as } \ n \to \infty.$$

(2) Q **maps bounded sets into bounded sets in** $\mathcal{C}(\mathcal{J}, \mathcal{R})$.

In fact, it suffices to verify that for $\xi^* > 0$ there exists a fixed $R_1 > 0$ satisfying $u \in \mathcal{B}_{\xi^*} = \{u \in \mathcal{C}(\mathcal{J}, \mathcal{R}) : ||u||_\infty \leq \xi^*\}$ we have $||Q(u)||_\infty \leq R_1$. From (C3), for $t \in \mathcal{J}$, we have

$$
\begin{aligned}
|Qu(t)| \leq & \frac{1}{\Gamma(\nu)} \int_0^t (t-\sigma)^{\nu-1} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{b^2}{2\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{b^3}{2d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{b}{d\Gamma(\nu)} \int_0^b (b-\sigma)^{\nu-1} |(\sigma, u(\sigma))| d\sigma \\
& + \frac{b^2}{d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{b}{d\Gamma(\nu-1)} \int_0^b (b-\sigma)^{\nu-2} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{b^2}{2d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{1}{d\Gamma(\nu)} \int_0^b (b-\sigma)^{\nu-1} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{b}{d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} |f(\sigma, u(\sigma))| d\sigma \\
& + \frac{1}{d\Gamma(\nu-1)} \int_0^b (b-\sigma)^{\nu-2} |f(\sigma, u(\sigma))| d\sigma \\
& + |u_0| \Big(\frac{b}{d} + 1 + \frac{1}{d} \Big) + \frac{|u_b|}{d}(1+b) + |u_b^*| \left[\frac{b^2}{2} + \frac{b^3}{2d} + \frac{b^2}{d} + \frac{b^2}{2d} + \frac{b}{d} \right] \\
\leq & \frac{N_2}{\Gamma(\nu)} \int_0^t (t-\sigma)^{\nu-1} d\sigma + \frac{b^2 N_2}{2\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} d\sigma \\
& + \frac{b^3 N_2}{2d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} d\sigma + \frac{bN_2}{d\Gamma(\nu)} \int_0^b (b-\sigma)^{\nu-1} d\sigma \\
& + \frac{b^2 N_2}{d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} d\sigma + \frac{bN_2}{d\Gamma(\nu-1)} \int_0^b (b-\sigma)^{\nu-2} d\sigma \\
& + \frac{b^2 N_2}{2d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} d\sigma + \frac{N_2}{d\Gamma(\nu)} \int_0^b (b-\sigma)^{\nu-1} d\sigma \\
& + \frac{bN_2}{d\Gamma(\nu-2)} \int_0^b (b-\sigma)^{\nu-3} d\sigma + \frac{N_2}{d\Gamma(\nu-1)} \int_0^b (b-\sigma)^{\nu-2} d\sigma \\
& + |u_0| \Big(\frac{b}{d} + 1 + \frac{1}{d} \Big) + \frac{|u_b|}{d}(1+b) + |u_b^*| \left[\frac{b^2}{2} + \frac{b^3}{2d} + \frac{b^2}{d} + \frac{b^2}{2d} + \frac{b}{d} \right] \\
\leq & \frac{N_2}{\Gamma(\nu+1)} b^\nu + \frac{N_2}{2\Gamma(\nu-1)} b^\nu + \frac{N_2}{2\Gamma(\nu-1)} b^\nu + \frac{N_2}{\Gamma(\nu+1)} b^\nu + \frac{N_2}{\Gamma(\nu-1)} b^\nu \\
& + \frac{N_2}{\Gamma(\nu)} b^\nu + \frac{N_2}{2\Gamma(\nu-1)} b^\nu + \frac{N_2}{\Gamma(\nu+1)} b^\nu + \frac{N_2}{\Gamma(\nu-1)} b^\nu + \frac{N_2}{\Gamma(\nu)} b^\nu \\
& + |u_0| \Big(\frac{b}{d} + 1 + \frac{1}{d} \Big) + \frac{|u_b|}{d}(1+b) + |u_b^*| \Big[\frac{b^2}{2} + \frac{b^3}{2d} + \frac{b^2}{d} + \frac{b^2}{2d} + \frac{b}{d} \Big],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
||Qu||_\infty \leq & \frac{3N_2}{\Gamma(\nu+1)} b^\nu + \frac{2N_2}{\Gamma(\nu)} b^\nu + \frac{7N_2}{2\Gamma(\nu-1)} b^\nu \\
& + \frac{|u_b^*|}{2d}(2b^3 + 5b^2 + 2b) + \frac{|u_b|}{d}(b+1) + \frac{|u_0|}{d}(2b+3) := R_1.
\end{aligned}
$$

(3) Q maps bounded sets into equicontinuous sets of $\mathcal{C}(\mathcal{J},\mathcal{R})$.

