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# Some Properties of the semigroup $PG_Y(X)$: Green's relations, ideals, isomorphism theorems and ranks

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**Abstract:** Let $T(X)$ be the full transformation semigroup on the set $X$. For a fixed nonempty subset $Y$ of $X$, let

$$PG_Y(X) = \{\alpha \in T(X) : \alpha|_Y \in G(Y)\}$$

where $G(Y)$ is the permutation group on $Y$. It is known that $PG_Y(X)$ is a regular subsemigroup of $T(X)$. In this paper, we give a simpler description of Green's relations and characterize the ideals of $PG_Y(X)$. Moreover, we prove some isomorphism theorems for $PG_Y(X)$. For finite sets, we investigate the cardinalities of $PG_Y(X)$ and of its subsets of idempotents, and we also calculate their ranks.

**Key words:** Green's relations, ideal, isomorphism theorem, rank

## 1. Introduction

The study of semigroups of full transformations has been fruitful over years. As far back in 1952, Malcev [10] determined ideals of $T(X)$. Later in 1955, Miller and Doss [3] proved that $T(X)$ is a regular semigroup and described its Green's relations. And in 1959, Hall [4] showed that every semigroup is isomorphic to a subsemigroup of $T(X)$ for some an appropriate set $X$. It is well-known that $T(X)$ is isomorphic to $T(Y)$ if and only if $|X| = |Y|$. In fact, each isomorphism $\Phi\colon T(X) \to T(Y)$ is induced by a bijection $g\colon X \to Y$ in the sense that $\alpha\Phi = g^{-1}\alpha g$ for every $\alpha \in T(X)$.

The *rank* of a semigroup $S$ is the minimal size of a generating set of $S$.

For a positive integer $n$, let $T_n$ denote the full transformation semigroup on the set $X = \{1, 2, \ldots, n\}$. For $n \geq 3$, the rank of $T_n$ is equal to 3, see [6]. Let $1 \leq r \leq n$ and $K(n,r) = \{\alpha \in T_n\colon |X\alpha| \leq r\}$. Then $K(n,r)$ is an ideal of $T_n$. In 1990, Howie and McFadden [7] proved that the rank of $K(n,r)$ is $S(n,r)$, for $2 \leq r \leq n-1$ where $S(n,r)$ is the Stirling number of the second kind.

For a nonempty subset $Y$ of $X$, let $S(X,Y) = \{\alpha \in T(X)\colon Y\alpha \subseteq Y\}$. The semigroup $S(X,Y)$ was introduced and studied by Magill [9] in 1966. In 2005, Nenthein, Youngkhong and Kemprasit [11] gave a necessary and sufficient condition for $S(X,Y)$ to be regular. Later in 2011, Honyam and Sanwong [5] described Green's relations on $S(X,Y)$ and characterized its ideals.

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In 1994, Umar [13] constructed the subsemigroup of $T(X)$ as follows:

$$F_Y(X) = \{\alpha \in T(X) \colon C(\alpha)\alpha \subseteq Y = Y\alpha \text{ and } \alpha|_Y \text{ is injective}\}$$

where $C(\alpha) = \bigcup\{t\alpha^{-1} : t \in X\alpha, |t\alpha^{-1}| \geq 2\}$. The author determined Green's relation on $F_Y(X)$ and proved that it is an $\mathcal{R}$-unipotent subsemigroup of $T(X)$. Later in 2018, Billhardt, Sanwong and Sommanee [1] modified the semigroup $F_Y(X)$ as follows:

$$F_Y(X) = \{\alpha \in T(X) \colon \alpha|_Y \in G(Y) \text{ and } \alpha|_{A_\alpha} \text{ is injective}\} \tag{2.1}$$

where $G(Y)$ is the permutation group on $Y$ and $A_\alpha = \{x \in X \colon x\alpha \notin Y\}$. Moreover, they determined all maximal inverse subsemigroups of $F_Y(X)$ when $|Y| \geq 2$. And for finite sets, the authors proved when two semigroups of the type $F_Y(X)$ are isomorphic and described its ideals. They also computed the rank of $F_Y(X)$ when $X$ is finite.

In 2016, Laysirikul [8] defined

$$PG_Y(X) = \{\alpha \in T(X) \colon \alpha|_Y \in G(Y)\}.$$

The author proved that $PG_Y(X)$ is a regular semigroup and investigated a (left, right, completely) regular element of $PG_Y(X)$.

From the definitions of $S(X,Y), F_Y(X)$ and $PG_Y(X)$, we have

$$F_Y(X) \subseteq PG_Y(X) \subseteq S(X,Y).$$

For a fixed nonempty subset $Y$ of a set $X$, let

$$T_{(X,Y)} = \{\alpha \in T(X) \colon Y\alpha = Y\}.$$

Then $T_{(X,Y)}$ is a subsemigroup of $T(X)$. In general, $PG_Y(X) \subseteq T_{(X,Y)}$. We note that if $Y$ is finite, then $PG_Y(X) = T_{(X,Y)}$. In 2018, Toker and Ayik [12] studied generating sets and the rank of $T_{(X,Y)}$ when $X$ is finite.

Here, in Section 3, we describe Green's relations on $PG_Y(X)$ and characterize its ideals. In Section 4, we find the cardinalities of $PG_Y(X)$ and of its subsets of idempotents when $X$ is finite. In Section 5, we investigate some isomorphism theorems for $PG_Y(X)$. Finally, in Section 6, we calculate the rank of $PG_Y(X)$ when $X$ is finite. Although the rank of $PG_Y(X) = T_{(X,Y)}$ was done by Toker and Ayik [12], but in this paper, we use a different technique to obtain a minimal generating set and the rank of $PG_Y(X)$, independently.

## 2. Preliminaries and notations

For all undefined notions, the reader is referred to [6].

