Turkish Journal of Mathematics

Volume 45 | Number 4 Article 27

1-1-2021

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 1847 – 1870
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doi:10.3906/mat-2104-40

# Half inverse problems for the impulsive quadratic pencil with the discontinouty coefficient

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**Received:** 09.04.2021 • **Accepted/Published Online:** 09.06.2021 • **Final Version:** 27.07.2021

**Abstract:** In this paper, we study the inverse spectral problem for the quadratic differential pencils with discontinuity coefficient on $[0,\pi]$ with separable boundary conditions and the impulsive conditions at the point $x=\frac{\pi}{2}$. We prove that two potential functions on the interval $[0,\pi]$, and the parameters in the boundary and impulsive conditions can be determined from a sequence of eigenvalues for two cases: (i) The potentials are given on $\left(0,\frac{\pi}{4}(1+\alpha)\right)$, (ii) The potentials are given on $\left(\frac{\pi}{4}(1+\alpha),\pi\right)$, where $0<\alpha<1$, respectively.

**Key words:** Inverse spectral problems, Sturm–Liouville operator, spectrum, uniqueness

## 1. Introduction

We consider the quadratic pencils of Sturm–Liouville operator $L(p,q,h,H,\alpha,\beta,\gamma)$ of the form

$$\ell y := -y'' + [2\lambda p(x) + q(x)]\, y = \lambda^2 \rho(x)\, y, \qquad x \in \left[0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right] \tag{1.1}$$

with the boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \tag{1.2}$$

$$V(y) := y'(\pi) + Hy(\pi) = 0 \tag{1.3}$$

and with the impulsive conditions

$$\begin{aligned} y\left(\frac{\pi}{2}+0\right) &= \beta y\left(\frac{\pi}{2}-0\right), \\ y'\left(\frac{\pi}{2}+0\right) &= \beta^{-1} y'\left(\frac{\pi}{2}-0\right) + \gamma y\left(\frac{\pi}{2}-0\right), \end{aligned} \tag{1.4}$$

where $\lambda$ is the spectral parameter, $p(x) \in W_2^1[0,\pi]$, $q(x) \in L_2[0,\pi]$ are real-valued functions, $h, H, \in \mathbb{R}, \alpha, \beta, \gamma$ are real numbers, $\beta > 0, |\beta-1|^2 + \gamma^2 \neq 0$ and

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2010 *AMS Mathematics Subject Classification:* 34A55, 34B24, 34L05

$$\rho(x) = \begin{cases} 1, & 0 < x < \dfrac{\pi}{2} \\ \alpha^2, & \dfrac{\pi}{2} < x < \pi, \end{cases} \quad , \quad 0 < \alpha < 1.$$

Here we denote by $W_2^m[0,\pi]$ the space of functions $f(x), x \in [0,\pi]$, such that the derivatives $f^{(m)}(x)$ $(m = \overline{0, n-1})$ are absolute continuous and $f^{(n)}(x) \in L_2[0,\pi]$.

We can get $p(0) = 0$ without general exposure; otherwise, if $c_0 = p(0) \neq 0$ by direct calculation, we note that equation (1.1) is equivalent to

$$L_\lambda y := -y'' + \left[q(x) + 2p(x)c_0 - c_0^2 + 2(\lambda - c_0)(p(x) - c_0)\right] y = (\lambda - c_0)^2 \rho(x)\, y. \tag{1.5}$$

Let

$$\widehat{q}(x) = q(x) + 2p(x)c_0 - c_0^2\rho(x), \widehat{p}(x) = p(x) - c_0\rho(x), \widehat{\lambda} = \lambda - c_0$$

then for the problem with the form (1.1)–(1.4), we have $\widehat{p}(0) = 0$.

Quadratic pencils of Sturm–Liouville operator with discontinuous coefficient appear frequently in various models of classical mechanics and quantum (see [12–14, 20, 31] and the references therein). For instance, the evolution equations which are used to model interactions between colliding relativistic spineless particles can be reduced to the form (1.1), here the parameter $\lambda^2$ can be regarded as the energy of this system (see [12, 20, 31]).

Boundary value problems with discontinuous coefficients and discontinuous conditions at the inner point of the finite part have an important place, especially in applied mathematics, quantum physics, and engineering (applied sciences) sciences. An important application area of such problems is earth sciences. Determining the refraction coefficients and the refraction coefficients of the waves passing through the medium consisting of different layers is an important problem of geophysics, and learning the diffraction coefficients of the wave propagating under different effects in a wire made of different materials is also an important problem of classical mechanics. It is necessary to examine most of the natural events that occur in this way and to create their mathematical models to be able to intervene in these events. Information about the event can be obtained by examining the mathematical models created. For example, when the wavelengths and refraction coefficients of the wave passing through the crystal are known, information about the formation of the crystal and its shape (form) can be obtained. These types of problems are called inverse problems for singular differential operators in mathematics. The problems in [23] and [19] appear in all areas of applied sciences. In general, these types of problems are reduced to the inverse problems posed for the operators produced by the one-dimensional or multidimensional Shrödinger equation. Similar problems are encountered in earth sciences [17].

There are many studies on the determination of singular operators (coefficients of differential expression and other parameters) according to spectral characteristics (data) of the given problem recently [5, 22].These types of problems create different classes according to the location [1, 9, 28]. The inverse spectral problem for the boundary value problem given in (1.1)–(1.4) were previously studied in [24] in general.

One of the most important problems recently introduced and studied regarding the determination of singular Shrödinger and diffusion operators are the semiinverse problems ([25, 26, 36, 37]). This problem, which was first examined by Hochstandt and Liberman (([10],[37])) for the Sturm–Liouville operator, has also been studied for different operators ([11, 32, 33]) and continues to be studied.

In this paper, unlike other similar studies, we consider the problem $L(p, q, h, H, \alpha, \beta, \gamma)$ and prove that if the potentials $q(x)$ and $p(x)$ on $(0, c)$ and $h$ are given, then only a single spectrum is sufficient to determine

$q(x)$ and $p(x)$ on $(0,\pi)$, $\rho(x), H, \beta$, and $\alpha$. We also consider the case that the $q(x)$ and $p(x)$ are given on the $(c,\pi)$, and prove a uniqueness theorem. Moreover, it is shown that potentials $q(x)$ and $p(x)$ on $(0,\pi)$, $\rho(x)$, $\beta$, $\alpha$, $h$, and $H$ can be uniquely determined by one spectrum and some information on eigenfunctions at the internal point $c=\dfrac{1+\alpha}{4}\pi$. In Section 4, new results about the interior inverse problem are given. In Section 5, a reconstruction method is given for half inverse problems. Unlike the methods in similar articles, this method is given by using the main equation. Finally, some examples are given in Section 6.

## 2. Preliminaries

Let $\varphi(x,\lambda)$ function be the solution of the equation (1.1) that satisfies the initial conditions

$$\varphi(0,\lambda)=1,\ \varphi'(0,\lambda)=h \tag{2.1}$$

and the impulse condition (1.4). It is shown in [2] if $q(x)\in L_2[0,\pi]$ and $p(x)\in W_2^1[0,\pi]$ that there exist functions $A(x,t)$, $B(x,t)$ and $A_x, A_t, B_x, B_t \in L_1[0,\pi]$ for each $x\in[0,\pi]$ such that

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+\int\limits_0^{\mu^+(x)} A(x,t)\cos\lambda t dt+\int\limits_0^{\mu^+(x)} B(x,t)\sin\lambda t dt, \tag{2.2}$$

$$\beta^+\omega^+(x)=\beta^+ x p(0)+2\int\limits_0^{x}\left[A(\xi,\xi)\sin\omega^+(\xi)-B(\xi,\xi)\cos\omega^+(\xi)\right]d\xi, \tag{2.3}$$

$$2\frac{d}{dx}\left[A(x,x)\cos\omega^+(x)+B(x,x)\sin\omega^+(x)\right]=\beta^+\left[q(x)+p^2(x)\right], \tag{2.4}$$

$$2\frac{d}{dx}\left[A(x,t)\cos\omega^-(x)-B(x,t)\sin\omega^-(x)\right]\Big|_{t=\mu^+(x)-0}^{t=\mu^+(x)+0}=\beta^-\left[q(x)+p^2(x)\right], \tag{2.5}$$

$$A_t(x,t)\left.\right|_{t=0}=B(x,0)=0 \tag{2.6}$$

are held.

Next, suppose that $q(x)\in W_2^1[0,\pi], p(x)\in W_2^2[0,\pi]$, then the functions $A(x,t)$ and $B(x,t)$ satisfy the system of partial differential equations

$$\begin{aligned} A_{xx}(x,t)-q(x)A(x,t)-2p(x)B_t(x,t)&=A_{tt}(x,t),\\ B_{xx}(x,t)-q(x)B(x,t)+2p(x)A_t(x,t)&=B_{tt}(x,t). \end{aligned} \tag{2.7}$$

Conversely, if the second order derivatives of functions $A(x,t), B(x,t)$ are summable on $[0,\pi]$ for each $x\in[0,\pi]$ and $A(x,t), B(x,t)$ satisfy equalities (2.7) and conditions (2.3)–(2.6), then the function $\varphi(x,\lambda)$ which is defined by (2.2) is a solution of (1.1) satisfying initial conditions (2.1) and impulsive conditions (1.4)
where

$$\varphi_0(x,\lambda) = \beta^+ \cos\left[\lambda\mu^+(x) - \frac{\omega^+(x)}{\sqrt{\rho(x)}}\right] + \beta^- \cos\left[\lambda\mu^-(x) + \frac{\omega^-(x)}{\sqrt{\rho(x)}}\right] \tag{2.8}$$
$$+\frac{h}{\lambda}\left\{\beta^+ \sin\left[\lambda\mu^+(x) - \frac{\omega^+(x)}{\sqrt{\rho(x)}}\right] + \beta^- \sin\left[\lambda\mu^-(x) + \frac{\omega^-(x)}{\sqrt{\rho(x)}}\right]\right\}$$

and

$$\beta^\pm = \frac{1}{2}\left(\beta \pm \frac{1}{\alpha\beta}\right), \mu^\pm(x) = \pm\sqrt{\rho(x)}x + \frac{\pi}{2}(1 \mp \sqrt{\rho(x)}), \tag{2.9}$$
$$\omega^+(x) = \int_0^x p(t)dt, \omega^-(x) = \int_{\frac{\pi}{2}}^x p(t)dt$$

