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New results on derivatives of the shape operator of a real hypersurface in a complex projective space

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Abstract: We consider real hypersurfaces M in complex projective space equipped with both the Levi-Civita and generalized Tanaka-Webster connections. For any nonnull real number k and any symmetric tensor field of type (1,1) L on M we can define a tensor field of type (1,2) on M, $L_F^{(k)}$, related to both connections. We study symmetry and skewsymmetry of the tensor $A_F^{(k)}$ associated to the shape operator A of M.

Key words: g-Tanaka-Webster connection, complex projective space, real hypersurface, k-th Cho operator

1. Introduction

Consider a real hypersurface without boundary M of the complex projective space $\mathbb{C}P^m$, $m \geq 2$, endowed with the Fubini-Study metric g of constant holomorphic sectional curvature 4. We will denote by ∇ the Levi-Civita connection on M and by J the Kaehlerian structure of $\mathbb{C}P^m$. Take a locally defined unit normal vector field N on M and denote by $\xi = -JN$. This tangent vector field to M is called the structure vector field on M. From the Kahlerian structure of $\mathbb{C}P^m$, we can induce on M an almost contact metric structure (φ, ξ, η, g), where φ is the tangent component of J, η is an one-form given by $\eta(X) = g(X, \xi)$ for any X tangent to M and g is the induced metric on M. The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$ was obtained by Takagi, see [6], [14], [15], [16]. His classification contains 6 types of real hypersurfaces. Among them, we find type (A_1) real hypersurfaces that are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$ and type (A_2) real hypersurfaces that are tubes of radius r, $0 < r < \frac{\pi}{2}$, over totally geodesic complex projective spaces $\mathbb{C}P^n$, $0 < n < m-1$. Type (A_1) real hypersurfaces have two distinct constant principal curvatures and type (A_2) have three distinct constant principal curvatures. We will call both types of real hypersurfaces type (A) real hypersurfaces. Type (B) real hypersurfaces are tubes of radius r, $0 < r < \frac{\pi}{4}$, over totally geodesic real projective space $\mathbb{R}P^m$. This kind of real hypersurfaces has three distinct constant principal curvatures.

Kimura, [6], proved that any real hypersurface M in $\mathbb{C}P^m$ whose structure vector field is principal for the shape operator A of M and all whose principal curvatures are constant must be one in Takagi's list.

A ruled real hypersurface of $\mathbb{C}P^m$ satisfies that the maximal holomorphic distribution on M, $\mathbb{D}$, given at any point by the vectors orthogonal to ξ, is integrable, and its integral manifolds are totally geodesic $\mathbb{C}P^{m-1}$, or, equivalently, $g(A\mathbb{D}, \mathbb{D}) = 0$. For the examples of ruled real hypersurfaces, see [7] or [9].

We will say that a type (1,1) tensor field L defined on M is parallel if $\nabla_X L = 0$ for any X tangent to

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M, where $(\nabla_X L)Y = \nabla_X LY - L\nabla_X Y$, for any Y tangent to M.

The notion of L being parallel can be generalized by the concept of L being Codazzi, which means that $(\nabla_X L)Y = (\nabla_Y L)X$ for any X, Y tangent to M. Due to Codazzi equation (see Section 2) for the case $L = A$ we conclude that there does not exist any real hypersurface in $\mathbb{C}P^m$ whose shape operator is Codazzi, and, therefore, it cannot be parallel.

Blair, [1], also generalized the notion of L being parallel, giving the definition of L being Killing if $(\nabla_X L)X = 0$ for any X tangent to M, which is equivalent to the fact that $(\nabla_X L)Y + (\nabla_Y L)X = 0$ for any X, Y tangent to M. Codazzi equation also yields non-existence of real hypersurfaces in $\mathbb{C}P^m$ whose shape operator is Killing.