For $m_1, m_2 \in \mathcal{J}$, $m_1 < m_2$, suppose that $\mathcal{B}_{\xi^*}$ is a bounded set of $\mathcal{C}(\mathcal{J},\mathcal{R})$ like in (2). For any $u \in \mathcal{B}_{\xi^*}$, we have

$$
\begin{aligned}
|Qu(m_2) - Qu(m_1)| =& \Bigg| \frac{1}{\Gamma(\nu)} \int_0^{m_1} [(m_2-\varsigma)^{\nu-1} - (m_1-\varsigma)^{\nu-1}] f(\varsigma, u(\varsigma)) d\varsigma \\
&+ \frac{(m_2-m_1)^2}{\Gamma(\nu-2)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-3} f(\varsigma, u(\varsigma)) d\varsigma + \frac{(m_2-m_1)}{\Gamma(\nu-2)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-3} f(\varsigma, u(\varsigma)) d\varsigma + \frac{(m_2-m_1)}{\Gamma(\nu)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} f(\varsigma, u(\varsigma)) d\varsigma + \frac{(m_2-m_1)}{\Gamma(\nu-2)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-3} f(\varsigma, u(\varsigma)) d\varsigma + \frac{(m_2-m_1)}{\Gamma(\nu-1)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-2} f(\varsigma, u(\varsigma)) d\varsigma + \frac{1}{\Gamma(\nu-2)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-3} f(\varsigma, u(\varsigma)) d\varsigma + \frac{1}{\Gamma(\nu)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} f(\varsigma, u(\varsigma)) d\varsigma + \frac{1}{\Gamma(\nu-2)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-3} f(\varsigma, u(\varsigma)) d\varsigma + \frac{1}{\Gamma(\nu-1)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-2} f(\varsigma, u(\varsigma)) d\varsigma \Bigg| + \frac{|u_b^*|}{2}(m_2-m_1)^2 \\
&+ \frac{|u_b|}{d}(m_2-m_1) + \frac{|u_0|}{d}(m_2-m_1) + \frac{|u_b^*|}{2d} b^2 (m_2-m_1) \\
&+ \frac{|u_b^*|}{d} b(m_2-m_1) + |u_0|\Big(1+\frac{1}{d}\Big) + \frac{|u_b|}{d} + |u_b^*|\Big(\frac{b^2}{2d}+\frac{b}{d}\Big) \\
\le& \frac{N_2}{\Gamma(\nu)} \int_0^{m_1} [(m_1-\varsigma)^{\nu-1} - (m_2-\varsigma)^{\nu-1}] d\varsigma + \frac{N_2(m_2-m_1)^2}{\Gamma(\nu-2)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma + \frac{N_2(m_2-m_1)}{\Gamma(\nu-2)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma \\
&+ \frac{N_2(m_2-m_1)}{\Gamma(\nu)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma + \frac{N_2(m_2-m_1)}{\Gamma(\nu-2)} \\
&\times \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma + \frac{N_2(m_2-m_1)}{\Gamma(\nu-1)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma \\
&+ \frac{N_2}{\Gamma(\nu-2)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma + \frac{N_2}{\Gamma(\nu)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma \\
&+ \frac{N_2}{\Gamma(\nu-2)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma + \frac{N_2}{\Gamma(\nu-1)} \int_{m_1}^{m_2} (m_2-\varsigma)^{\nu-1} d\varsigma \\
&+ \frac{|u_b^*|}{2}(m_2-m)^2 + \frac{|u_b|}{d}(m_2-m_1) + \frac{|u_0|}{b+2}(m_2-m_1) \\
&+ \frac{|u_b^*|}{2d} b^2(m_2-m_1) + \frac{|u_b^*|}{d} b(m_2-m_1) \\
&+ |u_0|\Big(1+\frac{1}{d}\Big) + \frac{|u_b|}{d} + |u_b^*|\Big(\frac{b^2}{2d}+\frac{b}{d}\Big)
\end{aligned}
$$