An element $e$ of a semigroup $S$ is said to be *idempotent* if $e^2 = e$. As usual, we denote by $E(U)$ the set of all idempotents of $U \subseteq S$. For any set $A$, $|A|$ means the cardinality of the set $A$. If $A$ is a subset of a semigroup $S$, then $\langle A \rangle$ denotes the subsemigroup of $S$ generated by $A$. The *rank* of a semigroup $S$ is the smallest number of elements required to generate $S$, defined by

$$\text{rank}(S) = \min\{|A| : A \subseteq S \text{ and } \langle A \rangle = S\}.$$

Let $X$ be a nonempty set and let $T(X)$ be the set of all functions from $X$ into $X$. Then $T(X)$ is a semigroup under the composition of functions. We call $\alpha \in T(X)$ a *transformation* and $T(X)$ is called the *full transformation semigroup on* $X$. In this paper, we will multiply functions from the left to the right and use the corresponding notation for the left to right composition of functions: $x(\alpha\beta) = (x\alpha)\beta$.

For $\alpha \in T(X)$ and $x \in X$, the image of $x$ under $\alpha$ is written as $x\alpha$ and the image of a subset $A$ of $X$ under $\alpha$ is denoted by $A\alpha$. If $A = X$, then $X\alpha$ is the range (image) of $\alpha$. We denote by $x\alpha^{-1}$ the set of all inverse images of $x$ under $\alpha$, that is, $x\alpha^{-1} = \{z \in X : z\alpha = x\}$.

For $\alpha \in T(X)$ and $A \subseteq X$, the restriction of $\alpha$ to $A$ is denoted by $\alpha|_A$, that is, $\alpha|_A : A \to X$ with $x(\alpha|_A) = x\alpha$ for all $x \in A$. We let $\mathrm{id}_A$ denote the identity function on $A$. Then $\mathrm{id}_X$ is the identity element of $T(X)$. Let $G(A)$ be the set of all bijections from $A$ onto $A$. We called $G(A)$ the *permutation group on the set* $A$. If $A$ is a finite set and $|A| = n$, we write $S_n$ instead of $G(A)$, and call $S_n$ the *symmetric group of order* $n$. It is well-known that $|S_n| = n!$.

As in Clifford and Preston [2], we shall use the notation

$$\alpha = \begin{pmatrix} A_i \\ a_i \end{pmatrix} \tag{2.2}$$

to mean $\alpha \in T(X)$ and take as understood that the subscript $i$ belongs to some (unmentioned) index set $I$, $X\alpha = \{a_i : i \in I\}$ and $A_i = a_i\alpha^{-1}$ for all $i \in I$.

We note that for any $\alpha \in T(X)$, the symbol $\pi_\alpha$ denotes the partition of $X$ induced by the transformation $\alpha$, namely,

$$\pi_\alpha = \{x\alpha^{-1} : x \in X\alpha\}.$$

In [3], the authors gave a complete description of Green's relations on $T(X)$ as follows: For $\alpha, \beta \in T(X)$,

(1) $\alpha\mathcal{L}\beta$ if and only if $X\alpha = X\beta$;

(2) $\alpha\mathcal{R}\beta$ if and only if $\pi_\alpha = \pi_\beta$;

(3) $\alpha\mathcal{D}\beta$ if and only if $|X\alpha| = |X\beta|$;

(4) $\mathcal{D} = \mathcal{J}$.

Throughout the paper, we assume that $Y$ is a nonempty subset of a set $X$ and define

$$PG_Y(X) = \{\alpha \in T(X) : \alpha|_Y \in G(Y)\}.$$

Then $PG_Y(X)$ is a regular subsemigroup of $T(X)$, see [8, Theorem 2.2]. It is easy to see that if $X = Y$, then $PG_Y(X) = G(X)$. We may regard $PG_Y(X)$ as a generalization of $G(X)$. For $\alpha \in PG_Y(X)$, we see that $Y \subseteq X\alpha \subseteq X$ and so $|Y| \le |X\alpha| \le |X|$. By [8, Theorem 2.3] proved that

$$F_Y(X) = PG_Y(X) \text{ if and only if } |X \setminus Y| \le 1.$$

Let $G(X, Y) = \{g \in G(X) : g|_Y \in G(Y)\}$. Then $G(X, Y)$ is a subgroup of $G(X)$ and we establish the following proposition.

**Proposition 2.1** *$G(X,Y)$ is the group of units of $PG_Y(X)$.*

**Proof** Since $G(X,Y)$ is a subgroup of $PG_Y(X)$, it is clear that all elements of $G(X,Y)$ are units in $PG_Y(X)$. Let $g$ be a unit of $PG_Y(X)$. Then there exists $g' \in PG_Y(X)$ such that $gg' = g'g = \mathrm{id}_X$. Thus, $g\colon X \to X$ is bijective and so $g \in G(X)$. Since $g \in PG_Y(X)$, we obtain $g|_Y \in G(Y)$. Hence, $g \in G(X,Y)$ and therefore $G(X,Y)$ is the group of units of $PG_Y(X)$. □

**Remark 2.1** *$G(X,Y) \cong G(Y) \times G(X \setminus Y)$ via $g \mapsto (g|_Y, g|_{X\setminus Y})$.*

With the notation (2.2), if $Y = \{a_i : i \in I\}$, then for any $\alpha \in PG_Y(X)$ we can write

$$\alpha = \begin{pmatrix} A_i & B_j \\ a_i\sigma & b_j \end{pmatrix} \tag{2.3}$$

where $\sigma \in G(Y), A_i \cap Y = \{a_i\}$ for all $i \in I, B_j \subseteq X \setminus Y$ and $b_j \in X \setminus Y$ for all $j \in J$. Notice that $Y \subseteq \bigcup_{i\in I} A_i$ and $\alpha|_Y = \sigma \in G(Y)$.

## 3. Green's relations and ideals

In this section, we let $Y = \{a_i : i \in I\} \subseteq X$. As a consequence of Green's relations on $T(X)$, we have the following description of Green's relations on $PG_Y(X)$.