The Tanaka-Webster connection, [17], [19], is the canonical affine connection defined on a non-degenerate, pseudo-Hermitian CR-manifold. As a generalization of this connection, Tanno, [18], defined the generalized Tanaka-Webster connection for contact metric manifolds by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\varphi Y. \tag{1.1}$$

Using the naturally extended affine connection of Tanno's generalized Tanaka–Webster connection, Cho defined the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for a real hypersurface M in $\mathbb{C}P^m$ given, see [4], [5], by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y \tag{1.2}$$

for any X, Y tangent to M, where k is a non-zero real number. Then the four elements of the almost contact metric structure on M are parallel for this connection, that is, $\hat{\nabla}^{(k)}\eta = 0$, $\hat{\nabla}^{(k)}\xi = 0$, $\hat{\nabla}^{(k)}g = 0$, $\hat{\nabla}^{(k)}\varphi = 0$. In particular, if the shape operator of a real hypersurface satisfies $\varphi A + A\varphi = 2k\varphi$, the real hypersurface is contact and the g-Tanaka-Webster connection coincides with the Tanaka-Webster connection.

Here, we can consider the tensor field of type (1,2) given by the difference of both connections $F^{(k)}(X, Y) = g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y$, for any X, Y tangent to M, see [8] Proposition 7.10, pages 234–235. We will call this tensor the k-th Cho tensor on M. Associated to it, for any X tangent to M and any nonnull real number k, we can consider the tensor field of type (1,1) $F^{(k)}_X$, given by $F^{(k)}_X Y = F^{(k)}(X, Y)$ for any $Y \in TM$. This operator will be called the k-th Cho operator corresponding to X. The torsion of the connection $\hat{\nabla}^{(k)}$ is given by $\hat{T}^{(k)}(X, Y) = F^{(k)}_X Y - F^{(k)}_Y X$ for any X, Y tangent to M.

Let now L be a symmetric tensor of type (1,1) defined on M. We can consider then the type (1,2) tensor $L^{(k)}_F$ associated to L in the following way: $L^{(k)}_F(X, Y) = [F^{(k)}_X, L]Y = F^{(k)}_X LY - LF^{(k)}_X Y$, for any X, Y tangent to M.The corresponding operator $L^{(k)}_{F_X} Y = L^{(k)}_F(X, Y)$ gives a measure of how far are $F^{(k)}_X$ and L of being commutative. We will say that L is $(\hat{\nabla}^{(k)}, \nabla)$-parallel if $(\hat{\nabla}^{(k)}_X - \nabla_X)L = 0$, for any X tangent to M. This condition is equivalent to the fact that $L^{(k)}_F = 0$.

Generalizing such a concept, we will say that L is $(\hat{\nabla}^{(k)}, \nabla)$-Codazzi if $(\hat{\nabla}^{(k)}_X L)Y - (\hat{\nabla}^{(k)}_Y L)X = (\nabla_X L)Y - (\nabla_Y L)X$ for any X, Y tangent to M. This condition is equivalent to $L^{(k)}_F$ being symmetric.

On the other hand, we will say that L is $(\hat{\nabla}^{(k)}, \nabla)$-Killing if $(\hat{\nabla}^{(k)}_X L)Y + (\hat{\nabla}^{(k)}_Y L)X - (\nabla_X L)Y - (\nabla_Y L)X = 0$ for any X, Y tangent to M. This condition is equivalent to $L^{(k)}_F$ being skewsymmetric.

In [13] we proved non-existence of real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that the shape operator is $(\hat{\nabla}^{(k)}, \nabla)$-parallel, that is, $A_F^{(k)} = 0$, for any nonnull real number k.