$$\begin{aligned}
\leq & \frac{N_2}{\Gamma(\nu+1)}[(m_2-m_1)^\nu + m_1^\nu - m_2^\nu] + \frac{N_2}{\Gamma(\nu-1)}(m_2-m_1)^\nu \\
& + \frac{N_2}{\Gamma(\nu-1)}(m_2-m_1)^\nu + \frac{N_2}{\Gamma(\nu+1)}(m_2-m_1)^\nu \\
& + \frac{N_2}{\Gamma(\nu-1)}(m_2-m_1)^\nu + \frac{N_2}{\Gamma(\nu)}(m_2-m_1)^\nu \\
& + \frac{N_2}{\Gamma(\nu-1)}(m_2-m_1)^\nu + \frac{N_2}{\Gamma(\nu+1)}(m_2-m_1)^\nu \\
& + \frac{N_2}{\Gamma(\nu-1)}(m_2-m_1)^\nu + \frac{N_2}{\Gamma(\nu)}(m_2-m_1)^\nu \\
& + \frac{|u_b^*|}{2}(m_2-m_1)^2 + \frac{|u_b|}{d}(m_2-m_1) + \frac{|u_0|}{d}(m_2-m_1) \\
& + \frac{|u_b^*|}{2d}b^2(m_2-m_1) + \frac{|u_b^*|}{d}b(m_2-m_1) \\
& + |u_0|\Big(1+\frac{1}{d}\Big) + \frac{|u_b|}{d} + |u_b^*|\Big(\frac{b^2}{2d}+\frac{b}{d}\Big) \to 0 \text{ as } m_1 \to m_2.
\end{aligned}$$

In view of (1) to (3), it follows by Arzela-Ascoli theorem that $Q : \mathcal{C}(\mathcal{J},\mathcal{R}) \to \mathcal{C}(\mathcal{J},\mathcal{R})$ is completely continuous.

(4) A priori bounds.

Finally we will verify that the set

$$\mathbf{\Omega} = \{u \in \mathcal{C}(\mathcal{J},\mathcal{R}) \mid u = \lambda Q u, \quad \lambda \in (0,1)\}$$

is bounded.

Let $u \in \mathbf{\Omega}$, $u = \lambda Q u$ for $\lambda \in (0,1)$. Therefore, for $t \in \mathcal{J}$, we have

$$\begin{aligned}
u(t) \quad = \quad & \frac{\lambda}{\Gamma(\nu)}\int_0^t (t-\sigma)^{\nu-1} f(\sigma,u(\sigma))d\sigma - \frac{\lambda}{2\Gamma(\nu-2)}t^2\int_0^b (b-\sigma)^{\nu-3}f(\sigma,u(\sigma))d\sigma \\
& + \frac{\lambda b^2}{2d\Gamma(\nu-2)}t\int_0^b (b-\sigma)^{\nu-3}f(\sigma,u(\sigma))d\sigma - \frac{\lambda}{d\Gamma(\nu)}t \\
& \times \int_0^b (b-\sigma)^{\nu-1}f(\sigma,u(\sigma))d\sigma + \frac{\lambda b}{d\Gamma(\nu-2)}t\int_0^b (b-\sigma)^{\nu-3}f(\sigma,u(\sigma)d\sigma \\
& - \frac{\lambda}{d\Gamma(\nu-1)}t\int_0^b (b-\sigma)^{\nu-2}f(\sigma,u(\sigma))d\sigma + \frac{\lambda b^2}{2d\Gamma(\nu-2)} \\
& \times \int_0^b (b-\sigma)^{\nu-3}f(\sigma,u(\sigma))d\sigma - \frac{\lambda}{d\Gamma(\nu)}\int_0^b (b-\sigma)^{\nu-1}f(\sigma,u(\sigma))d\sigma \\
& + \frac{\lambda b}{d\Gamma(\nu-2)}\int_0^b (b-\sigma)^{\nu-3}f(\sigma,u(\sigma))d\sigma - \frac{\lambda}{d\Gamma(\nu-1)} \\
& \times \int_0^b (b-\sigma)^{\nu-2}f(\sigma,u(\sigma))d\sigma + \lambda\frac{u_b^*}{2}t^2 \\
& + \frac{\lambda t}{d}\Big[-u_0 + u_b - \frac{u_b^* b^2}{2} - u_b^* b\Big] + \lambda u_0\Big(1-\frac{1}{d}\Big) + \lambda\frac{u_b}{d} - \lambda u_b^*\Big(\frac{b^2}{2d}+\frac{b}{d}\Big).
\end{aligned}$$