**Theorem 3.1** *Let $\alpha, \beta \in PG_Y(X)$. Then*

(1) *$\alpha\mathcal{L}\beta$ if and only if $X\alpha = X\beta$;*

(2) *$\alpha\mathcal{R}\beta$ if and only if $\pi_\alpha = \pi_\beta$;*

(3) *$\alpha\mathcal{D}\beta$ if and only if $|X\alpha| = |X\beta|$;*

(4) *$\mathcal{D} = \mathcal{J}$.*

**Proof** Since $PG_Y(X)$ is a regular subsemigroup of $T(X)$, we have by Hall's Theorem [6, Proposition 2.4.2] that the $\mathcal{L}$ and $\mathcal{R}$ relations on $PG_Y(X)$ are the restrictions to $PG_Y(X)$ of the corresponding relations on $T(X)$. Thus,

$$\alpha\mathcal{L}\beta \text{ if and only if } X\alpha = X\beta, \text{ and } \alpha\mathcal{R}\beta \text{ if and only if } \pi_\alpha = \pi_\beta.$$

Therefore, we obtain (1) and (2). To prove (3), it is clear that if $\alpha\mathcal{D}\beta$ on $PG_Y(X)$, then $\alpha\mathcal{D}\beta$ on $T(X)$, that is $|X\alpha| = |X\beta|$. Now, assume that $|X\alpha| = |X\beta|$. By (2.3), we can write

$$\alpha = \begin{pmatrix} A_i & B_j \\ a_i\sigma & b_j \end{pmatrix} \text{ and } \beta = \begin{pmatrix} C_i & D_j \\ a_i\delta & d_j \end{pmatrix}$$

where $\sigma, \delta \in G(Y)$; $A_i \cap Y = \{a_i\} = C_i \cap Y$ for all $i \in I$; $B_j, D_j \subseteq X \setminus Y$ and $b_j, d_j \in X \setminus Y$ for all $j \in J$. Then we define

$$\gamma = \begin{pmatrix} C_i & D_j \\ a_i\sigma & b_j \end{pmatrix}.$$

Thus, $\gamma \in PG_Y(X)$ such that $X\alpha = X\gamma$ and $\pi_\alpha = \pi_\beta$. So, $\alpha\mathcal{L}\gamma$ and $\gamma\mathcal{R}\beta$ by (1) and (2). Hence, $\alpha\mathcal{D}\beta$ on $PG_Y(X)$. Now, we prove (4) by assuming that $\alpha\mathcal{J}\beta$ on $PG_Y(X)$. Then $\alpha\mathcal{J}\beta$ on $T(X)$ and thus $|X\alpha| = |X\beta|$. Whence, $\alpha\mathcal{D}\beta$ on $PG_Y(X)$ by (3). In general, $\mathcal{D} \subseteq \mathcal{J}$. Therefore, $\mathcal{D} = \mathcal{J}$. □

To determine the ideals of $PG_Y(X)$, we need the following lemma.

**Lemma 3.2** *If $\alpha, \beta \in PG_Y(X)$ and $|X\alpha| \le |X\beta|$, then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in PG_Y(X)$.*

**Proof** Assume that $\alpha, \beta \in PG_Y(X)$ and $|X\alpha| \le |X\beta|$. Then by (2.3), we can write

$$\alpha = \begin{pmatrix} A_i & B_j \\ a_i\sigma & b_j \end{pmatrix} \text{ and } \beta = \begin{pmatrix} C_i & D_j & E_k \\ a_i\delta & d_j & e_k \end{pmatrix}$$

where $\sigma, \delta \in G(Y)$; $A_i \cap Y = \{a_i\} = C_i \cap Y$ for all $i \in I$; $B_j, D_j, E_k \subseteq X \setminus Y$ and $b_j, d_j, e_k \in X \setminus Y$ for all $j \in J, k \in K$. We choose and fix $d_{j_0} \in X\beta$ for some $j_0 \in J$, define

$$\lambda = \begin{pmatrix} A_i & B_j \\ a_i & d'_j \end{pmatrix} \text{ and } \mu = \begin{pmatrix} a_i\delta & d_j & X \setminus (Y \cup \{d_j : j \in J\}) \\ a_i\sigma & b_j & d_{j_0} \end{pmatrix}$$

where $d'_j \in D_j$ for all $j \in J$. Then $\lambda, \mu \in PG_Y(X)$ and $\alpha = \lambda\beta\mu$. □

Let $p$ be any cardinal number and let

$$p' = \min\{q : q > p\}.$$

Note that $p'$ always exits since the cardinals are well-ordered and $p'$ is an immediate successor of $p$. For the case when $p$ is finite, we have $p' = p + 1$.

**Theorem 3.3** *The proper ideals of $PG_Y(X)$ are precisely the sets*

$$Q(k) = \{\alpha \in PG_Y(X) : |X\alpha| < k\}$$

*where $|Y|' \le k \le |X|$.*

**Proof** Let $\alpha \in Q(k)$ where $|Y|' \le k \le |X|$, and $\beta \in PG_Y(X)$. Then $|X\alpha| < k$, $|X\alpha\beta| \le |X\alpha| < k$ and $|X\beta\alpha| \le |X\alpha| < k$. Hence, $\alpha\beta, \beta\alpha \in Q(k)$ and so $Q(k)$ is an ideal of $PG_Y(X)$. Since $|X\mathrm{id}_X| = |X| \ge k$, $\mathrm{id}_X \notin Q(k)$. Thus, $Q(k)$ is proper.

Conversely, let $I$ be a proper ideal of $PG_Y(X)$. Define the class of cardinal numbers as follows.

$$C = \{q : |X\alpha| < q \text{ for all } \alpha \in I\}.$$

Since $|X\alpha| \le |X| < |X|'$ for all $\alpha \in I$, we obtain $C \ne \emptyset$. Using the well-ordering theorem, we let $k$ be the least element of $C$. That is, $k$ is the least cardinal number such that $|X\alpha| < k$ for all $\alpha \in I$. Since $|Y| \le |X\alpha|$ for all $\alpha \in I$, we have $|Y| < k$. We prove $I = Q(k)$. It is clear that $I \subseteq Q(k)$. Now, let $\beta \in Q(k)$. Then $|X\beta| < k$. If $|X\alpha| < |X\beta|$ for all $\alpha \in I$, then $k \le |X\beta|$ by the property of $k$, a contradiction. That is, $|X\beta| \le |X\alpha|$ for some $\alpha \in I$. Then by Lemma 3.2, there are $\lambda, \mu \in PG_Y(X)$ such that $\beta = \lambda\alpha\mu$. It follows from the fact that $I$ is an ideal of $PG_Y(X)$ and $\alpha \in I$, we get that $\beta \in I$. Hence, $Q(k) \subseteq I$ and so $I = Q(k)$. Since $Q(k) = I$ is a proper subset of $PG_Y(X)$, there is $\gamma \in PG_Y(X) \setminus Q(k)$. This implies that $k \le |X\gamma| \le |X|$. Since $k > |Y|$ and $|Y|' = \min\{q : q > |Y|\}$, we obtain $|Y|' \le k$. Therefore, $|Y|' \le k \le |X|$. □

## 4. The finite case

In this section, let $X$ be a finite set with $n$ elements such that $|Y| = r \leq n$, and we write $PG_r(n)$ instead of $PG_Y(X)$.