The purpose of the present paper is to study real hypersurfaces M in $\mathbb{C}P^m$ such that the shape operator is either $(\hat{\nabla}^{(k)}, \nabla)$-Codazzi or $(\hat{\nabla}^{(k)}, \nabla)$-Killing. In fact we will obtain the following

Theorem 1 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Let k be a nonnull real number. Then $A_F^{(k)}(X,Y) = A_F^{(k)}(Y,X)$ for any $X, Y \in \mathbb{D}$ if and only if M is locally congruent to a ruled real hypersurface.*

and

Corollary 1 *There does not exist any real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, such that for a nonnull real number k $A_F^{(k)}$ is symmetric.*

On the other hand, we also have

Theorem 2 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, and k a nonnull real number. Then $A_F^{(k)}(X,Y) = -A_F^{(k)}(Y,X)$ for any $X, Y \in \mathbb{D}$ if and only if M is locally congruent to either a real hypersurface of type (A) or to a tube of radius $\frac{\pi}{4}$ over a complex submanifold of $\mathbb{C}P^m$ or to a ruled real hypersurface.*

and

Corollary 2 *There does not exist any real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, such that for a nonnull real number k the tensor field $A_F^{(k)}$ is skewsymmetric.*

2. Preliminaries

Throughout this paper, all manifolds, vector fields, etc. will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let N be a locally defined unit normal vector field on M. Let ∇ be the Levi-Civita connection on M and (J, g) the Kaehlerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M, we write $JX = \varphi X + \eta(X)N$ and $-JN = \xi$. Then, (φ, ξ, η, g) is an almost contact metric structure on M, see [2]. That is, we have

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any tangent vectors X, Y to M. From (2.1), we obtain

$$\varphi\xi = 0, \quad \eta(X) = g(X, \xi). \tag{2.2}$$

From the parallelism of J, we get

$$(\nabla_X \varphi)Y = \eta(Y)AX - g(AX,Y)\xi \tag{2.3}$$

and

$$\nabla_X \xi = \varphi AX \tag{2.4}$$

for any X, Y tangent to M, where A denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given, respectively, by

$$\begin{aligned} R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y \\ -2g(\varphi X, Y)\varphi Z + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.5}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi \tag{2.6}$$

for any tangent vectors X, Y, Z to M, where R is the curvature tensor of M. We will call the maximal holomorphic distribution $\mathbb{D}$ on M to the following one: at any $p \in M$, $\mathbb{D}(p) = \{X \in T_pM | g(X, \xi) = 0\}$. We will say that M is Hopf if ξ is principal, that is, $A\xi = \alpha\xi$ for a certain function α on M.

In the sequel, we need the following results:

Theorem 2.1, [12] *Let M be a real hypersurface of $\mathbb{C}P^m$, $m \geq 2$. Then, the following are equivalent:*

1. *M is locally congruent to either a geodesic hypersphere or a tube of radius r, $0 < r < \frac{\pi}{2}$, over a totally geodesic $\mathbb{C}P^n$, $0 < n < m-1$.*
2. *$\varphi A = A\varphi$.*

Theorem 2.2, [10] *If ξ is a principal curvature vector with corresponding principal curvature α, this is locally constant, and if $X \in \mathbb{D}$ is principal with principal curvature λ, then $2\lambda - \alpha \neq 0$ and φX is principal with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

3. Proofs of Theorem 1 and Corollary 1

If $A_F^{(k)}(X,Y) = A_F^{(k)}(Y,X)$, for any X, Y tangent to M we get

$$\begin{array}{c} g(\varphi AX, AY)\xi - \eta(AY)\varphi AX - k\eta(X)\varphi AY - g(\varphi AX, Y)A\xi + \eta(Y)A\varphi AX \\ +k\eta(X)A\varphi Y = g(\varphi AY, AX)\xi - \eta(AX)\varphi AY - k\eta(Y)\varphi AX \\ -g(\varphi AY, X)A\xi + \eta(X)A\varphi AY + k\eta(Y)A\varphi X. \end{array} \tag{3.1}$$

If we suppose that $X, Y \in \mathbb{D}$, (3.1) becomes

$$\begin{array}{c} g(\varphi AX, AY)\xi - \eta(AY)\varphi AX - g(\varphi AX, Y)A\xi = g(\varphi AY, AX)\xi \\ -\eta(AX)\varphi AY - g(\varphi AY, X)A\xi. \end{array} \tag{3.2}$$