This indicates by (C3) that for $t \in \mathcal{J}$ we have

$$\begin{aligned}
|u(t)| \leq & \frac{N_2}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} d\xi + \frac{b^2 N_2}{2\Gamma(\nu-2)} \int_0^b (b-\xi)^{\nu-3} d\xi \\
& + \frac{b^3 N_2}{2d\Gamma(\nu-2)} \int_0^b (b-\xi)^{\nu-3} d\xi + \frac{b N_2}{d\Gamma(\nu)} \int_0^b (b-\xi)^{\nu-1} d\xi \\
& + \frac{b^2 N_2}{d\Gamma(\nu-2)} \int_0^b (b-\xi)^{\nu-3} d\xi + \frac{b N_2}{d\Gamma(\nu-1)} \int_0^b (b-\xi)^{\nu-2} d\xi \\
& + \frac{b^2 N_2}{2d\Gamma(\nu-2)} \int_0^b (b-\xi)^{\nu-3} d\xi + \frac{N_2}{d\Gamma(\nu)} \int_0^b (b-\xi)^{\nu-1} d\xi \\
& + \frac{b N_2}{d\Gamma(\nu-2)} \int_0^b (b-\xi)^{\nu-3} d\xi + \frac{N_2}{d\Gamma(\nu-1)} \int_0^b (b-\xi)^{\nu-2} d\xi \\
& + |u_0| \Big(\frac{b}{d} + 1 + \frac{1}{d} \Big) + \frac{|u_b|}{d}(1+b) + |u_b^*| \left[\frac{b^2}{2} + \frac{b^3}{2d} + \frac{b^2}{d} + \frac{b^2}{2d} + \frac{b}{d} \right] \\
\leq & \frac{N_2}{\nu\Gamma(\nu)} b^\nu + \frac{N_2}{2(\nu-2)\Gamma(\nu-2)} b^\nu + \frac{N_2}{2(\nu-2)\Gamma(\nu-2)} b^\nu + \frac{N_2}{\nu\Gamma(\nu)} b^\nu \\
& + \frac{N_2}{(\nu-2)\Gamma(\nu-2)} b^\nu + \frac{N_2}{(\nu-1)\Gamma(\nu-1)} b^\nu + \frac{N_2}{2(\nu-2)\Gamma(\nu-2)} b^\nu \\
& + \frac{N_2}{\nu\Gamma(\nu)} b^\nu + \frac{N_2}{(\nu-2)\Gamma(\nu-2)} b^\nu + \frac{N_2}{(\nu-1)\Gamma(\nu-1)} b^\nu \\
& + |u_0| \Big(\frac{b}{d} + 1 + \frac{1}{d} \Big) + \frac{|u_b|}{d}(1+b) + |u_b^*| \left[\frac{b^2}{2} + \frac{b^3}{2d} + \frac{b^2}{d} + \frac{b^2}{2d} + \frac{b}{d} \right].
\end{aligned}$$

Hence for every $t \in \mathcal{J}$ we have

$$\begin{aligned}
||u||_\infty \quad \leq \quad & \frac{3N_2}{\Gamma(\nu+1)} b^\nu + \frac{2N_2}{\Gamma(\nu)} b^\nu + \frac{7N_2}{2\Gamma(\nu-1)} b^\nu \\
& + \frac{|u_b^*|}{2d}(2b^3 + 5b^2 + 2b) + \frac{|u_b|}{d}(b+1) + \frac{|u_0|}{d}(2b+3) := R_2.
\end{aligned}$$

We have shown that the set $\mathbf{\Omega}$ is bounded. We can complete that Q has a fixed point which is a solution of the BVP (1.1)-(1.2) by using Schaefer's fixed point theorem. □

Now we establish our existence results for the BVP (1.1) and (1.2) by employing the Leray-Schauder fixed point theorem, here the assumption (C3) is diminished.

Theorem 3.6 *Suppose that assumption (C2) holds. Define* $N_2 = D_1 + 1$, *where*

$$\begin{aligned}
D_1 = & ||I^q \varphi_f||_L^1 \psi(N_2) + \frac{b^2}{2}(I^{\nu-2}\varphi_f)(b)\psi(N_2) + \frac{b^3}{2d}(I^{\nu-2}\varphi_f)(b)\psi(N_2) \\
& + \frac{b}{d}(I^\nu \varphi_f)(b)\psi(N_2) + \frac{b^2}{d}(I^{\nu-2}\varphi_f)(b)\psi(N_2) + \frac{b}{d}(I^{\nu-1}\varphi_f)(b) \\
& \times \psi(N_2) + \frac{b^2}{2d}(I^{\nu-2}\varphi_f)(b)\psi(N_2) + \frac{1}{d}(I^\nu \varphi_f)(b)\psi(N_2) \\
& + \frac{b}{d}(I^{\nu-2}\varphi_f)(b)\psi(N_2) + \frac{1}{d}(I^{\nu-1}\varphi_f)(b)\psi(N_2) \\
& + |u_0| \Big(\frac{b}{d} + 1 + \frac{1}{d} \Big) + \frac{|u_b|}{d}(1+b) + |u_b^*| \left(\frac{b^2}{2} + \frac{b^3}{2d} + \frac{b^2}{d} + \frac{b^2}{2d} + \frac{b}{d} \right).
\end{aligned}$$