It is clear that the characteristic function of the problem $L(p,q,h,H,\alpha,\beta,\gamma)$ can be

$$\Delta(\lambda) = \varphi'(\pi,\lambda) + H\varphi(\pi,\lambda) \tag{2.10}$$

and

$$\Delta_0(\lambda) = \varphi_0'(\pi,\lambda) + H\varphi_0(\pi,\lambda)$$
$$= -\beta^+\left(\lambda\alpha - \frac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^+(\pi) - \frac{\omega^+(\pi)}{\alpha}\right)$$
$$+\beta^-\left(\lambda\alpha - \frac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^-(\pi) + \frac{\omega^-(\pi)}{\alpha}\right)$$
$$+\frac{h}{\lambda}\left\{\beta^+\left(\lambda\alpha - \frac{p(\pi)}{\alpha}\right)\cos\left(\lambda\mu^+(\pi) - \frac{\omega^+(\pi)}{\alpha}\right)\right.$$
$$\left.+\beta^-\left(\lambda\alpha - \frac{p(\pi)}{\alpha}\right)\cos\left(\lambda\mu^-(\pi) + \frac{\omega^-(\pi)}{\alpha}\right)\right\}$$
$$+H\left\{\beta^+\cos\left(\lambda\mu^+(\pi) - \frac{\omega^+(\pi)}{\alpha}\right) + \beta^-\cos\left(\lambda\mu^-(\pi) + \frac{\omega^-(\pi)}{\alpha}\right) + \right.$$
$$\left.+\frac{h}{\lambda}\left[\beta^+\sin\left(\lambda\mu^+(x) - \frac{\omega^+(x)}{\sqrt{\rho(x)}}\right) + \beta^-\sin\left(\lambda\mu^-(x) + \frac{\omega^-(x)}{\sqrt{\rho(x)}}\right)\right]\right\}$$

or $\Delta(\lambda) =< \psi(x,\lambda), \varphi(x,\lambda) >= V(\varphi) = -U(\psi)$ where $< y,z >:= y'z - yz'$ and $\psi(x,\lambda)$ be a solution of (1.1) with the initial conditions $\psi(\pi,\lambda) = 1$, $\psi'(\pi,\lambda) = -H$ and the impulsive conditions (1.4).

Next, suppose that the function $q(x)$ satisfies the additional condition

$$h|y(0)|^2 + H|y(\pi)|^2 + \int_0^\pi \left\{|y'(x)|^2 + q(x)|y(x)|^2\right\}dx > 0 \tag{2.11}$$

for all $y(x) \in W_2^2\left(\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]\right)$ such that $y(x) \neq 0$ and

$$y'(0)\overline{y(0)} - y'(\pi)\overline{y(\pi)} = 0. \tag{2.12}$$

Then it is shown in [2] that the eigenvalues of the boundary value problem $L(p, q, h, H, \alpha, \beta, \gamma)$ are real nonzero, simple and does not have associated functions. Additionally, eigenfunctions corresponding to different eigenvalues of the problem $L(p, q, h, H, \alpha, \beta, \gamma)$ are orthogonal in the sense of the equality

$$(\lambda_1 + \lambda_2)(\rho(x) y_1, y_2) - 2(py_1, y_2) = 0, \tag{2.13}$$

where $(.,.)$ denotes the inner product in $L_2[0, \pi]$.

The function $\Delta(\lambda)$ is entire in $\lambda$. Zeros $\{\lambda_n\}_{n\in\mathbb{Z}}$ of $\Delta(\lambda)$ coincide with the eigenvalues of the problem $L(p, q, h, H, \alpha, \beta, \gamma)$. We note that for $\lambda \in G_\delta = \{\lambda : |\lambda - \lambda_n| > \delta, \delta > 0\}$,where $\delta$ is sufficiently small positive number $\left(\delta \ll \frac{\beta}{2}\right)$, then [4] there exists a constant $C_\delta > 0$ such that

$$|\Delta(\lambda)| \geq \beta^+ (|\lambda| \alpha - C_\delta) \exp\left(\tau \mu^+(\pi)\right), \text{ for all } \lambda \in \overline{G}_\delta, (\tau = \operatorname{Im} \lambda). \tag{2.14}$$

**Lemma 2.1** *The following asymptotic relations hold as* $|\lambda| \to \infty$. *For* $\frac{\pi}{2} < x < \pi$

$$\varphi(x, \lambda) = \beta^+ \cos\left[\lambda\mu^+(x) - \frac{\omega^+(x)}{\sqrt{\rho(x)}}\right] + \beta^- \cos\left[\lambda\mu^-(x) + \frac{\omega^-(x)}{\sqrt{\rho(x)}}\right] + O\left(|\lambda|^{-1} e^{|\tau|\mu^+(x)}\right) \tag{2.15}$$

$$\varphi'(x, \lambda) = -\beta^+ \left[\lambda\sqrt{\rho(x)} - \frac{p(x)}{\sqrt{\rho(x)}}\right] \sin\left[\lambda\mu^+(x) - \frac{\omega^+(x)}{\sqrt{\rho(x)}}\right] - \\ - \beta^- \left[\lambda\sqrt{\rho(x)} - \frac{p(x)}{\sqrt{\rho(x)}}\right] \sin\left[\lambda\mu^-(x) + \frac{\omega^-(x)}{\sqrt{\rho(x)}}\right] \\ + O\left(e^{|\tau|\mu^+(x)}\right) \tag{2.16}$$

For $0 < x < \frac{\pi}{2}$,

$$\psi(x, \lambda) = A^+ \cos\left[\lambda(\pi - \mu^+(x)) - \frac{1}{\alpha}\omega^+(x) + \alpha^+\omega^-(x)\right] + \\ + A^- \cos\left[\lambda(\pi - \mu^-(x)) + \frac{1}{\alpha}\omega^+(x) - \alpha^+\omega^-(x)\right] + O\left(|\lambda|^{-1} e^{|\tau|(\pi - \mu^+(x))}\right), \tag{2.17}$$

$$\psi'(x, \lambda) = A^+ (\lambda - \alpha p(x)) \sin\left[\lambda(\pi - \mu^+(x)) - \frac{1}{\alpha}\omega^+(x) + \alpha^+\omega^-(x)\right] - \\ - A^- \left(\lambda - (\alpha - \frac{2}{\alpha})p(x)\right) \sin\left[\lambda(\pi - \mu^-(x)) + \frac{1}{\alpha}\omega^+(x) - \alpha^+\omega^-(x)\right] \\ + O\left(e^{|\tau|(\pi - \mu^+(x))}\right), \tag{2.18}$$

where $A^{\pm}=\frac{1}{2}\left(\frac{1}{\beta}\pm\alpha\beta\right)$.

**Proof** The accuracy of the equations (2.15) and (2.16) were obtained in the study of [24], Similarly, the accuracy of the (2.17) and (2.18) behavior is easily obtained.

It is easy to verify that if $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of equation (1.1) and satisfy the impulsive conditions (1.4), then $<\varphi,\psi>$ is independent $x$, and

$$W(\varphi,\psi)\Big|_{x=\frac{\pi}{2}-0}=W(\varphi,\psi)\Big|_{x=\frac{\pi}{2}+0}.$$

It is clear from (2.10) that

$$\Delta(\lambda)=W(\varphi,\psi)=V(\varphi)=-U(\psi). \tag{2.19}$$

□

**Lemma 2.2** *The zeros $\{\lambda_n\}$ of the characteristic function $\Delta(\lambda)$ coincide with the eigenvalues of the boundary value problem $L(p,q,h,H,\alpha,\beta,\gamma)$ and there exists a sequence $\{\beta_n\}$ such that $\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n)$, $\beta_n\neq 0, n\in\mathbb{Z}$.*

The proof of Lemma is made similar to the proof of Lemma 8 in [2].

Next, we denote by $L_2((0,\pi);\rho(x))$ a space which has the inner product

$$(\varphi,\psi)=\int_0^{\pi}\rho(x)\varphi(x)\psi(x)\,dx.$$

Let $\alpha_n\ (n\in\mathbb{Z})$ be the sequence of normalized constants, which are defined as

$$\alpha_n:=\int_0^{\pi}\rho(x)\varphi^2(x,\lambda_n)\,dx-\frac{1}{\lambda_n}\int_0^{\pi}p(x)\varphi^2(x,\lambda_n)\,dx.$$

**Lemma 2.3** *The equality $\dot{\Delta}(\lambda_n)=-2\lambda_n\alpha_n\beta_n$ holds. Here $\dot{\Delta}(\lambda)=(d/d\lambda)\Delta(\lambda)$.*

**Lemma 2.4** *The problem $L(p,q,h,H,\alpha,\beta,\gamma)$ has countable set of eigenvalues. If one denotes by $\lambda_1,\lambda_2,\ldots$ the positive eigenvalues arranged in increasing order and by $\lambda_{-1},\lambda_{-2},\ldots$ the negative eigenvalues arranged in decreasing order, then eigenvalues and the normalized constans $\alpha_n$ of the problem $L(p,q,h,H,\alpha,\beta,\gamma)$ have the asymptotic behavior respectively*

$$\lambda_n=\lambda_n^o+\frac{d_n}{\lambda_n^o}+\frac{\delta_n}{\lambda_n^o},\quad n\to\pm\infty,$$

$$\alpha_n=\frac{\pi}{2}\left[\left(\beta^+\right)^2+\left(\beta-\right)^2\right]+\frac{\beta_{11}}{\lambda_n^0}+\frac{\beta_{1n}}{n},$$

*where* $\{\delta_n\},\{\beta_{1n}\}\in l_2$, *and* $\{d_n\}$ *is a bounded sequence,* $\lambda_n^o=\dfrac{n}{\mu^+(\pi)}+\dfrac{\omega^+(\pi)}{\mu^+(\pi)}+h_n$, $\sup_n|h_n|<\infty$, $\beta_{11}=-\dfrac{\beta\pi}{2}p(0)$.