If M is Hopf (3.2) gives $g(\varphi AX, AY)\xi - \alpha g(\varphi AX, Y)\xi = g(\varphi AY, AX)\xi - \alpha g(\varphi AY, X)\xi$ for any $X, Y \in \mathbb{D}$, where we suppose $A\xi = \alpha\xi$. This yields $g(\varphi AX, AY) - \alpha g(\varphi AX, Y) = g(\varphi AY, AX) - \alpha g(\varphi AY, X)$ for any $X, Y \in \mathbb{D}$. Therefore, for any $X \in \mathbb{D}$, we obtain

$$2A\varphi AX = \alpha\varphi AX + \alpha A\varphi X. \tag{3.3}$$

Let $X \in \mathbb{D}$ be a unit vector field such that $AX = \lambda X$. Then, from Theorem 2.2, $A\varphi X = \mu\varphi X$ with $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. From (3.3) for such an X we get $2\lambda\mu\varphi X = \alpha(\lambda+\mu)\varphi X$. That is, $2\lambda\mu = \alpha(\lambda+\mu)$ or $\frac{2\alpha\lambda^2+4\lambda}{2\lambda-\alpha} = \alpha(\lambda + \frac{\alpha\lambda+2}{2\lambda-\alpha}) = \alpha(\frac{2\lambda^2+2}{2\lambda-\alpha})$. Thus, we have $\alpha\lambda^2 + 2\lambda = \alpha\lambda^2 + \alpha$. This means that $2\lambda = \alpha$, which is impossible by Theorem 2.2

Now we suppose M is non Hopf. Thus, at least on a neighbourhood of a certain point of M, we can write $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in $\mathbb{D}$ and β a non-vanishing function. All the computations are made on such a neighbourhood. From now on, we will denote $\mathbb{D}_U = \{X \in \mathbb{D} | g(X, U) = g(X, \varphi U) = 0\}$. Taking the scalar product of (3.2) and φU, we obtain $-\eta(AY)g(AX,U) = -\eta(AX)g(AY,U)$ for any $X,Y \in \mathbb{D}$. Taking $Y \in \mathbb{D}$ orthogonal to U we get $-\eta(AX)g(AY,U) = 0$ for any $X \in \mathbb{D}$. If $X = U$ we arrive at $-\beta g(AY,U) = 0$ for any $Y \in \mathbb{D}$ orthogonal to U. That is

$$AU = \beta\xi + \gamma U \tag{3.4}$$

for a certain function γ. The scalar product of (3.2) and U yields

$$-\eta(AY)g(\varphi AX,U) - \beta g(\varphi AX,Y) = -\eta(AX)g(\varphi AY,U) - \beta g(\varphi AY,X) \tag{3.5}$$

for any $X,Y \in \mathbb{D}$. Taking $X = U$ in (3.5) and bearing in mind (3.4) it follows $-\beta g(\varphi AU,Y) = -2\beta g(\varphi AY,U) = 2\beta g(A\varphi U,Y)$ for any $Y \in \mathbb{D}$. This yields $-\varphi AU = 2A\varphi U$ or $2A\varphi U = -\gamma\varphi U$. Therefore,

$$A\varphi U = -\tfrac{\gamma}{2}\varphi U. \tag{3.6}$$

The scalar product of (3.2) and ξ implies

$$g(\varphi AX,AY) - \alpha g(\varphi AX,Y) = g(\varphi AY,AX) - \alpha g(\varphi AY,X). \tag{3.7}$$

for any $X,Y \in \mathbb{D}$. If $X = U$, $Y = \varphi U$ it follows $g(\varphi AU,A\varphi U) - \alpha g(\varphi AU,\varphi U) = g(\varphi A\varphi U,AU) - \alpha g(\varphi A\varphi U,U)$. Therefore, $g(\varphi AU,A\varphi U) - \alpha g(AU,U) = -g(A\varphi U,\varphi AU) + \alpha g(A\varphi U,\varphi U)$. From (3.4) and (3.6) we have $-\frac{\gamma^2}{2} - \alpha\gamma = \frac{\gamma^2}{2} - \frac{\alpha\gamma}{2}$. It follows $\gamma(\gamma + \frac{\alpha}{2}) = 0$ and this yields either $\gamma = 0$ or $\gamma = -\frac{\alpha}{2}$.