Suppose that

(C4) there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ *and a function* $\varphi_f \in L^1(\mathcal{J},\mathcal{R}^+)$ *satisfying*

$$|f(t,w)| \le \varphi_f(t)\psi(|w|), \qquad t \in \mathcal{J}, \qquad w \in \mathcal{R}.$$

Then the BVP (1.1) *and* (1.2) *has at least one solution on* $\mathcal{J}$*.*

Proof We recognize the operator Q described in Theorems 3.4, 3.5. It is easy to prove that Q is continuous, completely continuous. Let $\lambda \in [0,1]$. Suppose u is such that for $t \in \mathcal{J}$ we have $u(t) = \lambda(Qu)(t)$. By (C4) for $t \in \mathcal{J}$ we have

$$\begin{aligned}
|u(t)| \le & \frac{1}{\Gamma(\nu)}\int_0^t (t-\eta)^{\nu-1}\varphi_f(\eta)\psi(|u(\eta)|)d\eta + \frac{b^2}{2\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)\psi(|u(\eta)|)d\eta \\
& + \frac{b^3}{2d\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)\psi(|u(\eta)|)d\eta + \frac{b}{d\Gamma(\nu)} \\
& \times \int_0^b (b-\eta)^{\nu-1}\varphi_f(\eta)\psi(|u(\eta)|)d\eta + \frac{b^2}{d\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)\psi(|u(\eta)|)d\eta \\
& + \frac{b}{d\Gamma(\nu-1)}\int_0^b (b-\eta)^{\nu-2}\varphi_f(\eta)\psi(|u(\eta)|)d\eta + \frac{b^2}{2d\Gamma(\nu-2)} \\
& \times \int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)\psi(|u(\eta)|)d\eta + \frac{1}{d\Gamma(\nu)}\int_0^b (b-\eta)^{\nu-1}\varphi_f(\eta)\psi(|u(\eta)|)d\eta \\
& + \frac{b}{d\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)\psi(|u(\eta)|)d\eta + \frac{1}{d\Gamma(\nu-1)} \\
& \times \int_0^b (b-\eta)^{\nu-2}\varphi_f(\eta)\psi(|u(\eta)|)d\eta + |u_0|\Big(\frac{b}{d}+1+\frac{1}{d}\Big) \\
& + \frac{|u_b|}{d}(1+b) + |u_b^*|\left(\frac{b^2}{2}+\frac{b^3}{2d}+\frac{b^2}{d}+\frac{b^2}{2d}+\frac{b}{d}\right) \\
\le & \psi(||u||_\infty)\frac{1}{\Gamma(\nu)}\int_0^t (t-\eta)^{\nu-1}\varphi_f(\eta)d\eta + \psi(||u||_\infty)\frac{b^2}{2\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)d\eta \\
& + \psi(||u||_\infty)\frac{b^3}{2d\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)d\eta + \psi(||u||_\infty)\frac{b}{d\Gamma(\nu)} \\
& \times \int_0^b (b-\eta)^{\nu-1}\varphi_f(\eta)d\eta + \psi(||u||_\infty)\frac{b^2}{d\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)d\eta \\
& + \psi(||u||_\infty)\frac{b}{d\Gamma(\nu-1)}\int_0^b (b-\eta)^{\nu-2}\varphi_f(\eta)d\eta + \psi(||u||_\infty)\frac{b^2}{2d\Gamma(\nu-2)} \\
& \times \int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)d\eta + \psi(||u||_\infty)\frac{1}{d\Gamma(\nu)}\int_0^b (b-\eta)^{\nu-1}\varphi_f(\eta)d\eta \\
& + \psi(||u||_\infty)\frac{b}{d\Gamma(\nu-2)}\int_0^b (b-\eta)^{\nu-3}\varphi_f(\eta)d\eta + \psi(||u||_\infty)\frac{1}{d\Gamma(\nu-1)} \\
& \times \int_0^b (b-\eta)^{\nu-2}\varphi_f(\eta)d\eta + |u_0|\Big(\frac{b}{d}+1+\frac{1}{d}\Big) \\
& + \frac{|u_b|}{d}(1+b) + |u_b^*|\left(\frac{b^2}{2}+\frac{b^3}{2d}+\frac{b^2}{d}+\frac{b^2}{2d}+\frac{b}{d}\right).
\end{aligned}$$