The following theorem is shown in [3]:

**Theorem 2.5** *The functions* $A(x,t)$ *and* $B(x,t)$ *satisfy the following system of linear integral equations:*

$$\begin{aligned}&\beta^+(F_{11}(x,t)\cos\omega^+(x)+F_{12}(x,t)\sin\omega^+(x))+\beta^-(F_{11}(2a-x,t)\cos\omega^-(x)+F_{12}(2a-x,t)\sin\omega^-(x))\\&+A(x,t)+\int_0^x A(x,\xi)F_{11}(\xi,t)d\xi+\int_0^x B(x,\xi)F_{12}(\xi,t)d\xi=0,\ 0\le t<x,\end{aligned}\tag{2.20}$$

$$\begin{aligned}&\beta^+(F_{21}(x,t)\cos\omega^+(x)+F_{22}(x,t)\sin\omega^+(x))+\beta^-(F_{21}(2a-x,t)\cos\omega^-(x)+F_{22}(2a-x,t)\sin\omega^-(x))\\&+B(x,t)+\int_0^x A(x,\xi)F_{21}(\xi,t)d\xi+\int_0^x B(x,\xi)F_{22}(\xi,t)d\xi=0,\ 0\le t<x,\end{aligned}\tag{2.21}$$

*where*

$$F_{11}(x,t)=\frac{1}{\pi}\cos c_0x\cos c_0t+\sum_{n\in\mathbb{Z}}\left[\frac{1}{2\alpha_n}\cos\lambda_nx\cos\lambda_nt-\frac{1}{\pi}\cos(n+c_0)x\cos(n+c_0)t\right],\tag{2.22}$$

$$F_{12}(x,t)=\frac{1}{\pi}\sin c_0x\cos c_0t+\sum_{n\in\mathbb{Z}}\left[\frac{1}{2\alpha_n}\sin\lambda_nx\cos\lambda_nt-\frac{1}{\pi}\sin(n+c_0)x\cos(n+c_0)t\right],\tag{2.23}$$

$$F_{21}(x,t)=\frac{1}{\pi}\cos c_0x\sin c_0t+\sum_{n\in\mathbb{Z}}\left[\frac{1}{2\alpha_n}\cos\lambda_nx\sin\lambda_nt-\frac{1}{\pi}\cos(n+c_0)x\sin(n+c_0)t\right],\tag{2.24}$$

$$F_{22}(x,t)=\frac{1}{\pi}\sin c_0x\sin c_0t+\sum_{n\in\mathbb{Z}}\left[\frac{1}{2\alpha_n}\sin\lambda_nx\sin\lambda_nt-\frac{1}{\pi}\sin(n+c_0)x\sin(n+c_0)t\right],\tag{2.25}$$

$c_0=\omega^+(\pi)$.

*The equations (2.20), (2.21) are called the fundamental equations of the inverse problem.*

## 3. Main results

Now we state the main result of this work. It is assumed in what follows that if a certain symbol $s$ denotes an object related to $L(p,q,h,H,\alpha,\beta,\gamma)$, then the corresponding symbol $\widetilde{s}$ with tilde denotes the analogous object related to $\tilde{L}(\tilde{p},\tilde{q},\widetilde{h},\widetilde{H},\tilde{\alpha},\widetilde{\beta},\widetilde{\gamma})$. Let us denote by $\varphi(x,\lambda_n)$, the eigenfunction which corresponds to $\lambda_n$.

**Lemma 3.1** *If* $\lambda_n=\tilde{\lambda}_n$, $n\in\mathbb{Z}$, *then* $\omega^+(\pi)=\widetilde{\omega}^+(\pi)$, $\omega^-(\pi)=\widetilde{\omega}^-(\pi)$ *and* $\alpha=\widetilde{\alpha}$, *that is the sequence* $\{\lambda_n\}$ *uniquely determines* $\omega^\pm(\pi)$ *and* $\alpha$.

Proof of Lemma is easily obtained from the asymptotic expression of $\lambda_n$.

**Lemma 3.2** *If* $\lambda_n = \tilde{\lambda}_n$, $n = 0, \pm 1, \pm 2, ...$ *then* $\beta = \widetilde{\beta}$ *and* $H = \widetilde{H}$.

**Proof** Since $\lambda_n = \tilde{\lambda}_n$ and $\Delta(\lambda), \widetilde{\Delta}(\lambda)$ are entire functions in $\lambda$ of order one by Hadamard factorization theorem, for $\lambda \in \mathbb{C}$

$$\Delta(\lambda) \equiv C\widetilde{\Delta}(\lambda). \tag{3.1}$$

On the other hand, (3.1) can be written as

$$\Delta_0(\lambda) - C\widetilde{\Delta}_0(\lambda) = C\left[\widetilde{\Delta}(\lambda) - \widetilde{\Delta}_0(\lambda)\right] - \left[\Delta(\lambda) - \Delta_0(\lambda)\right]. \tag{3.2}$$

Hence,

$$\begin{aligned}
&C\left[\widetilde{\Delta}(\lambda) - \widetilde{\Delta}_0(\lambda)\right] - \left[\Delta(\lambda) - \Delta_0(\lambda)\right] \qquad (3.3)\\
&= -\beta^+\left(\lambda\alpha - \frac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^+(\pi) - \frac{\omega^+(\pi)}{\alpha}\right)\\
&+ \beta^-\left(\lambda\alpha - \frac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^-(\pi) + \frac{\omega^-(\pi)}{\alpha}\right)\\
&+ \frac{h}{\lambda}\left\{\beta^+\left(\lambda\alpha - \frac{p(\pi)}{\alpha}\right)\cos\left(\lambda\mu^+(\pi) - \frac{\omega^+(\pi)}{\alpha}\right)\right.\\
&\left.- \beta^-\left(\lambda\alpha - \frac{p(\pi)}{\alpha}\right)\cos\left(\lambda\mu^-(\pi) + \frac{\omega^-(\pi)}{\alpha}\right)\right\}\\
&+ H\left\{\beta^+\cos\left(\lambda\mu^+(\pi) - \frac{\omega^+(\pi)}{\alpha}\right) + \beta^-\cos\left(\lambda\mu^-(\pi) + \frac{\omega^-(\pi)}{\alpha}\right)\right\}\\
&+ H\frac{h}{\lambda}\left\{\beta^+\sin\left(\lambda\mu^+(\pi) - \frac{\omega^+(\pi)}{\alpha}\right) + \beta^-\sin\left(\lambda\mu^-(\pi) + \frac{\omega^-(\pi)}{\alpha}\right)\right\}\\
&- C\left[-\widetilde{\beta}^+\left(\lambda\widetilde{\alpha} - \frac{\widetilde{p}(\pi)}{\widetilde{\alpha}}\right)\sin\left(\lambda\widetilde{\mu}^+(\pi) - \frac{\widetilde{\omega}^+(\pi)}{\widetilde{\alpha}}\right)\right.\\
&\left.+\widetilde{\beta}^-\left(\lambda\widetilde{\alpha} - \frac{\widetilde{p}(\pi)}{\widetilde{\alpha}}\right)\sin\left(\lambda\widetilde{\mu}^-(\pi) + \frac{\widetilde{\omega}^-(\pi)}{\widetilde{\alpha}}\right)\right]\\
&- C\frac{\widetilde{h}}{\lambda}\left\{\widetilde{\beta}^+\left(\lambda\widetilde{\alpha} - \frac{\widetilde{p}(\pi)}{\widetilde{\alpha}}\right)\cos\left(\lambda\widetilde{\mu}^+(\pi) - \frac{\widetilde{\omega}^+(\pi)}{\widetilde{\alpha}}\right)\right.\\
&\left.-\widetilde{\beta}^-\left(\lambda\widetilde{\alpha} - \frac{\widetilde{p}(\pi)}{\widetilde{\alpha}}\right)\cos\left(\lambda\widetilde{\mu}^-(\pi) + \frac{\widetilde{\omega}^-(\pi)}{\widetilde{\alpha}}\right)\right\}\\
&- C\left[\widetilde{H}\widetilde{\beta}^+\cos\left(\lambda\widetilde{\mu}^+(\pi) - \frac{\widetilde{\omega}^+(\pi)}{\widetilde{\alpha}}\right) + \widetilde{H}\widetilde{\beta}^-\cos\left(\lambda\widetilde{\mu}^-(\pi) + \frac{\widetilde{\omega}^-(\pi)}{\widetilde{\alpha}}\right)\right]\\
&- C\widetilde{H}\frac{\widetilde{h}}{\lambda}\left\{\widetilde{\beta}^+\sin\left(\lambda\widetilde{\mu}^+(\pi) - \frac{\widetilde{\omega}^+(\pi)}{\widetilde{\alpha}}\right) + \widetilde{\beta}^-\sin\left(\lambda\widetilde{\mu}^-(\pi) + \frac{\widetilde{\omega}^-(\pi)}{\widetilde{\alpha}}\right)\right\}
\end{aligned}$$

if we multiply both sides of (3.3) with $\sin\left(\lambda\mu^{+}(\pi)-\dfrac{\omega^{+}(\pi)}{\alpha}\right)$ and integrate with respect to $\lambda$ in $(\varepsilon,T)$ ($\varepsilon$ is sufficiently small positive number) for any positive real number $T$, then we get