Suppose now $X,Y \in \mathbb{D}_U$. Then (3.5) yields $g(\varphi AX,Y) = g(\varphi AY,X)$. From (3.4) and (3.6) we obtain $A\varphi X + \varphi AX = 0$ for any $X \in \mathbb{D}_U$. Then, if $X \in \mathbb{D}_U$ is unit and $AX = \lambda X$, $A\varphi X = -\lambda\varphi X$. Now from (3.7) we get $A\varphi AX - \alpha\varphi AX = -A\varphi AX + \alpha A\varphi X$. That is, $2A\varphi AX = \alpha(\varphi A + A\varphi)X = 0$ for any $X \in \mathbb{D}_U$. This implies $-2\lambda^2 = 0$. Therefore, on $\mathbb{D}_U$ the unique principal curvature is 0.

Then, if $\gamma = 0$, we obtain that M is locally congruent to a ruled real hypersurface.

If $\gamma = -\frac{\alpha}{2}$, $AX = 0$ for any $X \in \mathbb{D}_U$. Take a unit $X \in \mathbb{D}_U$. The Codazzi equation gives $(\nabla_X A)\varphi X - (\nabla_{\varphi X} A)X = -2\xi$. As $AX = A\varphi X = 0$ this yields $-A\nabla_X\varphi X + A\nabla_{\varphi X}X = -2\xi$. Its scalar product with ξ gives $-g(\nabla_X\varphi X, \alpha\xi + \beta U) + g(\nabla_{\varphi X}X, \alpha\xi + \beta U) = -2$. This implies

$$g([\varphi X,X],U) = -\tfrac{2}{\beta}. \tag{3.8}$$

From its scalar product with U we obtain $-g(\nabla_X\varphi X, \beta\xi - \frac{\alpha}{2}U) + g(\nabla_{\varphi X}X, \beta\xi - \frac{\alpha}{2}U) = 0$. That is, $-\frac{\alpha}{2}g([\varphi X,X],U) = 0$. But from (3.8) $g([\varphi X,X],U) \neq 0$. Thus $\alpha = 0$ and M should be locally congruent to a minimal ruled real hypersurface. This finishes the proof of Theorem 1.

In order to prove Corollary 1, take $X = \xi$, $Y \in \mathbb{D}$ in (3.1). We get

$$\begin{aligned} &g(\varphi A\xi,AY)\xi - \eta(AY)\varphi A\xi - k\varphi AY - g(\varphi A\xi,Y)A\xi + kA\varphi Y \\ &= g(\varphi AY,A\xi)\xi - \eta(A\xi)\varphi AY + A\varphi AY \end{aligned} \tag{3.9}$$

for any $Y \in \mathbb{D}$. From Theorem 1, we suppose that M is ruled. Then, (3.9) yields

$$\begin{aligned} -\beta\eta(AY)\varphi U - k\varphi AY - \beta g(\varphi U, Y)A\xi + kA\varphi Y \\ = -\alpha g(\varphi AY, U)\xi - \alpha\varphi AY + A\varphi AY \end{aligned} \tag{3.10}$$

for any $Y \in \mathbb{D}$. The scalar product of (3.10) and φU gives $-\beta\eta(AY) = 0$. Taking $Y = U$ we obtain $\beta^2 = 0$, which is impossible and proves the Corollary.