Since $|G(Y)| = |S_r| = r!$ and the number of all functions from $X \setminus Y$ to $X$ is $n^{n-r}$, by the definition of $PG_r(n)$ we obtain

$$|PG_r(n)| = r! \cdot n^{n-r}. \tag{4.1}$$

For $r \leq k \leq n$, define

$$J(k) = \{\alpha \in PG_r(n) \colon |X\alpha| = k\}.$$

Then by Theorem 3.1, we get that $J(k)$ is a $\mathcal{J}$-class of the semigroup $PG_r(n)$ such that $J(n)$ is the maximum $\mathcal{J}$-class of $PG_r(n)$. Let

$$Q(n;k) = J(r) \cup J(r+1) \cup \cdots \cup J(k)$$

where $r \leq k \leq n$. It is clear that $Q(n;k) = \{\alpha \in PG_r(n) \colon |X\alpha| \leq k\}$ and $Q(n;n) = PG_r(n)$.

**Corollary 4.1** *The ideals of $PG_r(n)$ are of the form*

$$Q(n;k) = \{\alpha \in PG_r(n) \colon |X\alpha| \leq k\}$$

*where $r \leq k \leq n$.*

**Proof** It follows from Theorem 3.3 and the fact that $Q(n;n) = PG_r(n)$ is an ideal of itself. □

**Lemma 4.2** *For $r \leq k \leq n$, $|E\big(J(k)\big)| = \binom{n-r}{k-r} k^{n-k}$.*

**Proof** Let $r \leq k \leq n$ and $\varepsilon \in E\big(J(k)\big)$. Then $|X\varepsilon| = k$ and $\varepsilon|_Y = \mathrm{id}_Y$. If $r = k$, then the number of elements of $E\big(J(r)\big)$ is equivalent to the number of functions from $X \setminus Y$ into $Y$. Hence, $|E\big(J(r)\big)| = r^{n-r} = \binom{n-r}{k-r} k^{n-k}$. Now, assume that $r < k$. Then $X\varepsilon \cap (X \setminus Y) \neq \emptyset$ such that $|X\varepsilon \cap (X \setminus Y)| = k - r$, and $\varepsilon|_{X\varepsilon \cap (X \setminus Y)} = \mathrm{id}_{X\varepsilon \cap (X \setminus Y)}$. Since there are $\binom{n-r}{k-r}$ ways of choosing the $(k-r)$-element subsets of $X \setminus Y$ and there are $k^{n-k}$ functions from $X \setminus \big(Y \cup (X\varepsilon \cap (X \setminus Y))\big)$ into $Y \cup (X\varepsilon \cap (X \setminus Y))$, we obtain $|E\big(J(k)\big)| = \binom{n-r}{k-r} k^{n-k}$. □

**Theorem 4.3** $|E\big(PG_r(n)\big)| = \sum\limits_{k=r}^{n} \binom{n-r}{k-r} k^{n-k}$.

**Proof** It follows directly from Lemma 4.2. □

## 5. Isomorphism theorems

Recall that the *natural partial order* on $E(S)$ defined by for $e, f \in E(S)$,

$$e \leq f \text{ if and only if } e = ef = fe.$$

**Proposition 5.1** *Let $M = \{\alpha \in E\big(PG_Y(X)\big) : X\alpha = Y\}$. Then $M$ is the set of all minimal idempotents in $PG_Y(X)$.*

**Proof** For any $\alpha \in M$ and for any $\beta \in E\big(F_Y(X)\big)$ where $\beta \leq \alpha$, since $x\alpha \in Y$ for all $x \in X$ and since $y\beta = y$ for all $y \in Y$, it follows that $\beta = \beta\alpha = \alpha\beta$ if and only if $x\beta = (x\alpha)\beta = x\alpha$ for all $x \in X$, or equivalently, $\alpha = \beta$. Thus, every element in $M$ is a minimal idempotent.

On the other hand, we assume that $Y = \{a_i : i \in I\} \subseteq X$ and let $\alpha$ be a minimal idempotent of $PG_Y(X)$. We can write

$$\alpha = \begin{pmatrix} A_i & B_j \\ a_i & b_j \end{pmatrix}$$

where $A_i \cap Y = \{a_i\}$ for all $i \in I$ and $b_j \in B_j \subseteq X \setminus Y$ for all $j \in J$. Let $a_{i_0}$ be a fixed element of $Y$ for some $i_0 \in I$ and define

$$\varepsilon = \begin{pmatrix} A_i & B_j \\ a_i & a_{i_0} \end{pmatrix}.$$

Then $\varepsilon \in E\big(F_Y(X)\big)$ and $X\varepsilon = Y$, that is, $\varepsilon \in M$. It is easy to see that $\varepsilon\alpha = \varepsilon = \alpha\varepsilon$. Hence, $\varepsilon \leq \alpha$ and so $\alpha = \varepsilon \in M$ by the minimality of $\alpha$. □

Notice that the cardinality of the set $M$ as defined in Proposition 5.1 is equal to the cardinality of $Y^{X\setminus Y}$, the set of all functions from $X \setminus Y$ into $Y$, that is, $|M| = |Y^{X\setminus Y}|$.

Recall that the image set of the set of all minimal idempotents is also the set of all minimal idempotents under an isomorphism.