$$
\begin{aligned}
&\int_{\varepsilon}^{T}\left(C\left[\widetilde{\Delta}(\lambda)-\widetilde{\Delta}_0(\lambda)\right]-\left[\Delta(\lambda)-\Delta_0(\lambda)\right]\right)\sin\left(\lambda\mu^{+}(\pi)-\frac{\omega^{+}(\pi)}{\alpha}\right)d\lambda=\\
&=\int_{\varepsilon}^{T}\left\{-\beta^{+}\left(\lambda\alpha-\frac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^{+}(\pi)-\frac{\omega^{+}(\pi)}{\alpha}\right)\right.\\
&+\beta^{-}\left(\lambda\alpha-\frac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^{-}(\pi)+\frac{\omega^{-}(\pi)}{\alpha}\right)\\
&+\frac{h}{\lambda}\left[\beta^{+}\left(\lambda\alpha-\frac{p(\pi)}{\alpha}\right)\cos\left(\lambda\mu^{+}(\pi)-\frac{\omega^{+}(\pi)}{\alpha}\right)\right.\\
&\left.-\beta^{-}\left(\lambda\alpha-\frac{p(\pi)}{\alpha}\right)\cos\left(\lambda\mu^{-}(\pi)+\frac{\omega^{-}(\pi)}{\alpha}\right)\right]\\
&+H\beta^{+}\cos\left(\lambda\mu^{+}(\pi)-\frac{\omega^{+}(\pi)}{\alpha}\right)+H\beta^{-}\cos\left(\lambda\mu^{-}(\pi)+\frac{\omega^{-}(\pi)}{\alpha}\right)\\
&+H\frac{h}{\lambda}\left[\beta^{+}\sin\left(\lambda\mu^{+}(\pi)-\frac{\omega^{+}(\pi)}{\alpha}\right)+\beta^{-}\sin\left(\lambda\mu^{-}(\pi)+\frac{\omega^{-}(\pi)}{\alpha}\right)\right]\\
&-C\left[-\widetilde{\beta}^{+}\left(\lambda\widetilde{\alpha}-\frac{\widetilde{p}(\pi)}{\widetilde{\alpha}}\right)\sin\left(\lambda\widetilde{\mu}^{+}(\pi)-\frac{\widetilde{\omega}^{+}(\pi)}{\widetilde{\alpha}}\right)\right.\\
&\left.+\widetilde{\beta}^{-}\left(\lambda\widetilde{\alpha}-\frac{\widetilde{p}(\pi)}{\widetilde{\alpha}}\right)\sin\left(\lambda\widetilde{\mu}^{-}(\pi)+\frac{\widetilde{\omega}^{-}(\pi)}{\widetilde{\alpha}}\right)\right]\\
&-C\frac{\widetilde{h}}{\lambda}\left\{\widetilde{\beta}^{+}\left(\lambda\widetilde{\alpha}-\frac{\widetilde{p}(\pi)}{\widetilde{\alpha}}\right)\cos\left(\lambda\widetilde{\mu}^{+}(\pi)-\frac{\widetilde{\omega}^{+}(\pi)}{\widetilde{\alpha}}\right)\right.\\
&\left.-\widetilde{\beta}^{-}\left(\lambda\widetilde{\alpha}-\frac{\widetilde{p}(\pi)}{\widetilde{\alpha}}\right)\cos\left(\lambda\widetilde{\mu}^{-}(\pi)+\frac{\widetilde{\omega}^{-}(\pi)}{\widetilde{\alpha}}\right)\right\}\\
&-C\left[\widetilde{H}\widetilde{\beta}^{+}\cos\left(\lambda\widetilde{\mu}^{+}(\pi)-\frac{\widetilde{\omega}^{+}(\pi)}{\widetilde{\alpha}}\right)+\widetilde{H}\widetilde{\beta}^{-}\cos\left(\lambda\widetilde{\mu}^{-}(\pi)+\frac{\widetilde{\omega}^{-}(\pi)}{\widetilde{\alpha}}\right)\right]\\
&\left.-C\widetilde{H}\frac{\widetilde{h}}{\lambda}\left[\widetilde{\beta}^{+}\sin\left(\lambda\widetilde{\mu}^{+}(\pi)-\frac{\widetilde{\omega}^{+}(\pi)}{\widetilde{\alpha}}\right)+\widetilde{\beta}^{-}\sin\left(\lambda\widetilde{\mu}^{-}(\pi)+\frac{\widetilde{\omega}^{-}(\pi)}{\widetilde{\alpha}}\right)\right]\right\}\\
&.\sin\left(\lambda\mu^{+}(\pi)-\frac{\omega^{+}(\pi)}{\alpha}\right)d\lambda
\end{aligned}
$$

and so

$$\int_{\varepsilon}^{T}\left(C\left[\widetilde{\Delta}(\lambda)-\widetilde{\Delta}_0(\lambda)\right]-\left[\Delta(\lambda)-\Delta_0(\lambda)\right]\right)\sin\left(\lambda\mu^+(\pi)-\frac{\omega^+(\pi)}{\alpha}\right)d\lambda=$$

$$\int_{\varepsilon}^{T}\left[-\beta^+(\lambda\alpha-p(\pi))\sin^2\left(\lambda\mu^+(\pi)-\frac{\omega^+(\pi)}{\alpha}\right)\right]d\lambda-$$

$$-C\int_{\varepsilon}^{T}\left[\widetilde{\beta}^+(\lambda\alpha-\widetilde{p}(\pi))\sin\left(\lambda\mu^+(\pi)-\frac{\widetilde{\omega}^+(\pi)}{\alpha}\right)\sin\left(\lambda\mu^+(\pi)-\frac{\omega^+(\pi)}{\alpha}\right)\right]d\lambda$$

$$=\int_{\varepsilon}^{T}\left[-\frac{1}{2}\beta^+(\lambda\alpha-p(\pi))+\frac{1}{2}\beta^+(\lambda\alpha-p(\pi))\cos\left(2\lambda\mu^+(\pi)-\frac{2\omega^+(\pi)}{\alpha}\right)\right]d\lambda-$$

$$-C\int_{\varepsilon}^{T}\left\{\frac{1}{2}\widetilde{\beta}^+(\lambda\alpha-\widetilde{p}(\pi))-\cos\left(\frac{\omega^+(\pi)-\widetilde{\omega}^+(\pi)}{\alpha}\right)\right.$$

$$\left.+\cos\left(2\lambda\mu^+(\pi)-\frac{\omega^+(\pi)-\widetilde{\omega}^+(\pi)}{\alpha}\right)\right\}d\lambda.$$

Since

$$\Delta(\lambda)-\Delta_0(\lambda)=O\left(\frac{1}{\lambda}e^{|\operatorname{Im}\lambda|\mu^+(\pi)}\right),\widetilde{\Delta}(\lambda)-\widetilde{\Delta}_0(\lambda)=O\left(\frac{1}{\lambda}e^{|\operatorname{Im}\lambda|\mu^+(\pi)}\right)$$

for all $\lambda$ in $(\varepsilon,T)$

$$\frac{C\alpha}{4}\widetilde{\beta}^+-\frac{\alpha}{4}\beta^+=O(\frac{1}{T}).$$

By letting $T$ tend to infinity, we see that

$$C=\frac{\beta^+}{\widetilde{\beta}^+}. \tag{3.4}$$

Similarly, if we multiply both sides of (3.3) with $\sin\left(\lambda\mu^-(\pi)-\dfrac{\omega^-(\pi)}{\alpha}\right)$ and integrate again with respect to $\lambda$ in $(\varepsilon,T)$, and by letting $T$ tend to infinity, then we get

$$C=\frac{\beta^-}{\widetilde{\beta}^-}. \tag{3.5}$$

However, since $\alpha,\beta$ and $\widetilde{\alpha},\widetilde{\beta}$ are positive, since $\omega^+(\pi)-\widetilde{\omega}^+(\pi)=\omega^-(\pi)-\widetilde{\omega}^-(\pi)$, we conclude that $C=1$. Here $\dfrac{\beta^+}{\widetilde{\beta}^+}=\dfrac{\beta^-}{\widetilde{\beta}^-}$ is obtained. We have therefore proved, since $\alpha=\widetilde{\alpha}$, that $\beta=\widetilde{\beta}$.

Considering Lemma 3.1 and Lemma 3.2 and that $\beta=\widetilde{\beta}$, if both sides of the last expression are multiplied by the $\cos(\lambda\mu^{+}(\pi)-\dfrac{\omega^{+}(\pi)}{\alpha})$ and integrate with respect to $\lambda$ in $(\varepsilon, T)$, then we get

$$\frac{\beta^{+}\left(H-\widetilde{H}\right)}{2}+O(\frac{1}{T})=O(\frac{1}{T}).$$

Finally, by letting $T$ tend to infinity, $H=\widetilde{\widetilde{H}}$ is obtained. Similarly, it is shown when $h=\widetilde{h}$. □

Let $\Phi(x,\lambda)$ and $S(x,\lambda)$ be the solutions of (1.1) under the conditions $U(\Phi)=1,\ V(\Phi)=0, S(0,\lambda)=0, S'(0,\lambda)=1$ and under the impulse conditions (1.4). One set $M(\lambda):=\Phi(0,\lambda)$. The functions $\Phi(x,\lambda)$ and $M(\lambda)$ are called the Weyl solution and Weyl function for the boundary value problem $L(p,q,h,H,\alpha,\beta,\gamma)$, respectively. Using the solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$, we defined

$$\Phi(x,\lambda)=-\frac{\psi(x,\lambda)}{\Delta(\lambda)}=S(x,\lambda)+M(\lambda)\varphi(x,\lambda), \tag{3.6}$$

$$M(\lambda)=-\frac{\psi(0,\lambda)}{\Delta(\lambda)},\quad (-\frac{\varphi(\pi,\lambda)}{\varphi'(\pi,\lambda)}=-\frac{\varphi(\pi,\lambda)}{\Delta(\lambda)}),$$

and $S(x,\lambda)$ is defined from the equality

$$\psi(x,\lambda)=\psi(0,\lambda)\varphi(x,\lambda)-\Delta(\lambda)S(x,\lambda). \tag{3.7}$$

By virtue of equalities $\langle\varphi(x,\lambda),S(x,\lambda)\rangle\equiv 1$ and (3.3), one has

$$\langle\Phi(x,\lambda),\varphi(x,\lambda)\rangle\equiv 1, \tag{3.8}$$

$$\langle\varphi(x,\lambda),\psi(x,\lambda)\rangle\equiv -\Delta(\lambda)\ \ for\ x\neq\frac{\pi}{2}.$$

The theorem we proved above shows that the Weyl function uniquely determines the potentials and the coefficients of the problem $L(p,q,h,H,\alpha,\beta,\gamma)$.

**Theorem 3.3** *If* $M(\lambda)=\tilde{M}(\lambda)$, *then* $L(p,q,h,H,\alpha,\beta,\gamma)=\tilde{L}(\tilde{p},\tilde{q},\widetilde{h},\widetilde{H},\tilde{\alpha},\widetilde{\beta},\widetilde{\gamma})$.