4. Proofs of Theorem 2 and Corollary 2

If $A_F^{(k)}(X,Y) + A_F^{(k)}(Y,X) = 0$, for any X, Y tangent to M we have

$$\begin{aligned} -\eta(AY)\varphi AX - k\eta(X)\varphi AY - g(\varphi AX, Y)A\xi + \eta(Y)A\varphi AX + k\eta(X)A\varphi Y \\ -\eta(AX)\varphi AY - k\eta(Y)\varphi AX - g(\varphi AY, X)A\xi + \eta(X)A\varphi AY + k\eta(Y)A\varphi X = 0. \end{aligned} \tag{4.1}$$

If $X, Y \in \mathbb{D}$ (4.1) becomes

$$-\eta(AY)\varphi AX - g(\varphi AX, Y)A\xi - \eta(AX)\varphi AY - g(\varphi AY, X)A\xi = 0. \tag{4.2}$$

Let us suppose that M is Hopf, and write $A\xi = \alpha\xi$. Then (4.2) gives

$$\alpha g(\varphi AX, Y)\xi + \alpha g(\varphi AY, X)\xi = 0 \tag{4.3}$$

for any $X, Y \in \mathbb{D}$. If $X \in \mathbb{D}$ is unit and principal with principal curvature λ, as φX is principal with principal curvature $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$, (4.3) yields $\alpha\lambda g(\varphi X, Y) - \alpha\mu g(\varphi X, Y) = 0$ for any $Y \in \mathbb{D}$. Thus $\alpha(\lambda - \mu)\varphi X = 0$ and either $\alpha = 0$ or $\lambda = \mu$. If $\alpha = 0$, from [3], M must be locally congruent to a tube of radius $\frac{\pi}{4}$ over a complex submanifold of $\mathbb{C}P^m$. If $\lambda = \mu$, $\varphi A = A\varphi$ and from Theorem 2.1 M is locally congruent to a real hypersurface of type (A).

If M is non Hopf with $A\xi = \alpha\xi + \beta U$, the scalar product of (4.2) and φU gives $-\eta(AY)g(AX, U) - \eta(AX)g(AY, U) = 0$. If we take $Y \in \mathbb{D}$ and orthogonal to U we get $-\eta(AX)g(AU, Y) = 0$ and taking $X = U$ we obtain $-\beta g(AU, Y) = 0$. Therefore, $g(AU, Y) = 0$ for any $Y \in \mathbb{D}$ orthogonal to U and

$$AU = \beta\xi + \gamma U \tag{4.4}$$

for a certain function γ. Taking $Y = U$ in (4.2) we have $-\beta\varphi AX - g(\varphi AX, U)A\xi - \eta(AX)\varphi AU - g(\varphi AU, X)A\xi = 0$ for any $X \in \mathbb{D}$. Its scalar product with U yields $2\beta g(A\varphi U, X) - \beta g(\varphi AU, X) = 0$ for any $X \in \mathbb{D}$. Thus $2A\varphi U = \varphi AU = \gamma\varphi U$ and

$$A\varphi U = \tfrac{\gamma}{2}\varphi U. \tag{4.5}$$

The scalar product of (4.2) and ξ gives $-\alpha g(\varphi AX, Y) - \alpha g(\varphi AY, X) = 0$. Thus, either $\alpha = 0$ or $\alpha \neq 0$ and $g(\varphi AX, Y) + g(\varphi AY, X) = 0$ for any $X, Y \in \mathbb{D}$.

In the second case, taking $X = U$, $Y = \varphi U$ we have $g(\varphi AU, \varphi U) + g(\varphi A\varphi U, U) = 0$. Then, $g(AU, U) = g(A\varphi U, \varphi U)$, that is, $\gamma = \frac{\gamma}{2}$ and $\gamma = 0$. Therefore $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi$, $A\varphi U = 0$ and $\mathbb{D}_U$ is A-invariant.

Taking $X \in \mathbb{D}_U$, $Y = U$ in (4.2) we get $-\beta\varphi AX = 0$. This yields $AX = 0$ for any $X \in \mathbb{D}_U$ and M must be ruled. Any ruled real hypersurface satisfies (4.2).