**Theorem 5.2** *Let $X_1$ and $X_2$ be two sets, and let $Y_1$ and $Y_2$ be nonempty subsets of $X_1$ and $X_2$, respectively. If $PG_{Y_1}(X_1)$ is isomorphic to $PG_{Y_2}(X_2)$, then $|Y_1^{X_1\setminus Y_1}| = |Y_2^{X_2\setminus Y_2}|$.*

**Proof** Assume that $PG_{Y_1}(X_1) \cong PG_{Y_2}(X_2)$. Then there exists an isomorphism $\Phi\colon PG_{Y_1}(X_1) \to PG_{Y_2}(X_2)$. Let

$$M_1 = \{\alpha \in E\big(PG_{Y_1}(X_1)\big) \colon X_1\alpha = Y_1\}$$

and

$$M_2 = \{\alpha \in E\big(PG_{Y_2}(X_2)\big) \colon X_2\alpha = Y_2\}.$$

By Proposition 5.1, we get $M_1$ and $M_2$ are the sets of all minimal idempotents of $PG_{Y_1}(X_1)$ and $PG_{Y_2}(X_2)$, respectively. It follows that $M_1\Phi = M_2$ and whence $|Y_1^{X_1\setminus Y_1}| = |M_1| = |M_2| = |Y_2^{X_2\setminus Y_2}|$. □

The converse of Theorem 5.2 is not true as shown in the following example.

**Example 5.1** *Let $X = \{1,2,3,4,5,6\}, Y_1 = \{1,2\}$ and $Y_2 = \{1,2,3,4\}$. We see that $|Y_1^{X\setminus Y_1}| = 2^{6-2} = 2^4 = 4^2 = 4^{6-4} = |Y_2^{X\setminus Y_2}|$. While, $|PG_{Y_1}(X)| = 2!(6^{6-2}) = 2(6^4) \neq 24(6^2) = 4!(6^{6-4}) = |PG_{Y_2}(X)|$ by (4.1). Thus, $PG_{Y_1}(X)$ is not isomorphic to $PG_{Y_2}(X)$.*

**Theorem 5.3** *Let $X_1$ and $X_2$ be two sets, and let $Y_1$ and $Y_2$ be nonempty subsets of $X_1$ and $X_2$, respectively. If $|Y_1| = |Y_2|$ and $|X_1 \setminus Y_1| = |X_2 \setminus Y_2|$, then $PG_{Y_1}(X_1) \cong PG_{Y_2}(X_2)$.*

**Proof** Assume that $|Y_1| = |Y_2|$ and $|X_1 \setminus Y_1| = |X_2 \setminus Y_2|$. Then there exist bijective functions $\theta_1\colon Y_1 \to Y_2$ and $\theta_2\colon (X_1 \setminus Y_1) \to (X_2 \setminus Y_2)$. Let $\theta = \theta_1 \cup \theta_2$. It is clear that $\theta\colon X_1 \to X_2$ is a bijection. Now define

$$\Phi\colon PG_{Y_1}(X_1) \to PG_{Y_2}(X_2) \text{ by } \alpha\Phi = \theta^{-1}\alpha\theta \text{ for all } \alpha \in PG_{Y_1}(X_1).$$

Clearly $\theta^{-1}\alpha\theta \in PG_{Y_2}(X_2)$ for all $\alpha \in PG_{Y_1}(X_1)$ and $\theta\beta\theta^{-1} \in PG_{Y_1}(X_1)$ for all $\beta \in PG_{Y_2}(X_2)$. It is a routine matter to show that $\Phi$ is an isomorphism. Therefore, $PG_{Y_1}(X_1) \cong PG_{Y_2}(X_2)$. □

**Corollary 5.4** *Let $Y_1$ and $Y_2$ be nonempty finite subsets of a (finite or infinite) set $X$. If $|Y_1| = |Y_2|$, then $PG_{Y_1}(X) \cong PG_{Y_2}(X)$.*

**Proof** Suppose that $|Y_1| = |Y_2|$. Since $Y_1$ and $Y_2$ are finite subsets of $X$, we get $|X \setminus Y_1| = |X \setminus Y_2|$. It follows from Theorem 5.3 that $PG_{Y_1}(X) \cong PG_{Y_2}(X)$. □

## 6. Ranks

In this section, let $X$ be a finite set with $n$ elements and $|Y| = r \leq n$. And, we follow the notations and some results from Section 4.

Let $F_Y(X)$ be as defined in (2.1), and we write $F_r(n)$ instead of $F_Y(X)$. Then the following result was shown in [1, Section 4] by Billhardt et al.

**Lemma 6.1** *[1, Corollary 4.2] For $1 \leq r \leq n-1$,*

(1) $\operatorname{rank}(S_r \times S_{n-r}) = 1$ *if and only if* $n = 2$ *or* $n = 3$.

(2) $\operatorname{rank}(S_r \times S_{n-r}) = 2$ *if and only if* $n \geq 4$.

**Lemma 6.2** *[1, Theorem 4.8] For $1 \leq r \leq n-1$,*

$$\operatorname{rank}\big(F_r(n)\big) = \begin{cases} 2 & \text{if } n \in \{2,3\}, \\ 3 & \text{if } n \geq 4. \end{cases}$$

Notice that if $|X \setminus Y| = 1$, then $PG_Y(X) = F_Y(X)$ by [8, Theorem 2.3]. So, $\operatorname{rank}\big(PG_r(n)\big) = \operatorname{rank}(F_r(n))$ when $n - r = 1$. Moreover, if $X = Y$, then $PG_Y(X) = G(X)$. Hence, $\operatorname{rank}\big(PG_r(n)\big) = \operatorname{rank}(S_n)$ when $n = r$.

Now, we consider the case $1 \leq r \leq n-2$ with $n \geq 3$.

It is easy to verify that $J(n) = G(X,Y)$, where $G(X,Y)$ is the group of units in $PG_r(n)$, see Proposition 2.1. Then by Remark 2.1, we obtain

$$J(n) = G(X,Y) \cong G(Y) \times G(X \setminus Y) = S_r \times S_{n-r}.$$

Therefore, $\operatorname{rank}\big(J(n)\big) = \operatorname{rank}(S_r \times S_{n-r})$.