**Proof** Since

$$\psi^{(\nu)}(x,\lambda)=O(|\lambda|^{\nu-1}\exp(|\operatorname{Im}\lambda|(\pi-x))),\ \lambda\in\tilde{G}_{\delta}, \tag{3.9}$$

$$|\Delta(\lambda)|\geq C_{\delta}\exp(|\operatorname{Im}\lambda|\pi),\ \lambda\in\tilde{G}_{\delta},\ C_{\delta}>0,\ \nu=0,1, \tag{3.10}$$

it is easy to observe that

$$\left|\Phi^{(\nu)}(x,\lambda)\right|\leq C_{\delta}|\lambda|^{\nu-1}\exp(-|\operatorname{Im}\lambda|x),\ \lambda\in G_{\delta}. \tag{3.11}$$

Let us define the matrix $P(x,\lambda)=[P_{jk}(x,\lambda)]_{j,k=1,2}$, where

$$P_{j1}(x,\lambda)=\varphi^{(j-1)}(x,\lambda)\tilde{\Phi}'(x,\lambda)-\Phi^{(j-1)}(x,\lambda)\tilde{\varphi}'(x,\lambda), \tag{3.12}$$

$$P_{j2}(x,\lambda)=\Phi^{(j-1)}(x,\lambda)\tilde{\varphi}'(x,\lambda)-\varphi^{(j-1)}(x,\lambda)\tilde{\Phi}'(x,\lambda).$$

Then we have

$$\varphi(x,\lambda) = P_{11}(x,\lambda)\tilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}'(x,\lambda), \tag{3.13}$$
$$\Phi(x,\lambda) = P_{11}(x,\lambda)\tilde{\Phi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\Phi}'(x,\lambda).$$

According to (3.6) and (3.11), for each fixed $x$, the functions $P_{jk}(x,\lambda)$ are meromorphic in $\lambda$ with poles and points $\lambda_n$ and $\tilde{\lambda}_n$. Denote $G_\delta^0 = G_\delta \cap \tilde{G}_\delta$. By virtue of (3.11), (3.13), and

$$\varphi^{(\nu)}(x,\lambda) = O\left(|\lambda|^\nu \exp(|\operatorname{Im}\lambda|\,\mu^+(x)\right),\ \lambda \in G_\delta^0,\ \nu = 0,1, \tag{3.14}$$

we get

$$|P_{12}(x,\lambda)| \le C_\delta\,|\lambda|^{-1},\ \ |P_{12}(x,\lambda)| \le C_\delta,\ \lambda \in G_\delta^0. \tag{3.15}$$

It follows from (3.6) and (3.12) that if $M(\lambda) \equiv \tilde{M}(\lambda)$, then for each fixed $x$, the functions $P_{1k}$ are entire in $x$. Together with (3.15), this yields $P_{12}(x,\lambda) \equiv 0,\ P_{11}(x,\lambda) \equiv A(x)$. Now using (3.13), we obtain

$$\varphi(x,\lambda) \equiv A(x)\tilde{\varphi}(x,\lambda),\ \Phi(x,\lambda) \equiv A(x)\tilde{\Phi}(x,\lambda). \tag{3.16}$$

Therefore, for $|\lambda| \to \infty,\ \arg\lambda \in [\varepsilon, \pi-\varepsilon]\ \ (\varepsilon > 0)$, we have

$$\varphi(x,\lambda) = \frac{B}{2}\exp\left(-i\left(\lambda\mu^+(x) - \omega^+(x)\right)\right)\left(1 + O\left(\frac{1}{|\lambda|}\right)\right), \tag{3.17}$$

where $B = 1$, for $x < \dfrac{\pi}{2}$ and $B = \beta^+$ for $x > \dfrac{\pi}{2}$. Similarly, one can calculate

$$\Phi(x,\lambda) = (i\lambda B)^{-1}\exp\left(i\left(\lambda\mu^+(x) - \omega^+(x)\right)\right)\left(1 + O\left(\frac{1}{|\lambda|}\right)\right) \tag{3.18}$$

$|\lambda| \to \infty, \arg\lambda \in [\varepsilon, \pi-\varepsilon]$.

Finally, taking into account the relations $\langle \Phi(x,\lambda), \varphi(x,\lambda)\rangle \equiv 1$ and (3.11), we have $\beta^+ = \widetilde{\beta}^+,\ A(x) \equiv 1$, that is, $\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda),\ \Phi(x,\lambda) \equiv \tilde{\Phi}(x,\lambda)$ for all $x$ and $\lambda$. Consequently, $L(p,q,h,H,\alpha,\beta,\gamma) = \tilde{L}(\tilde{p},\tilde{q},\widetilde{h},\widetilde{H},\tilde{\alpha},\widetilde{\beta},\widetilde{\gamma})$. The theorem is proved. □

**Theorem 3.4** *Let $\{\lambda_n\}$ be the spectrum of both $L$ and $\widetilde{L}$. If $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on $\left(0, \frac{1+\alpha}{4}\pi\right)$, then $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ almost every where on $(0,\pi)$ and $H = \widetilde{H}$, $h = \widetilde{h}$, $\rho(x) = \widetilde{\rho}(x)$, $\beta = \widetilde{\beta}, \gamma = \widetilde{\gamma}$.*

**Proof** Let $\psi(x,\lambda), \widetilde{\psi}(x,\lambda)$ be the solutions of the equations

$$-\psi''(x,\lambda) + [2\lambda p(x) + q(x)]\,\psi(x,\lambda) = \lambda^2\rho(x)\,\psi(x,\lambda), \tag{3.19}$$

$$-\widetilde{\psi}''(x,\lambda) + [2\lambda\tilde{p}(x) + \tilde{q}(x)]\,\widetilde{\psi}(x,\lambda) = \lambda^2\widetilde{\rho}(x)\widetilde{\psi}(x,\lambda) \tag{3.20}$$

with the initial valued conditions, respectively

$$\psi(0,\lambda)=1,\ \psi'(0,\lambda)=-H \tag{3.21}$$

$$\widetilde{\psi}(0,\lambda)=1,\ \widetilde{\psi}'(0,\lambda)=-\widetilde{H} \tag{3.22}$$

and the impulsive conditions (1.4). Multiplying (3.19) by $\widetilde{\psi}(x,\lambda)$ and (3.20) by $\psi(x,\lambda)$, respectively, and subtracting, we get

$$\widetilde{\psi}''(x,\lambda)\psi(x,\lambda)-\psi''(x,\lambda)\widetilde{\psi}(x,\lambda)=\left[2\lambda\left(p(x)-\tilde{p}(x)\right)+\left(q(x)-\tilde{q}(x)\right)\right]\psi(x,\lambda)\widetilde{\psi}(x,\lambda). \tag{3.23}$$

Integrating the above equality from 0 to $\pi$ with respect to $x$, using the initial conditions at $x=\pi$ and impulse conditions (1.4), we have

$$\int_c^\pi \left[2\lambda\left(\tilde{p}(x)-p(x)\right)+\left(\tilde{q}(x)-q(x)\right)\right]\psi(x,\lambda)\widetilde{\psi}(x,\lambda)dx \tag{3.24}$$
$$=\psi'(c,\lambda)\widetilde{\psi}(c,\lambda)-\widetilde{\psi}'(c,\lambda)\psi(c,\lambda)$$

from the hypothesis $\tilde{p}(x)=p(x)$, $\tilde{q}(x)=q(x)$ on $(0,c)$, where $c=\frac{1+\alpha}{4}\pi$.

Denote

$$P(x)=\tilde{p}(x)-p(x),\ Q(x)=\tilde{q}(x)-q(x)$$

and

$$F_0(\lambda)=2\lambda\int_c^\pi P(x)\psi(x,\lambda)\widetilde{\psi}(x,\lambda)dx+\int_c^\pi Q(x)\psi(x,\lambda)\widetilde{\psi}(x,\lambda)dx. \tag{3.25}$$

It follows from (2.1), (2.2), and (2.14) that $F_0(\lambda)$ is an entire function of exponential type, and there are some positive constants $C_1$ and $C_2$ such that

$$|F_0(\lambda)|\le\left(C_1+C_2|\lambda|\right)\exp\left(|\operatorname{Im}\lambda|\,\alpha\pi\right)\text{ for all }\lambda\in\mathbb{C}. \tag{3.26}$$

It is clear from the properties of $\psi(x,\lambda)$, $\psi'(x,\lambda)$, and the boundary conditions (1.2) that

$$F_0(\lambda_n)=0,\ n\in\mathbb{Z} \tag{3.27}$$

for each eigenvalue $\lambda_n$.

Define

$$F(\lambda):=\frac{F_0(\lambda)}{\Delta(\lambda)}.$$

It is clear that $F(\lambda)$ is an entire function, and it follows from (2.13) and (3.26) that

$$F(\lambda)=O(1)$$

for sufficiently large $|\lambda|$, $\lambda\in G_\delta$. Using Liouville's theorem [18], we obtain for all $\lambda$ that

$$F(\lambda)=C,$$

where $C$ is a constant.

Let us show that the $C=0$. Now, we can rewrite the equation $F_0(\lambda)=C\Delta(\lambda)$ as

$$2\lambda\int_c^\pi P(x)\psi(x,\lambda)\widetilde{\psi}(x,\lambda)dx+\int_c^\pi Q(x)\psi(x,\lambda)\widetilde{\psi}(x,\lambda)dx$$
$$=C\left(-\beta^+\left(\lambda\alpha-\frac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^+(\pi)-\frac{\omega^+(\pi)}{\alpha}\right)\right.$$
$$\left.+\beta^-\left(\lambda\alpha-\frac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^-(\pi)+\frac{\omega^-(\pi)}{\alpha}\right)\right)$$
$$+C\left(+H\beta^+\cos\left(\lambda\mu^+(\pi)-\frac{\omega^+(\pi)}{\alpha}\right)\right.$$
$$\left.+H\beta^-\cos\left(\lambda\mu^-(\pi)+\frac{\omega^-(\pi)}{\alpha}\right)\right)$$
$$+O\left(e^{|\operatorname{Im}\lambda|\pi}\right).$$

By use of Riemann–Lebesgue Lemma [18], we see that the limit of the left-hand side of the above equality exists as $\lambda\to\infty$, $\lambda\in\mathbb{R}$. Therefore, we get that $C=0$. Thus, we have

$$F_0(\lambda)=0,\ \text{for all } \lambda\in\mathbb{C}.$$

Then, from the equality (3.25), we obtain

$$\widetilde{\psi}'(c,\lambda)\psi(c,\lambda)-\psi'(c,\lambda)\widetilde{\psi}(c,\lambda)=0$$

for all $\lambda\in\mathbb{C}$. Since

$$-\frac{\psi(c,\lambda)}{\Delta(\lambda)}=-\frac{\widetilde{\psi}(c,\lambda)}{\widetilde{\Delta}(\lambda)}$$

we obtain

$$M(\lambda)=\tilde{M}(\lambda) \tag{3.28}$$

for all $\lambda\in\mathbb{C}$.

