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ABDELALI SAIDI

MOULAY BRAHIM SEDRA

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The Hirota–Satsuma coupled KdV hierarchy on noncommutative phase-space

Abdelali SAIDI^{1*}, Moulay Brahim SEDRA^{1,2}

¹LaboSimo, Department of Physics, Faculty of Sciences, Ibn Tofail University, Kenitra, Morocco

²FST, Moulay Ismail University, Errachidia, Morocco

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Abstract: In the present contribution, the Hirota–Satsuma coupled KdV hierarchy on noncommutative phase-space is investigated using the noncommutative extension of Lax pair generating technique. In particular, the explicit representation of its associated Lax pair operators is constructed. It is shown that the obtained results for phase-space noncommutativity case reduce back to the standard commutative case when θ goes to $\frac{1}{2}$.

Key words: Noncommutative geometry, Hirota–Satsuma coupled KdV system, Moyal deformation

1. Introduction

The following system of equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} \frac{1}{2}u_{xxx} + 3uu_x - 6vv_x \\ -v_{xxx} - 3uv_x \end{pmatrix}, \quad (1.1)$$

is known as the Hirota–Satsuma coupled KdV system (HS-coupled KdV) proposed by Hirota and Satsuma [1], which describes interactions of two long waves with different dispersion relations. In [2], Hirota and Satsuma showed that the HS-coupled KdV system is the four-reduction of the KP hierarchy and its soliton solutions can be derived from those of the KP equation. The Lax representation of this system has been constructed in [3] by Dodd and Fordy. Different properties of the HS-coupled KdV system have been constructed such as Backlund transformation [4], Darboux transformation [5–7], bilinear form [1, 8], and Painleve property [9, 10]

More recently, the noncommutative geometry (NCG) [11, 12] has attracted attention from many researchers in physics [13–25]. On noncommutative (NC) spaces, the noncommutativity of the coordinates is defined as follows

$$[x^i, x^j] = i\theta^{ij}, \quad (1.2)$$

where x^i, x^j are noncommuting coordinates, and θ^{ij} are real constants called *the NC parameters*.

On the other hand, the multiplication of two arbitrary functions $f(x, p)$ and $g(x, p)$ on the two dimensional NC phase-space has been shown to satisfy the star product law defined as follows: [20–25]

$$f(x, p) \star g(x, p) = \sum_{s=0}^{\infty} \sum_{i=0}^s \frac{\theta^s}{s!} (-)^i c_s^i (\partial_x^i \partial_p^{s-i} f) (\partial_x^{s-i} \partial_p^i g), \quad (1.3)$$

*Correspondence: saidi220@gmail.com

with $c_s^i = \frac{s!}{i!(s-i)!}$.

In this context, on NC spaces we define the Moyal bracket given by the following expression [20–25]:

$$\{ f(x, p), g(x, p) \}_\theta = \frac{f \star g - g \star f}{2\theta}. \tag{1.4}$$

In the following section, we make a short review of the noncommutative Lax pair generating technique [26–28] which is relevant for this work. This technique will be useful to derive the noncommutative extension of the HS-coupled KdV hierarchy and obtain its associated Lax pair operators. A more detailed analysis in the commutative case can be found in [29–32].

2. Noncommutative Hirota–Satsuma coupled KdV hierarchy

An NC partial differential equation which has the Lax representation can be reformulated as follows:

$$\{L, T + \partial_t\}_\theta = 0. \tag{2.1}$$

This equation and the associated pair of operators (L, T) are called *the NC Lax differential equation* and *the Lax pair*, respectively, with $\partial_t \equiv \frac{\partial}{\partial t}$.

The main idea of the NC Lax pair generating technique is to find a T- operator satisfying (2.1) for a given L- operator. For this we have to consider the following ansatz on the T- operator, namely

$$T = p^n \star L + \tilde{T}, \tag{2.2}$$

where p^n are nothing but the momentum operators. With this last ansatz, the problem reduces to find the \tilde{T} - operator.