Therefore

$$\frac{||u||_\infty}{D_2} \le 1,$$

where

$$\begin{aligned} D_2 \quad = \quad & \psi(||u||_\infty)||I^\nu \varphi_f||_L^1 + \frac{b^2}{2}(I^{\nu-2}\varphi_f)(b)\psi(||u||_\infty) + \frac{b^3}{2d}(I^{\nu-2}\varphi_f)(b)\psi(||u||_\infty) \\ & + \frac{b}{d}(I^\nu \varphi_f)(b)\psi(||u||_\infty) + \frac{b^2}{d}(I^{\nu-2}\varphi_f)(b)\psi(||u||_\infty) + \frac{b}{d}(I^{\nu-1}\varphi_f)(b) \\ & \times\psi(||u||_\infty) + \frac{b^2}{2d}(I^{\nu-2}\varphi_f)(b)\psi(||u||_\infty) + \frac{1}{d}(I^\nu \varphi_f)(b)\psi(||u||_\infty) \\ & + \frac{b}{d}(I^{\nu-2}\varphi_f)(b)\psi(||u||_\infty) + \frac{1}{d}(I^{\nu-1}\varphi_f)(b)\psi(||u||_\infty) \\ & + |u_0|\left(\frac{b}{d} + 1 + \frac{1}{d}\right) + \frac{|u_b|}{d}(1+b) + |u_b^*|\left(\frac{b^2}{2} + \frac{b^3}{2d} + \frac{b^2}{d} + \frac{b^2}{2d} + \frac{b}{d}\right). \end{aligned}$$

Put

$$W = \{u \in \mathcal{C}(\mathcal{J}, \mathcal{R}) : ||u||_\infty < D_1 + 1\}.$$

Clearly, $Q : \overline{W} \to \mathcal{C}(\mathcal{J}, \mathcal{R})$ is continuous, completely continuous. By the select of W, there is no $u \in \partial W$ satisfying $u = \lambda Q(u)$ for $0 < \lambda < 1$. We can complete that Q has a fixed point $u \in \overline{W}$, which is a solution of the BVP (1.1)-(1.2) by employing the Leray-Schauder fixed point theorem [15]. □

4. Applications

In this section, we present some examples to illustrate our results.

Example 4.1 *Examine the fractional BVP*

$$^cD^\nu u(t) = \frac{e^{-t}|u(t)|}{(9+e^t)(1+|u(t)|)}, \qquad t \in \mathcal{J} := [0,1], \qquad \nu \in (2,3], \tag{4.1}$$

$$u(0) - u'(0) = 0, \qquad u(1) + u'(1) = 1, \qquad u''(1) = 0. \tag{4.2}$$

Let

$$f(t,u) = \frac{e^{-t}u}{(9+e^t)(1+u)}, \quad (t,u) \in \mathcal{J} \times [0,\infty).$$

For $u_1, u_2 \in [0,\infty)$ *and* $t \in \mathcal{J}$, *one has*

$$\begin{aligned} |f(t,u_1) - f(t,u_2)| &= \frac{e^{-t}}{(9+e^t)}\left|\frac{u_1}{1+u_1} - \frac{u_2}{1+u_2}\right| \\ &= \frac{e^{-t}|u_1-u_2|}{(9+e^t)(1+u_1)(1+u_2)} \\ &\leq \frac{e^{-t}}{(9+e^t)}|u_1-u_2| \\ &\leq \frac{1}{10}|u_1-u_2|, \end{aligned}$$

where $N_1 = \frac{1}{10}$. *Thus, the assumption (C1) is satisfied. Choose* $b = 1$. *We will confirm that assumption* (3.5) *holds. In fact,*

$$N_1 b^\nu \left[\frac{3}{\Gamma(\nu+1)} + \frac{2}{\Gamma(\nu)} + \frac{7}{2\Gamma(\nu-1)}\right] < 1 \Leftrightarrow \frac{3}{\Gamma(\nu+1)} + \frac{2}{\Gamma(\nu)} + \frac{7}{2\Gamma(\nu-1)} < 10. \tag{4.3}$$