The function $M(\lambda)=-\dfrac{\psi(c,\lambda)}{\Delta(\lambda)}$ is the Weyl function for equation (1.1) on the $\left(\dfrac{\pi(1+\alpha)}{4},\pi\right)$, with jump conditions (1.4) and boundary conditions $y'(c)=0, V(y)=0$. Consequently, relation (3.28) implies $p(x)=\widetilde{p}(x)$, $\rho(x)=\widetilde{\rho}(x)$ and $q(x)=\widetilde{q}(x)$ i.e. on $(c,\pi)$, $\beta=\widetilde{\beta}, \gamma=\widetilde{\gamma}, H=\widetilde{H}$, so assertion of the theorem is proved. □

**Theorem 3.5** *If* $\lambda_n=\tilde{\lambda}_n$ *for all* $n\in\mathbb{Z}$ *and* $p(x)=\tilde{p}(x)$, $q(x)=\tilde{q}(x)$ *on* $(c,\pi)$, *then* $p(x)=\tilde{p}(x)$ *and* $q(x)=\tilde{q}(x)$ *almost every where on* $(0,\pi)$.

**Proof** If the operations in the proof of Theorem 3.3 are also performed for the function $\varphi(x,\lambda)$ which provides the initial conditions $\varphi(0,\lambda)=1, \varphi^{'}(0,\lambda)=h$ and (1.4) impulsive conditions of equation (1.1)

$$\int_0^c \left[2\lambda\left(\tilde{p}(x)-p(x)\right)+\left(\tilde{q}(x)-q(x)\right)\right]\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda)dx \\ = -\widetilde{\varphi}'(c,\lambda)\varphi(c,\lambda)+\varphi'(c,\lambda)\widetilde{\varphi}(c,\lambda) \tag{3.29}$$

equality is obtained. Denote

$$H(\lambda)=\int_0^c \left[2\lambda\left(\tilde{p}(x)-p(x)\right)+\left(\tilde{q}(x)-q(x)\right)\right]\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda)dx.$$

Similar to the proof of Theorem 3.3, we have that $H(\lambda)=0$, for all $\lambda\in\mathbb{C}$. Then, from the equality (3.29), we obtain

$$\widetilde{\varphi}'(c,\lambda)\varphi(c,\lambda)-\varphi'(c,\lambda)\widetilde{\varphi}(c,\lambda)=0, \quad \forall\lambda\in\mathbb{C}$$

or

$$\widetilde{\Delta}(\lambda)\varphi(c,\lambda)-\Delta(\lambda)\widetilde{\varphi}(c,\lambda)=0,$$

for all $\lambda\in\mathbb{C}$. From here for all $\lambda\in\mathbb{C}$,

$$\frac{\varphi(c,\lambda)}{\Delta(\lambda)}=\frac{\widetilde{\varphi}(c,\lambda)}{\widetilde{\Delta}(\lambda)}$$

equality is obtained. The function $M(\lambda)=-\dfrac{\varphi(c,\lambda)}{\Delta(\lambda)}$ is the Weyl function for equation (1.1) on the $(0,c)$ with boundary conditions $U(y)=0, y^{'}(c)=0$ and without impulsive conditions (1.4). The Weyl function uniquely determined $p(x)$ and $q(x)$ on $(0,\pi)$ and the coefficient $h$, so Theorem 3.4 is proved. □

## 4. An interior inverse problems

We consider the interior inverse problem for the same problem $L$ and obtain the corresponding results. For brevity, denote $c_1=\dfrac{\pi(1+\alpha)}{4}, c_2=\dfrac{\pi(1+\alpha)}{2}$.

**Theorem 4.1** *If* $\lambda_n=\widetilde{\lambda}_n$ *for all* $n\in\mathbb{Z}$ *and*

$$\frac{y(c_1,\lambda_n)}{y^{'}(c_1,\lambda_n)}=\frac{\widetilde{y}(c_1,\lambda_n)}{\widetilde{y}'(c_1,\lambda_n)} \tag{4.1}$$

*then* $p(x)=\widetilde{p}(x)$ *on* $[0,\pi], q(x)=\widetilde{q}(x)$ *a.e. on* $[0,\pi]$ *and* $\rho(x)=\widetilde{\rho}(x), a=\widetilde{a}, h=\widetilde{h}, H=\widetilde{H}$.

**Proof** Let $\varphi(x,\lambda)$ be the solution of the problem $L(p,q,h,H,\alpha,\beta,\gamma)$ satisfying the initial conditions $\varphi(0,\lambda)=1, \varphi^{'}(0,\lambda)=h$ and impulsive conditions (1.4). Firstly, the assumption that $\lambda_n=\widetilde{\lambda}_n$ can determine $\rho(x)=\widetilde{\rho}(x), a=\widetilde{a}, h=\widetilde{h}, H=\widetilde{H}$ by Lemma3.2. The oder hand, from (3.29) we see that

$$\frac{\varphi(c_1,\lambda_n)}{\varphi^{'}(c_1,\lambda_n)}=\frac{\widetilde{\varphi}(c_1,\lambda_n)}{\widetilde{\varphi}'(c_1,\lambda_n)}.$$

Then from (3.29), the entire function $H(\lambda)$ has zeros $\{\lambda_n\}, n\in\mathbb{Z}$, i.e. $H(\lambda_n)=0$. Similar to the proof of Theorem 3.4, we have that $p(x)=\widetilde{p}(x)$ and $q(x)=\widetilde{q}(x)$ on $(0,c_1)$. Once we get that $p(x)=\widetilde{p}(x)$ and $q(x)=\widetilde{q}(x)$, by corollary of Theorem 3.3 we have that $p(x)=\widetilde{p}(x)$ on $(0,\pi)$ and $q(x)=\widetilde{q}(x)$ a.e. on $(0,\pi)$. Theorem is proved. □

**Theorem 4.2** *Let $m(n)$ be a sequence of integers such that* $\inf\limits_{n\in\mathbb{Z}}\dfrac{m(n)}{\lambda_n}\geq 1$.

***i)*** *If for any $n\in\mathbb{Z}$,*

$$\lambda_{m(n)}=\widetilde{\lambda}_{m(n)},\frac{y(c_1,\lambda_{m(n)})}{y'\left(c_1,\lambda_{m(n)}\right)}=\frac{\widetilde{y}(c_1,\lambda_{m(n)})}{\widetilde{y}'(c_1,\lambda_{m(n)})}. \tag{4.2}$$

*Then $p(x)=\widetilde{p}(x)$ on $(0,c_1)$, $q(x)=\widetilde{q}(x)$ a.e. on $(0,c_1)$, and $\rho(x)=\widetilde{\rho}(x),,h=\widetilde{h},H=\widetilde{H}$, $a=\widetilde{a}$.*

***ii)*** *If for any $n\in\mathbb{Z}$,*

$$\lambda_{m(n)}=\widetilde{\lambda}_{m(n)},\frac{y(c_2,\lambda_{m(n)})}{y'\left(c_2,\lambda_{m(n)}\right)}=\frac{\widetilde{y}(c_2,\lambda_{m(n)})}{\widetilde{y}'(c_2,\lambda_{m(n)})}. \tag{4.3}$$

*Then $p(x)=\widetilde{p}(x)$ on $(c_2,\pi)$, $q(x)=\widetilde{q}(x)$ a.e. on $(c_2,\pi)$, and $\rho(x)=\widetilde{\rho}(x),a=\widetilde{a},h=\widetilde{h},H=\widetilde{H}$.*

**Proof** **i)** From the assumption (4.2) and (3.29), we have

$$\varphi'\left(c_1,\lambda_{m(n)}\right)\widetilde{\varphi}(c_1,\lambda_{m(n)})-\widetilde{\varphi}'(c_1,\lambda_{m(n)})\varphi\left(c_1,\lambda_{m(n)}\right)=0$$

which means

$$H(\lambda_{m(n)})=0,\qquad n\in\mathbb{Z}. \tag{4.4}$$

Next, we shall show that $H(\lambda)\equiv 0$ on the whole $\lambda$ plane. From (2.2) and (2.1), one has

$$|H(\lambda)|\leq (A+Br)\,e^{2c_1 r|\sin\theta|} \tag{4.5}$$

for some positive constants $A$ and $B$, where $\lambda=re^{i\theta}$. Furthermore, it is clear from the inequality of (4.5) that the entire function $H(\lambda)$ is an exponential type less than $2c_1$. Define the indicator of function $H(\lambda)$ by

$$h(\theta)=\lim_{r\to\infty}\sup\frac{\ln\left|H\left(re^{i\theta}\right)\right|}{r}. \tag{4.6}$$

One can obtain the following estimate from (4.5) and (4.6) that $h(\theta)\leq 2c_1|\sin\theta|$. Let us denote by $n(r)$ the number of zeros of $H(\lambda)$ in the disk $|\lambda|\leq r$. From the equations (4.4), the assumption of (4.2) and known asymptotic expression of the eigenvalues $\lambda_n$, we have the following estimate for the number of zeros of $H(\lambda)$ in the disk $|\lambda|\leq r$.

$$n(r)=1+2\left[\sigma r(1+\varepsilon(r))\right]=2\sigma r(1+\varepsilon(r)).$$

Here $\varepsilon(r)\to 0$ for $r\to\infty$, $\sigma$ is number such that $\sigma>\dfrac{\alpha+\beta}{2}=\dfrac{2c_1}{\pi}$ and $[x]$ is the integer part of $x$. It follows that in the case under consideration

$$\lim_{r\to\infty}\frac{n(r)}{r}=2\sigma>\frac{4c_1}{\pi}=\frac{c_1}{\pi}\int_0^{2\pi}|\sin\theta|\,d\theta\geq\frac{1}{2\pi}\int_0^{2\pi}h(\theta)d\theta. \tag{4.7}$$

To complete the proof we have to recall the following Theorem 3.5 the set of zeros of every entire function of the exponential type, not identically zero, satisfy the inequality

$$\lim_{r\to\infty}\inf\frac{n(r)}{r}\leq\frac{1}{2\pi}\int_{0}^{2\pi}h(\theta)d\theta. \tag{4.8}$$

Inequalities (4.7) and (4.8) imply that $H(\lambda)\equiv 0$ on the whole $\lambda$- plane. As already mentioned, if $H(\lambda)\equiv 0$, then from (4.2) and (3.29), we have

$$\widetilde{\varphi}(c_1,\lambda)\varphi^{'}(c_1,\lambda)-\widetilde{\varphi}'(c_1,\lambda)\varphi(c_1,\lambda)=0,$$

so

$$\frac{\varphi(c_1,\lambda)}{\varphi^{'}(c_1,\lambda)}=\frac{\widetilde{\varphi}(c_1,\lambda)}{\widetilde{\varphi}'(c_1,\lambda)}$$

on the whole $\lambda$- plane.