Recall that if $Y_1$ and $Y_2$ are nonempty finite subsets of a set $X$ such that $|Y_1| = |Y_2|$, then $PG_{Y_1}(X) \cong PG_{Y_2}(X)$ by Corollary 5.4. Thus, from now on, we assume that

$$X = \{1,2,\ldots,n\} \text{ and } Y = \{1,2,\ldots,r\}.$$

**Lemma 6.3** *Let $r \leq k \leq n-2$ and $\alpha \in J(k)$. Then $\alpha = \lambda\delta$ for some $\lambda, \delta \in J(k+1)$.*

**Proof** We write

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r & B_{r+1} & B_{r+2} & \cdots & B_k \\ 1\sigma & 2\sigma & \cdots & r\sigma & b_{r+1} & b_{r+2} & \cdots & b_k \end{pmatrix}$$

for some $\sigma \in G(\{1,2,\ldots,r\}) = S_r$, where $A_i \cap Y = \{i\}$ for all $1 \le i \le r$; $b_j \in X \setminus Y$ and $B_j \subseteq X \setminus Y$ for all $r+1 \le j \le k$. Since $|X\alpha| = k \le n-2$, we have two distinct elements $u, v \in X \setminus Y$ such that $u, v \notin X\alpha$. Let us consider two cases: (i) $A_i \cap (X \setminus Y) = \emptyset$ for all $1 \le i \le r$; (ii) $A_j \cap (X \setminus Y) \neq \emptyset$ for some $j \in \{1,2,\ldots,r\}$.

In the former case, we write

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & r & B_{r+1} & B_{r+2} & \cdots & B_k \\ 1\sigma & 2\sigma & \cdots & r\sigma & b_{r+1} & b_{r+2} & \cdots & b_k \end{pmatrix}.$$

Since $|\pi_\alpha| = |X\alpha| = k < n$, $|B_t| \ge 2$ for some $t \in \{r+1, r+2, \ldots, k\}$. Let $x \in B_t$ and define

$$\lambda = \begin{pmatrix} 1 & \cdots & r & B_{r+1} & \cdots & B_{t-1} & B_t \setminus \{x\} & B_{t+1} & \cdots & B_k & x \\ 1\sigma & \cdots & r\sigma & b_{r+1} & \cdots & b_{t-1} & b_t & b_{t+1} & \cdots & b_k & u \end{pmatrix}$$

and

$$\delta = \begin{pmatrix} 1\sigma & \cdots & r\sigma & b_{r+1} & \cdots & b_{t-1} & \{b_t, u\} & b_{t+1} & \cdots & b_k & \{v\} \cup C \\ 1\sigma & \cdots & r\sigma & b_{r+1} & \cdots & b_{t-1} & b_t & b_{t+1} & \cdots & b_k & v \end{pmatrix}$$

where $C = X \setminus (X\alpha \cup \{u, v\})$. Clearly, $\lambda, \delta \in PG_r(n)$ and $\alpha = \lambda\delta$ such that $|X\lambda| = k+1 = |X\delta|$.

In the latter case we let $z \in A_j \cap (X \setminus Y)$ and define

$$\lambda = \begin{pmatrix} A_1 & \cdots & A_{j-1} & A_j \setminus \{z\} & A_{j+1} & \cdots & A_r & B_{r+1} & \cdots & B_k & z \\ 1\sigma & \cdots & (j-1)\sigma & j\sigma & (j+1)\sigma & \cdots & r\sigma & b_{r+1} & \cdots & b_k & u \end{pmatrix}$$

and

$$\delta = \begin{pmatrix} 1\sigma & \cdots & (j-1)\sigma & \{j\sigma, u\} & (j+1)\sigma & \cdots & r\sigma & b_{r+1} & \cdots & b_k & \{v\} \cup C \\ 1\sigma & \cdots & (j-1)\sigma & j\sigma & (j+1)\sigma & \cdots & r\sigma & b_{r+1} & \cdots & b_k & v \end{pmatrix}$$

where $C = X \setminus (X\alpha \cup \{u, v\})$. Then $\lambda, \delta \in J(k+1)$ and $\alpha = \lambda\delta$.

□

Inductive application of Lemma 6.3 yields the following corollary.

**Corollary 6.4** *For $r \le k \le n-1$, $Q(n;k) = \langle J(k) \rangle$.*

We now describe the partition of $X$ induced by transformation $\alpha$ in $J(n-1)$. Let

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_r & B_{r+1} & \cdots & B_{n-1} \\ 1\sigma & \cdots & r\sigma & b_{r+1} & \cdots & b_{n-1} \end{pmatrix} \in J(n-1)$$

where $\sigma \in S_r$, $A_i \cap Y = \{i\}$ for all $1 \le i \le r$; $b_j \in X \setminus Y$ and $B_j \subseteq X \setminus Y$ for all $r+1 \le j \le n-1$. Then either (i) there exists unique $t \in \{1,2,\ldots,r\}$ such that $|A_t| = 2$ and $|A_i| = 1 = |B_j|$ for all $1 \le i \le r, i \neq t$ and for all $r+1 \le j \le n-1$, or (ii) there exists unique $s \in \{r+1, r+2, \ldots, n-1\}$ such that $|B_s| = 2$ and $|A_i| = 1 = |B_j|$ for all $1 \le i \le r$ and for all $r+1 \le j \le n-1, j \neq s$. Indeed, we can write $A_t = \{x, y\}$ or $B_s = \{z, w\}$ where $y \in Y$ and $x, z, w \in X \setminus Y$.

**Remark 6.1** *For $\alpha \in J(n-1)$, if there are $x, y \in C$ for some $C \in \pi_\alpha$ such that $x \neq y$, then $C = \{x, y\}$ since $\pi_\alpha$ contains exactly one element of cardinality 2.*

Here, we define

$$P_1 = \{\alpha \in J(n-1) \colon \{x,y\} \in \pi_\alpha \text{ for some } y \in Y \text{ and } x \in X \setminus Y\},$$

and $$P_2 = \{\alpha \in J(n-1) \colon \{z,w\} \in \pi_\alpha \text{ for some } z,w \in X \setminus Y,\ z \neq w\}.$$

Notice that since $|X \setminus Y| \geq 2$, we have $P_1 \neq \emptyset \neq P_2$ and $J(n-1)$ is a disjoint union of $P_1$ and $P_2$. Moreover, $P_1 \subseteq F_r(n)$ and $P_2 \cap F_r(n) = \emptyset$.