One has

$$\frac{1}{2} \leq \frac{3}{\Gamma(\nu+1)} < \frac{3}{2}, \qquad\qquad 1 \leq \frac{2}{\Gamma(\nu)} < 2, \tag{4.4}$$

and

$$\frac{7}{2} \leq \frac{7}{2\Gamma(\nu-1)} < x, \tag{4.5}$$

for an suitably selected constant x *that will be defined. From* (4.3)-(4.5), *we can write as*

$$\frac{3}{\Gamma(\nu+1)} + \frac{2}{\Gamma(\nu)} + \frac{7}{2\Gamma(\nu-1)} < \frac{3}{2} + 2 + x < 10. \tag{4.6}$$

From inequality (4.6), *it follows that*

$$x < \frac{13}{2}.$$

By (4.5) *we obtain*

$$\frac{7}{13} \approx 0.538462 < \Gamma(\nu-1), \tag{4.7}$$

which holds for some $2 < \nu \leq 3$. *Therefore, by Theorem 3.4, there exists a unique solution for the BVP* (4.1) *and* (4.2) *on* $[0,1]$. *It is easy to see that inequality* (4.7) *is satisfied for the values of* ν.

Example 4.2 *Consider the fractional BVP*

$$^cD^{\frac{5}{2}}u(t) = \frac{\cos^2 t}{(e^{-t}+3)^2|u(t)|}, \qquad \textit{for} \quad (t,u) \in ([0,1], \mathcal{R}_+), \tag{4.8}$$

$$u(0) - u'(0) = 0, \qquad u(1) + u'(1) = 1, \qquad u''(1) = 0, \tag{4.9}$$

where $\nu = \frac{5}{2}, \quad b = 1,$ *and* $f(t,u) := \dfrac{\cos^2 t}{(e^{-t}+3)^2|u|}.$

We have

$$|f(t,u)| = \Big|\frac{\cos^2 t}{(e^{-t}+3)^2|u|}\Big| \leq \frac{1}{9}.$$

Choosing $N_2 = \frac{1}{9}$, *then by Theorem 3.5, the BVP* (4.8)-(4.9) *has at least one solution on* $[0,1]$.

Example 4.3 *Consider the fractional BVP*

$$^{c}D^{\frac{5}{2}}u(t) = \Big(\frac{1}{5+3t+t^2}\Big)\Big(\frac{1}{3}\times\frac{|u|^7}{1+|u|^6}+4\Big), \qquad t\in[0,1], \tag{4.10}$$

$$u(0)-u'(0)=0, \qquad u(1)+u'(1)=1, \qquad u''(1)=0, \tag{4.11}$$

where $\nu = \frac{5}{2}, \quad b = 1,$ *and* $f(t,u) = \Big(\dfrac{1}{5+3t+t^2}\Big)\Big(\dfrac{1}{3}\times\dfrac{|u|^7}{1+|u|^6}+4\Big).$

We have

$$|f(t,u)| \leq \frac{1}{5+3t+t^2}\Big(\frac{1}{3}|u|+4\Big).$$

Choosing $\varphi_f(t) = \dfrac{1}{5+3t+t^2}$ *and* $\psi(|u|) = \dfrac{1}{3}|u|+4$, *then by Theorem 3.6, the BVP* (4.10)-(4.11) *has at least one solution on* $[0,1]$.

Acknowledgment

We would like to express our gratitude to the anonymous reviewers and editors for their valuable comments and suggestions, which improve the quality of present paper.

References

[1] Agarwal RP, Benchohra M, Hamani S. Boundary value problems for fractional differential equations. Georgian Mathematical Journal 2009; 16 (3): 401-411.

[2] Agarwal RP, O'Regan D, Stanek S. Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. Journal of Mathematical Analysis and Applications 2010; 371: 57-68. doi:10.1016/j.jmaa.2010.04.034

[3] Ahmad B, Nieto JJ. Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. Computers & Mathematics with Applications 2009; 58 (9): 1838-1843.doi:10.1016/j.camwa.2009.07.091

[4] Araya D, Lizama C. Almost automorphic mild solutions to fractional differential equations. Nonlinear Analysis 2008; 69: 3692-3705. doi:10.1016/j.na.2007.10.004

[5] Bai Z, Lü H. Positive solutions for boundary-value problem of nonlinear fractional differential equation. Journal of Mathematical Analysis and Applications 2005; 311: 495-505. doi:10.1016/j.jmaa.2005.02.052

[6] Bai Z, Zhang S, Sun S, Yin C. Monotone iterative method for fractional differential equations. Electronic Journal of Differential Equations 2016; 2016 (6): 1-8.