The function $M(\lambda):=\dfrac{\varphi(c_1,\lambda)}{\varphi^{'}(c_1,\lambda)}$ is the Weyl function of the boundary value problem for the equation (1.1) on $(0,c_1)$ with boundary conditions $U(y)=0, y^{'}(c_1)=0$ and without impulsive conditions (1.4) (see [24]). By ([24]), the Weyl function uniquely species $p(x)$ and $q(x)$ a.e. on $(0,c_1)$ and coefficient $h$.

**ii)** To prove that $p(x)=\widetilde{p}(x)$ on $[c_2,\pi]$, $q(x)=\widetilde{q}(x)$ a.e. on $[c_2,\pi]$, $\rho(x)=\widetilde{\rho}(x), h=\widetilde{h}, H=\widetilde{H}, a=\widetilde{a}$. We will consider the supplementary problem $L$

$$\begin{cases} -y^{''}+[q_1(x)+2\lambda p_1(x)]\,y=\lambda^2\rho(x)\,y, \quad x\in\left[0,\dfrac{\pi}{2}\right)\cup\left(\dfrac{\pi}{2},\pi\right] \\ y(0)-Hy(0)=0, \\ y^{'}(\pi)+hy(\pi)=0, \\ y(\frac{\pi}{2}+0)=a^{-1}y(\frac{\pi}{2}-0), \\ y^{'}(\frac{\pi}{2}+0)=ay^{'}(\frac{\pi}{2}-0)+\gamma y(\frac{\pi}{2}-0) \end{cases}$$

where $q_1(x)=q(\pi-x)$ and $p_1(x)=p(\pi-x)$. A direct calculation implies that $\widehat{y}_n(x):=y_n(\pi-x)$ is the solution to the supplementary problem $\widehat{L}$ and $\widehat{y}_n(\pi-c_2)=y_n(c_2)$. Note that $\pi-c_2\in\left(0,\dfrac{\pi}{2}\right)$. Thus, the assumption conditions for $\widehat{L}$ in the case (i) are still satisfied. Repeating the above arguments, we can obtain the proof of Theorem 3.4.

**Corollory:** If for any $n\in\mathbb{Z}, \lambda_n=\widetilde{\lambda}_n, p(x)=\widetilde{p}(x)$ and $q(x)=\widetilde{q}(x)$ on $(0,c_1)$, then $p(x)=\widetilde{p}(x)$ on $(0,\pi)$, $q(x)=\widetilde{q}(x)$ a.e. on $(0,\pi)$ and $\rho(x)=\widetilde{\rho}(x), a=\widetilde{a}, h=\widetilde{h}, H=\widetilde{H}$. □

## 5. Reconstruction

Let the functions $C(x,\lambda), C_1(x,\lambda)$, and $S_1(x,\lambda)$ be solutions of equation (1.1) under the conditions

$$C(0,\lambda)=C_1(\pi,\lambda)=-S_1'(\pi,\lambda)=1,$$
$$C'(0,\lambda)=C_1'(\pi,\lambda)=S_1(\pi,\lambda)=0.$$

For each fixed $x \in [0,\pi]$, the functions $C(x,\lambda), C_1(x,\lambda)$, and $S_1(x,\lambda)$ together with their derivatives with respect to $x$ are entire in $\lambda$. The function $\Delta^o(\lambda) := \varphi(\pi,\lambda)$ is a characteristic function of the boundary value problem for equation (1.1) with the boundary conditions

$$U(y) = 0, y(\pi) = 0, \tag{5.1}$$

we have

$$\Phi(x,\lambda) = \frac{\varphi(x,\lambda)}{\Delta(\lambda)} = S_1(x,\lambda) + M(\lambda)\psi(x,\lambda), M(\lambda) = -\frac{\Delta^o(\lambda)}{\Delta(\lambda)}. \tag{5.2}$$

Let $\{\lambda_n^o\}$ be the spectrum of (1.1), (5.1). Clearly, $\{\lambda_n\} \cap \{\lambda_n^o\} = \varnothing$. Thus, $M(\lambda)$ is a meromorphic function with the poles $\lambda_n$ and the zeros $\lambda_n^o$.

Denote

$$\Delta_1^o(\lambda) := \varphi(c_1,\lambda), \Delta_1(\lambda) := \varphi'(c_1,\lambda) \tag{5.3}$$

$$\Delta_2^o(\lambda) := \psi(c_1,\lambda), \Delta_2(\lambda) := -\psi'(c_1,\lambda) \tag{5.4}$$

We note that $\Delta_1^o(\lambda)$ and $\Delta_1(\lambda)$ are characteristic functions of the boundary value problems

$$\ell y(x) = 0, 0 < x < c_1, U(y) = y(c_1) = 0, \tag{5.5}$$

$$\ell y(x) = 0, 0 < x < c_1, U(y) = y'(c_1) = 0, \tag{5.6}$$

respectively.

The function

$$M_1(\lambda) := -\frac{\Delta_1^o(\lambda)}{\Delta_1(\lambda)} \tag{5.7}$$

is the Weyl function for (5.6). The functions $\Delta_2^o(\lambda)$ and $\Delta_2(\lambda)$ are, in turn, characteristic functions of the boundary value problems

$$\ell y(x) = 0, c_1 < x < \pi, y(c_1) = V(y) = 0, \tag{5.8}$$

$$\ell y(x) = 0, c_1 < x < \pi, y'(c_1) = V(y) = 0, \tag{5.9}$$

respectively.

Let $\{\xi_n\}_{n\in\mathbb{Z}}$ and $\{\eta_n\}_{n\in\mathbb{Z}}$ be the spectra of the boundary value problems (5.8) and (5.9), i.e. they are the sequences of zeros of the functions $\Delta_2(\lambda)$ and $\Delta_2^o(\lambda)$, respectively. Thus, we have

$$\xi_n = \frac{\pi n}{\pi - c_1} + \frac{\omega^+(\pi) - \omega^+(c_1)}{\pi - c_1} + O\left(\frac{1}{n}\right),$$

$$\eta_n = \frac{\pi}{\pi - c_1}\left(n + \frac{1}{2}\right) + \frac{\omega^+(\pi) - \omega^+(c_1)}{\pi - c_1} + O\left(\frac{1}{n}\right).$$

According to (5.2)–(5.4), we have

$$\Delta_1^o(\lambda) = \Delta(\lambda) S_1(c_1, \lambda) - \Delta^o(\lambda)\Delta_2^o(\lambda), \tag{5.10}$$
$$\Delta_1(\lambda) = \Delta(\lambda) S_1'(c_1, \lambda) - \Delta^o(\lambda)\Delta_2(\lambda).$$

If we use Lemma 4 in [6], we get the following for the $\Delta_1(\lambda)$ and $\Delta_1^o(\lambda)$:

$$\Delta_1(\lambda) = -(\lambda - p(c_1)) \sin\left(\lambda c_1 - \omega^+(c_1)\right) + d(\lambda),$$

$$\Delta_1^o(\lambda) = \cos\left(\lambda c_1 - \omega^+(c_1)\right) + d_o(\lambda),$$

where

$$d(\lambda) = \sum_{n\in\mathbb{Z}} d(\xi_n) \frac{\Delta_2(\lambda)}{(\lambda - \xi_n)\dot{\Delta}_2(\xi_n)},$$

$$d_o(\lambda) = \sum_{n\in\mathbb{Z}} d_o(\eta_n) \frac{\Delta_2^o(\lambda)}{(\lambda - \eta_n)\dot{\Delta}_2^o(\eta_n)}.$$

From the asymptotic expression of $\xi_n$, the value of $\omega^+(c_1)$ is determined as

$$\omega^+(c_1) = -(\pi - c_1) \lim_{n\to\infty} (\xi_n - \frac{\pi n + \omega^+(\pi)}{\pi - c_1}).$$

In this case, if we take $\Delta_{1o}(\lambda) = -(\lambda - p(c_1)) \sin(\lambda c_1 - \omega^+(c_1))$,

$$\lambda_{n1}^o = \frac{\pi n}{c_1} + \omega_1, \omega_1 = \frac{1}{c_1}\int_0^{c_1} p(t)\,dt = \frac{1}{c_1}\omega^+(c_1)$$

follows.

From here,

$$\lambda_{n1} = \lambda_{n1}^o + \frac{(-1)^n \sin[\lambda_{n1}^o(\pi - c_1) - \omega^+(\pi) + \omega^+(c_1)]}{c_1 \lambda_{n1}^o \dot{\Delta}_2^o(\lambda_{n1}^o)} \sum_{m\in\mathbb{Z}} \frac{d(\xi_m)}{\dot{\Delta}_2(\xi_m)} + \frac{\varepsilon_n}{\lambda_{n1}^o} \tag{5.11}$$

asymptotic expression is obtained for the roots of the characteristic equation $\Delta_1(\lambda) = 0$. Here $\varepsilon_n \in \ell_2$.

Similarly,

$$\lambda_{no} = \lambda_{no}^o + \frac{(-1)^n \cos[\lambda_{no}^o(\pi - c_1) - \omega^+(\pi) + \omega^+(c_1)]}{c_1 \lambda_{no}^o \dot{\Delta}_2^o(\lambda_{no}^o)} \sum_{m\in\mathbb{Z}} \frac{d(\eta_m)}{\dot{\Delta}_2^o(\eta_m)} + \frac{\varepsilon_n^o}{\lambda_{no}^o} \tag{5.12}$$

behavior is found for the roots of the equation $\Delta_1^o(\lambda) = 0$. Here $\varepsilon_n^o \in \ell_2$.