**Lemma 6.5** *If $\alpha, \alpha\beta \in J(n-1)$, then $\pi_{\alpha\beta} = \pi_\alpha$.*

**Proof** Assume that $\alpha, \alpha\beta \in J(n-1)$. Let $C \in \pi_\alpha$ and $D \in \pi_{\alpha\beta}$ such that $|C| = 2 = |D|$. Let $x, y \in C$ such that $x \neq y$. Then $x\alpha = y\alpha$ and so $x\alpha\beta = y\alpha\beta$. Thus, $x, y \in D$ and hence $D = \{x,y\} = C$. Since $|\pi_\alpha| = n-1 = |\pi_{\alpha\beta}|$ and $|C'| = 1 = |D'|$ for all $C' \in \pi_\alpha \setminus \{C\}$, for all $D' \in \pi_{\alpha\beta} \setminus \{D\}$, it follows that $\pi_\alpha = \pi_{\alpha\beta}$. □

**Lemma 6.6** *Let $\alpha \in J(n)$ and $i \in \{1,2\}$. Then the following statements hold:*

(1) *If $\alpha\beta \in J(n-1)$ and $\beta \in P_i$, then $\alpha\beta \in P_i$;*

(2) *If $\beta\alpha \in J(n-1)$ and $\beta \in P_i$, then $\beta\alpha \in P_i$.*

**Proof** (i) Assume that $\alpha\beta \in J(n-1)$ and $\beta \in P_i$. For convenient, we write

$$\alpha = \begin{pmatrix} 1 & \cdots & r & r+1 & \cdots & n \\ 1\sigma & \cdots & r\sigma & (r+1)\delta & \cdots & (r+1)\delta \end{pmatrix} \in J(n)$$

for some $\sigma \in S_r$ and $\delta \in G(\{r+1,\ldots,n\}) = S_{n-r}$. If $\beta \in P_1$, then there exist $y \in Y$ and $x \in X \setminus Y$ such that $y\beta = x\beta$, that is, $\{x,y\} \in \pi_\beta$. Since $y \in Y$ and $x \in X \setminus Y$, there exist $y' \in Y$ and $x' \in X \setminus Y$ such that $y'\sigma = y$ and $x'\delta = x$. It follows that $y'(\alpha\beta) = (y'\alpha)\beta = (y'\sigma)\beta = y\beta = x\beta = (x'\delta)\beta = (x'\alpha)\beta = x'(\alpha\beta)$. Hence, $\{x', y'\} \in \pi_{\alpha\beta}$ and so $\alpha\beta \in P_1$. Now, suppose that $\beta \in P_2$. Then there exist $z, w \in X \setminus Y$ such that $z \neq w$ and $z\beta = w\beta$. Since $z, w \in X \setminus Y$, there exist $z', w' \in X \setminus Y$ such that $z' \neq w', z'\delta = z$ and $w'\delta = w$. This implies that $z'(\alpha\beta) = (z'\alpha)\beta = (z'\delta)\beta = z\beta = w\beta = (w'\delta)\beta = (w'\alpha)\beta = w'(\alpha\beta)$. Thus, $\{z', w'\} \in \pi_{\alpha\beta}$ and that $\alpha\beta \in P_2$.

(ii) Assume that $\beta\alpha \in J(n-1)$ and $\beta \in P_i$. Since $\beta, \beta\alpha \in J(n-1)$, we have $\pi_{\beta\alpha} = \pi_\beta$ by Lemma 6.5. This implies that $\beta\alpha \in P_i$. □

**Lemma 6.7** *Let $A$ be a generating set of $PG_r(n)$. Then $A \cap P_i \neq \emptyset$ for all $i \in \{1,2\}$.*

**Proof** Let $i \in \{1,2\}$ and $\alpha \in P_i$. Then $\alpha \in J(n-1)$ and $\alpha = \alpha_1\alpha_2\cdots\alpha_k$ where $\alpha_1, \alpha_2, \ldots, \alpha_k \in A$. If $\alpha_1, \alpha_2, \ldots, \alpha_k \in J(n)$, then $\alpha = \alpha_1\alpha_2\cdots\alpha_k \in J(n)$ since $J(n)$ is a subgroup of $PG_r(n)$, which is a contradiction. Thus, there exists $\alpha_t \in A$ such that $\alpha_t \notin J(n)$, that is $\alpha_t \in Q(n;n-1)$. If $\alpha_t \in Q(n;n-2)$, we obtain $\alpha = \alpha_1\alpha_2\cdots\alpha_k \in Q(n;n-2)$ since $Q(n;n-2)$ is an ideal of $PG_r(n)$, a contradiction. Hence, $\alpha_t \in J(n-1)$. We may assume that $t$ is the least integer among $1, 2, \ldots, k$ in which $\alpha_t \in J(n-1)$, this means $\alpha_1, \alpha_2, \ldots, \alpha_{t-1} \in J(n)$. Now, we write $\alpha = \gamma\alpha_t\lambda$ where $\gamma = \alpha_1\alpha_2\cdots\alpha_{t-1} \in J(n)$ and $\lambda =$

$\alpha_{t+1}\alpha_{t+2}\cdots\alpha_k \in J(n) \cup J(n-1)$. We note that $\gamma\alpha_t \in Q(n;n-1)$ since $\alpha_t \in J(n-1)$. If $\gamma\alpha_t \in Q(n;n-2)$, then $\alpha = (\gamma\alpha_t)\lambda \in Q(n;n-2)$, this is a contradiction. So, $\gamma\alpha_t \in J(n-1)$. We consider two cases.