[7] Bekkai A, Rebiai B, Kirane M. On local existence and blowup of solutions for a time-space fractional diffusion equation with exponential nonlinearity. Mathematical Methods in the Applied Sciences 2019; 42: 1819-1830. doi: 10.1002/mma.5476

[8] Benchohra M, Henderson J, Ntouyas SK, Quahab A. Existence results for fractional order functional differential equations with infinite delay. Journal of Mathematical Analysis and Applications 2008; 338 (2): 1340-1350. doi:10.1016/j.jmaa.2007.06.021

[9] Bonilla B, Rivero M, Rodriguez-Germa L, Trujillo JJ. Fractional differential equations as alternative models to nonlinear differential equations. Applied Mathematics and Computation 2007; 187: 79-88. doi:10.1016/j.amc.2006.08.105

[10] Cabada A, Wang G. Positive solutions of nonlinear fractional differential equations with integral boundary value conditions. Journal of Mathematical Analysis and Applications 2012; 389: 403-411. doi:10.1016/j.jmaa.2011.11.065

[11] Chang YK, Nieto JJ. Some new existence results for fractional differential inclusions with boundary conditions. Mathematical and Computer Modelling 2009; 49: 605-609. doi:10.1016/j.mcm.2008.03.014

[12] Diethelm K, Ford NJ. Analysis of fractional differential equations. Journal of Mathematical Analysis and Applications 2002; 265: 229-248. doi:10.1006/jmaa.2000.7194

[13] Graef JR, Kong L, Kong Q, Wang M. Fractional boundary value problems with integral boundary conditions. Applicable Analysis 2013; 92(10): 2008-2020. doi:10.1080/00036811.2012.715151

[14] Graef JR, Kong L, Yang B. Positive solutions for a fractional boundary value problem. Applied Mathematics Letters 2016; 56: 49-55. doi:10.1016/j.aml.2015.12.006

[15] Granas A, Dugundji J. Fixed Point Theory. Springer Monographs in Mathematics. New York, NY, USA: Springer-Verlag, 2003.

[16] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. Amsterdam, Netherlands: Elsevier Science B.V., 2006.

[17] Kiryakova V. Generalized Fractional Calculus and Applications. Harlow, UK: Longman, 1994.

[18] Lakshmikantham V, Vatsala AS. Basic theory of fractional differential equations. Nonlinear Analysis 2008; 69(8): 2677-2682. doi:10.1016/j.na.2007.08.042

[19] Liu X, Jia M, Ge W. The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator. Applied Mathematics Letters 2017; 65: 56-62. doi:10.1016/j.aml.2016.10.001

[20] Miller KS, Ross B. An Introduction to the Fractional Calculus and Differential Equations. New York, NY, USA: Wiley, 1993.

[21] Oldham KB, Spanier J. The Fractional Calculus. New York, NY, USA: Academic Press, 1974.

[22] Podlubny I. Fractional Differential Equation. San Diego, USA: Academic Press, 1999.

[23] Samko SG, Kilbas AA, Marichev OI. Fractional Integrals and Derivatives: Theory and Applications. Yverdon, Switzerland: Gordon and Breach, 1993.

[24] Tuan NH, Ngoc TB, Huynh LN, Kirane M. Existence and uniqueness of mild solution of time-fractional semilinear differential equations with a nonlocal final condition. Computers & Mathematics with Applications 2019; 78 (5): 1651-1668. doi:10.1016/j.camwa.2018.11.007

[25] Yang X, Wei Z, Dong W. Existence of positive solutions for the boundary value problem of nonlinear fractional differential equations. Communications in Nonlinear Science and Numerical Simulation 2012; 17 (1): 85-92. doi:10.1016/j.cnsns.2011.05.007

[26] Zhang S. Positive solutions for boundary value problems of nonlinear fractional differential equations. Electronic Journal of Differential Equations 2006; 2006 (36): pp. 1-12.

[27] Zhang S. Monotone iterative method for initial value problem involving Riemann-Liouville fractional derivatives. Nonlinear Analysis 2009; 71: 2087-2093. doi:10.1016/j.na.2009.01.043

[28] Zhao Y, Sun S, Han Z, Li Q. The existence of multiple positive solutions for boundary value problems of nonlinear fractional differential equations. Communications in Nonlinear Science and Numerical Simulation 2011; 16 (4): 2086-2097. doi:10.1016/j.cnsns.2010.08.017