If we use the proof of Lemma 9 in [2],

$$-2\lambda_{n1}\alpha_{n1} = \dot{\Delta}_1(\lambda_{n1})\Delta_1^o(\lambda_{n1})$$

or

$$\alpha_{n1} = -\frac{1}{2\lambda_{n1}}\dot{\Delta}_1\left(\lambda_{n1}\right)\Delta_1^o\left(\lambda_{n1}\right) \tag{5.13}$$

is obtained. The sequences $\{\alpha_{n1}, \lambda_{n1}\}$ are called spectral data of the boundary value problem (5.6).

By using the expressions (5.11) and (5.12),

$$\alpha_{n_1} = \frac{c_1}{2} + \frac{\delta_{1n}}{\lambda_{n1}^o} \tag{5.14}$$

is obtained from the equation (5.13), where $\delta_{1n} \in \ell_2$.

If the solution of the system of integral equations (2.20) and (2.21) is searched as

$$A\left(x,t\right) = A_o\left(x,t\right)\cos\omega^+\left(x\right) + A_1\left(x,t\right)\sin\omega^+\left(x\right) \tag{5.15}$$

$$B\left(x,t\right) = B_o\left(x,t\right)\cos\omega^+\left(x\right) + B_1\left(x,t\right)\sin\omega^+\left(x\right) \tag{5.16}$$

the functions of $A_o\left(x,t\right)$ and $B_o\left(x,t\right)$

$$\begin{aligned}&\beta^+F_{11}\left(x,t\right) + A_o\left(x,t\right) + \beta^-F_{11}\left(2a-x,t\right)\\&+\int_0^x A_o\left(x,\xi\right)F_{11}\left(\xi,t\right)d\xi + \int_0^x B_o\left(x,\xi\right)F_{12}\left(\xi,t\right)d\xi = 0, 0 \le t < x\end{aligned} \tag{5.17}$$

$$\begin{aligned}&\beta^+F_{21}\left(x,t\right) + B_o\left(x,t\right) + \beta^-F_{21}\left(2a-x,t\right)\\&+\int_0^x A_o\left(x,\xi\right)F_{21}\left(\xi,t\right)d\xi + \int_0^x B_o\left(x,\xi\right)F_{22}\left(\xi,t\right)d\xi = 0, 0 \le t < x\end{aligned} \tag{5.18}$$

system, $A_1\left(x,t\right)$ and $B_1\left(x,t\right)$ functions are found as the solution of the system of

$$\begin{aligned}&\beta^+F_{12}\left(x,t\right) + A_1\left(x,t\right) + \beta^-F_{12}\left(2a-x,t\right)\\&+\int_0^x A_1\left(x,\xi\right)F_{11}\left(\xi,t\right)d\xi + \int_0^x B_1\left(x,\xi\right)F_{12}\left(\xi,t\right)d\xi = 0, 0 \le t < x\end{aligned} \tag{5.19}$$

$$\begin{aligned}&\beta^+F_{22}\left(x,t\right) + B_1\left(x,t\right) + \beta^-F_{22}\left(2a-x,t\right)\\&+\int_0^x A_1\left(x,\xi\right)F_{21}\left(\xi,t\right)d\xi + \int_0^x B_1\left(x,\xi\right)F_{22}\left(\xi,t\right)d\xi = 0, 0 \le t < x\end{aligned} \tag{5.20}$$

equations.

On the other hand, if we use the equations (5.15) and (5.16), we get the nonlinear Volterra type integral equation in the following form

$$\beta^+\omega^+\left(x\right) = \beta^+xp(0) + \int_0^x \Phi\left(\xi,\omega^+\left(\xi\right)\right)d\xi \tag{5.21}$$

for the function $\omega^{+}(x)$ from equation (2.3), where

$$R(\xi,z)=2A_1(\xi,\xi)\sin^2 z-2B_o(\xi,\xi)\cos^2 z+[A_o(\xi,\xi)-B_1(\xi,\xi)]\sin 2z \tag{5.22}$$

is.

Thus, we get the following algorithm for solution of the inverse problem:

Given a collection of numbers $\{\lambda_{n1},\alpha_{n1}\}$ satisfying the conditions (5.12) and (5.13), we construct the functions $F_{ij}(x,t)$ $(i,j=1,2)$ by (2.22)–(2.25) and consider the two systems of equations (5.17), (5.18) and (5.19),(5.20) with respect to $A_o(x,t),B_o(x,t)$ and $A_1(x,t),B_1(x,t)$, respectively. Solving these systems, we find $A_o(x,t),B_o(x,t)$ and $A_1(x,t),B_1(x,t)$. Then we form the function $R(\xi,z)$ by (5.22) and consider equation (5.21) for $\omega^{+}(x)$. Solving this equation we find $\omega^{+}(x)$ and then $p(x)$ by $p(x)=(\omega^{+}(x))'$ according to (2.8). Next, define $A(x,t),B(x,t)$ by (5.15),(5.16) and then $q(x)$ by (2.4).

## 6. Examples

**Example 6.1** *Define* $\rho(x)=\begin{cases}1, & for\ x<\dfrac{\pi}{2}\\ \alpha^2, & for\ x>\dfrac{\pi}{2}\end{cases}$ *and then consider the boundary value problem* $L$

$$-y''+[q(x)+2\lambda p(x)]\,y=\lambda^2\rho(x)\,y,\qquad x\in(0,\pi)$$

$$U(y):=y'(0)-hy(0)=0$$

$$V(y):=y'(\pi)+Hy(\pi)=0$$

*with the impulsive conditions*

$$y\left(\frac{\pi}{2}+0\right)=\beta y\left(\frac{\pi}{2}-0\right),$$

$$y'\left(\frac{\pi}{2}+0\right)=\beta^{-1}y'\left(\frac{\pi}{2}-0\right)+\gamma y\left(\frac{\pi}{2}-0\right).$$

*Also consider the boundary value problem* $L_0$,

$$-y''=\lambda^2\rho(x)\,y,\qquad x\in(0,\pi)$$

$$U(y):=y'(0)=0$$

$$V(y):=y'(\pi)=0$$

*with the some impulsive conditions. Let* $\lambda_n$ *and* $\lambda_n^0$ *be the eigenvalues and* $M(\lambda)$, $M_0(\lambda)$ *be the Weyl functions of boundary value problems* $L$ *and* $L_0$, *respectively.*

Using Theorem3.3, if $\lambda_n=\lambda_n^0$ and $p(x)=0,\ q(x)=0$ on $\left(0,\dfrac{1+\alpha}{4}\pi\right)$, and $h=0$, then $M(\lambda)=M_0(\lambda)$ i.e. $p(x)=0$ and $q(x)=0$ a.e. on $(0,\pi)$ and $H=0,\alpha=\widetilde{\alpha},\beta=\widetilde{\beta},\gamma=\widetilde{\gamma}$.

**Example 6.2** *Let* $q(x)=\begin{cases} q_1(x), & 0\le x<c_1 \\ 0, & c_1< x\le \pi \end{cases}$ *and* $p(x)=\begin{cases} p_1(x), & 0< x<c_1 \\ 0, & c_1< x\le \pi \end{cases}$ *be the differential equation (1.1), (1.2)-(1.3) boundary conditions and (1.4) discontinuity conditions. Since in this case* $\xi_n=\dfrac{\pi n}{\pi-c_1}, \eta_n=\dfrac{\pi}{\pi-c_1}(n+\dfrac{1}{2})$ *will be,* $d(\lambda_{n1}^o)=0$*,* $d_o(\lambda_{no}^o)=0$ *and* $\alpha_{n1}=\dfrac{c_1}{2}$*. From this, it is obtained that* $F_{ij}(x,t)=0,\ (i,j=1,2),\ 0\le t\le x$*. Thus, since* $A_1(x,t)=0,\ B_1(x,t)=0,\ A_o(x,t)=0,\ B_o(x,t)=0$ *are* $R(\xi,z)=0$*. Thus, for* $\forall x\in[0,\pi]$ $\omega^+(x)=0$ *or* $\int\limits_0^x p_1(t)dt=0$ *is obtained. From here, it becomes* $p_1(x)=0$*. Since* $q_1(x)=-p_1^2(x)$*,* $q_1(x)=0$ *is obtained.*

**Example 6.3** *Let* $q(x)=\begin{cases} q_1(x), & 0\le x\le c_1 \\ \dfrac{2a^2}{(1+ax)^2}, & c_1\le x\le \pi \end{cases}$ *and* $p(x)=0$ *be the differential equation (1.1), (1.2)-(1.3) boundary conditions* $\beta=1,\gamma=0$*. In this case, the function*

$$\psi(x,\lambda)=\cos\lambda(\pi-x)-\frac{a}{1+ax}.\frac{\sin\lambda(\pi-x)}{\lambda}$$

*is the solution of the differential equation*

$$-y^{''}+\frac{2a^2}{(1+ax)^2}y=\lambda^2 y$$

*that satisfies the initial conditions* $\psi(\pi,\lambda)=1,\psi'(\pi,\lambda)=\dfrac{a}{1+a\pi}$*. From here, if we take* $c_1=\dfrac{\pi}{2}$*, it becomes* $\lambda_{n1}^2=(2n)^2$*,* $\lambda_{no}^2=(2n+1)^2$*,* $\alpha_n=\dfrac{\pi}{2}$*. If we take* $a:=\dfrac{1}{\alpha_0}-\dfrac{1}{\pi}$*, it becomes*

$$F(x,t)=\sum_{n=0}^{\infty}\left(\frac{\cos\lambda_{n1}x\cos\lambda_{n1}t}{\alpha_n}-\frac{\cos{(2n+1)}\,x\cos{(2n+1)}\,t}{\alpha_n^o}\right)$$

*from equation (1.5.10) given in [7]. Therefore,* $K(x,t)=-\dfrac{a}{1+ax}$ *is obtained from the basic integral equation*

$$K(x,t)+F(x,t)+\int\limits_0^x K(x,s)F(s,t)ds=0, 0<t<x<\frac{\pi}{2}$$

*given in [7]. Since* $q_1(x)=2\dfrac{dK(x,x)}{dx}$*, it turns out that*

$$q_1(x)=2\frac{d}{dx}\left(-\frac{a}{1+ax}\right)=\frac{2a^2}{(1+ax)^2}, 0<x<\frac{\pi}{2}.$$

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