Suppose now $\alpha = 0$. Then, $A\xi = \beta U$ and (4.2) becomes $-\eta(AY)\varphi AX - \beta g(\varphi AX, Y)\ U - \eta(AX)\varphi AY - \beta g(\varphi AY, X)U = 0$ for any $X, Y \in \mathbb{D}$. Taking $Y = U$, $X \in \mathbb{D}_U$ we get $-\beta\varphi AX = 0$. Then $\varphi AX = 0$ for any $X \in \mathbb{D}_U$ and this yields $AX = 0$ for any $X \in \mathbb{D}_U$. For such an X Codazzi equation implies $-A\nabla_X\varphi X + A\nabla_{\varphi X}X = -2\xi$ and its scalar product with ξ yields

$$g([\varphi X, X], U) = -\tfrac{2}{\beta} \tag{4.6}$$

and its scalar product with U implies $\gamma g([\varphi X, X], U) = 0$. From (4.6) $g([\varphi X, X], U) \neq 0$ and then we should have $\gamma = 0$. In this case M is ruled and minimal and we conclude the proof of Theorem 2.

In order to prove Corollary 2, taking $X = \xi$, $Y \in \mathbb{D}$ in (4.1) we get

$$-\eta(AY)\varphi A\xi - k\varphi AY - g(\varphi A\xi, Y)A\xi + kA\varphi Y - \eta(A\xi)\varphi AY + A\varphi AY = 0 \tag{4.7}$$

for any $Y \in \mathbb{D}$. If M is Hopf with $A\xi = \alpha\xi$, (4.7) gives $-k\varphi AY + kA\varphi Y - \alpha\varphi AY + A\varphi AY = 0$ for any $Y \in \mathbb{D}$. Suppose $Y \in \mathbb{D}$ is unit and $AY = \lambda Y$. We know $A\varphi Y = \mu\varphi Y$ with $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. Then it follows $-k\lambda\varphi Y + k\mu\varphi Y - \alpha\lambda\varphi Y + \lambda\mu\varphi Y = 0$. That is, $-k\lambda + k\mu - \alpha\lambda + \lambda\mu = 0$ and bearing in mind the expression of μ we obtain

$$-(2k+\alpha)\lambda^2 + (2\alpha k + \alpha^2 + 2)\lambda + 2k = 0. \tag{4.8}$$

If $2k + \alpha = 0$ we have $2\lambda + k = 0$. Therefore $2\lambda - \alpha = 0$ and from Theorem 2.2 this is impossible. From (4.8) on M there are, at most, three distinct constant principal curvatures and then, [6], M must be locally congruent to a real hypersurface either of type (A) or of type (B).

Looking at Theorem 2, if $\alpha = 0$, (4.8) yields $k\lambda^2 - \lambda - k = 0$. If M is of type (A), $\lambda = \mu = \frac{1}{\lambda}$. As $\lambda^2 = 1$, it follows $\lambda = 0$, a contradiction. On the other hand, type (B) real hypersurfaces do not have $\alpha = 0$.

If $\alpha \neq 0$, M must be of type (A). In this case $\alpha = 2cot(2r)$ and one of the principal curvatures on $\mathbb{D}$ is $\lambda = cot(r)$. This principal curvature does not satisfy (4.8) and this case does not occur.

Then, M must be ruled and taking $X = \xi$, $Y \in \mathbb{D}$ in (4.1) we get

$$-\beta\eta(AY)\varphi U - k\varphi AY - \beta g(\varphi U, Y)A\xi + k\varphi AY - \alpha\varphi AY + A\varphi AY = 0 \tag{4.9}$$

for any $Y \in \mathbb{D}$. The scalar product of (4.9) and φU gives $-\beta\eta(AY) = 0$, for any $Y \in \mathbb{D}$. If, in particular, we take $Y = U$ we obtain $\beta^2 = 0$, which is impossible, finishing the proof.

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□

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