**Case 1**: $\lambda \in J(n)$. Assume that $\alpha_t \in P_j$ where $j \in \{1,2\} \setminus \{i\}$. Then by Lemma 6.6 (1) we get that $\gamma\alpha_t \in P_j$. This implies that $\alpha = (\gamma\alpha_t)\lambda \in P_j$ by Lemma 6.6 (2), this contradicts the fact that $\alpha \in P_i$. Thus, $\alpha_t \in P_i$.

**Case 2**: $\lambda \in J(n-1)$. Since $\gamma\alpha_t, (\gamma\alpha_t)\lambda \in J(n-1)$, it follows from Lemma 6.5 that $\pi_{\gamma\alpha_t} = \pi_{(\gamma\alpha_t)\lambda} = \pi_\alpha$. Hence, $\gamma\alpha_t \in P_i \subseteq J(n-1)$ since $\alpha \in P_i$. If $\alpha_t \in P_j$ where $j \in \{1,2\} \setminus \{i\}$. Then by Lemma 6.6 (1) we get that $\gamma\alpha_t \in P_j$, a contradiction. So, $\alpha_t \in P_i$.

In both cases, we have $\alpha_t \in A \cap P_i$. Therefore, $A \cap P_i \neq \emptyset$. □

Since $r \leq n-2$, $PG_r(n) \neq J(n) = \langle J(n) \rangle$. Moreover, an element of $J(n)$ can not be written as a product of some elements of $Q(n;n-1)$ since $Q(n;n-1)$ is an ideal. It is clear therefore that any generating set of $PG_r(n)$ must contain a generating set of $J(n)$. Then by Lemma 6.7, we obtain

$$\operatorname{rank}\big(PG_r(n)\big) \geq \operatorname{rank}\big(J(n)\big) + 2 = \operatorname{rank}(S_r \times S_{n-r}) + 2. \tag{6.1}$$

We note that $P_1 \cup J(n) \subseteq F_r(n)$. The following lemma follows immediately from [1, Lemma 4.7].

**Lemma 6.8** *Let $\xi$ be any an element of $P_1$. If $\alpha \in P_1$, then $\alpha = \lambda\xi\mu$ for some $\lambda, \mu \in J(n)$.*

**Lemma 6.9** *Let $\zeta$ be any an element of $P_2$. If $\alpha \in P_2$, then $\alpha = \lambda\zeta\mu$ for some $\lambda, \mu \in J(n)$.*

**Proof** Assume that $\alpha \in P_2$. We write

$$\zeta = \begin{pmatrix} 1 & \cdots & r & \{t,u\} & y_i \\ 1\sigma & \cdots & r\sigma & b & y_i\zeta \end{pmatrix} \text{ and } \alpha = \begin{pmatrix} 1 & \cdots & r & \{v,w\} & x_i \\ 1\delta & \cdots & r\delta & c & x_i\alpha \end{pmatrix}$$

where $\sigma, \delta \in S_r; t,u,v,w,b,c \in X \setminus Y; y_i \in (X \setminus Y) \setminus \{t,u\}; x_i \in (X \setminus Y) \setminus \{v,w\}; y_i\zeta \in (X \setminus Y) \setminus \{b\}$ and $x_i\alpha \in (X \setminus Y) \setminus \{c\}$ for all $1 \leq i \leq n-r-2$. Define

$$\lambda = \begin{pmatrix} 1 & \cdots & r & v & w & x_i \\ 1 & \cdots & r & t & u & y_i \end{pmatrix} \text{ and } \mu = \begin{pmatrix} 1\sigma & \cdots & r\sigma & b & y_i\zeta & z \\ 1\delta & \cdots & r\delta & c & x_i\alpha & z' \end{pmatrix}$$

where $z \in X \setminus X\zeta$ and $z' \in X \setminus X\alpha$. Then $\lambda, \mu \in J(n)$ and $\alpha = \lambda\zeta\mu$. □

**Corollary 6.10** *Let $n \geq 4$ and $J(n) = \langle \mu, \rho \rangle$. Then $\{\mu, \rho, \xi, \zeta\}$ is a generating set of $PG_r(n)$ where $\xi$ and $\zeta$ are any elements of $P_1$ and $P_2$, respectively.*

**Proof** We let $\xi$ and $\zeta$ be elements of $P_1$ and $P_2$, respectively. Then by Lemmas 6.8,6.9, we obtain $J(n-1) \subseteq \langle J(n) \cup \{\xi, \zeta\} \rangle = \langle \mu, \rho, \xi, \zeta \rangle$. It follows that $\langle J(n-1) \rangle \subseteq \langle \mu, \rho, \xi, \zeta \rangle$. By Corollary 6.4, we have $Q(n;n-1) = \langle J(n-1) \rangle$. This implies that

$$PG_r(n) = Q(n;n-1) \cup J(n) = \langle J(n-1) \rangle \cup \langle \mu, \rho \rangle \subseteq \langle \mu, \rho, \xi, \zeta \rangle \subseteq PG_r(n).$$

Hence, $PG_r(n) = \langle \mu, \rho, \xi, \zeta \rangle$. □

**Remark 6.2** *For the case $n=3$, we have $r=1$ and $J(n)=J(3)=\langle \left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right) \rangle$. Then $\{ \left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right), \xi, \zeta \}$ is a generating set of $PG_1(3)$ where $\xi$ and $\zeta$ are any elements of $P_1$ and $P_2$, respectively.*

**Theorem 6.11** *For $1 \leq r \leq n-2$,*

$$\operatorname{rank}\big(PG_r(n)\big) = \begin{cases} 3 & \text{if } n=3, \\ 4 & \text{if } n \geq 4. \end{cases}$$

**Proof** It is known that $\operatorname{rank}\big(PG_r(n)\big) \geq \operatorname{rank}(S_r \times S_{n-r}) + 2$ by (6.1). Then by Lemma 6.1, we get that $\operatorname{rank}\big(PG_r(n)\big) \geq 3$ if $n=3$, and $\operatorname{rank}\big(PG_r(n)\big) \geq 4$ if $n \geq 4$. An immediate consequence of Remark 6.2 and Corollary 6.10 is that $\operatorname{rank}\big(PG_r(n)\big) = 3$ if $n=3$, and $\operatorname{rank}\big(PG_r(n)\big) = 4$ if $n \geq 4$, as required. □


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