Now, let us recall that the momentum Lax operator of the θ - deformed HS-coupled KdV hierarchy is defined as [3]

$$\begin{aligned} L &= (p^2 + u + v) \star (p^2 + u - v) \\ &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \end{aligned} \tag{2.3}$$

The T- operator can be written as follows:

$$T = p^{2n-3} \star L + \tilde{T}. \tag{2.4}$$

Straightforward algebraic computations based on the NC Lax pair generating technique lead to the explicit forms of the set of Moyal HS-coupled KdV hierarchy. The results are as follows:

- For $n = 0$, the NC Lax differential equation (2.1) leads the 1st-order NC HS-coupled KdV equations.

The NC Lax pair is given by

$$\begin{aligned} L &= (p^2 + u + v) \star (p^2 + u - v) \\ &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \end{aligned} \tag{2.5}$$

and

$$T = p^{-3} \star L + \tilde{T} \tag{2.6}$$

From the NC Lax differential equation (2.1), one finds

$$-\{L, \tilde{T}\}_\theta = (-2u_x - \frac{\dot{u}}{2\theta})p^2 + (4\theta v_{xx} + \dot{v}_x)p - 2\theta^2 u_{xxx} - 2uu_x + 2vv_x - \frac{1}{2}\theta \dot{u}_{xx} - \frac{u\dot{u}}{2\theta} + \frac{v\dot{v}}{2\theta}, \tag{2.7}$$

with the following system of 1st-order NC HS-coupled KdV equations

$$-\frac{\dot{u}}{2\theta} = 2u_x, \tag{2.8}$$

$$-\frac{\dot{v}}{2\theta} = 2v_x, \tag{2.9}$$

where $\dot{u} \equiv \frac{\partial u}{\partial t}$, $u_x \equiv \frac{\partial u}{\partial x}$, $u_{xx} \equiv \frac{\partial^2 u}{\partial x^2}$ and so on.

Hence, the NC Lax pair for the 1st-order NC HS-coupled KdV equations is explicitly given by

$$\begin{cases} L = p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \\ T = p \end{cases} \tag{2.10}$$

- For $n = 1$, the NC Lax differential equation (2.1) represents the 3rd-order NC HS-coupled KdV equations.

The NC Lax pair is given by

$$\begin{aligned} L &= (p^2 + u + v) \star (p^2 + u - v) \\ &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \end{aligned} \tag{2.11}$$

and

$$T = p^{-1} \star L + \tilde{T} \tag{2.12}$$

Then, we can take the \tilde{T} - operator as the following form

$$\tilde{T} = A.p + B, \tag{2.13}$$

where A and B are polynomials of $u, u_x, u_{xx} \dots etc.$ The NC Lax differential equation (2.1) takes the form

$$\begin{aligned} -\{L, \tilde{T}\}_\theta &= 2u_x p^4 - (8\theta u_{xx} + 4\theta v_{xx})p^3 + (6vv_x - 2uu_x - \frac{\dot{u}}{2\theta})p^2 - (8\theta^3 u_{4x} + 12\theta^3 v_{4x} \\ &\quad + 8\theta uu_{xx} + 8\theta uv_{xx} + 8\theta u_x v_x + \dot{v}_x)p - 2\theta^4 u_{5x} - 6\theta^2 uu_{3x} + 8\theta^2 v_x u_{xx} \\ &\quad + 22\theta^2 v_x v_{xx} - 10\theta^2 u_x u_{xx} + 2\theta^2 vv_{3x} - 4u_x u^2 + 4uvv_x - \frac{1}{2}\theta \dot{u}_{xx} - \frac{u\dot{u}}{2\theta} + \frac{v\dot{v}}{2\theta}. \end{aligned} \tag{2.14}$$

The left-hand side of Eq. (2.14) is explicitly given by

$$\begin{aligned}
 -\{L, \tilde{T}\}_\theta &= -4A_x p^4 - 4B_x p^3 + (2Au_x - 4uA_x - 4\theta^2 A_{xxx})p^2 + (4\theta A_x v_x - 4\theta A v_{xx} - 4\theta^2 B_{xxx} \\
 &\quad - 4uB_x)p + 2\theta^2 Au_{xxx} + 2\theta^2 u_x A_{xx} + 2Auu_x - 2Avv_x + 4\theta B_x v_x
 \end{aligned} \tag{2.15}$$

Identifying Eqs. (2.14) and (2.15) leads to the following constrains equations:

$$A = -\frac{1}{2}u \quad ; \quad B = 2\theta u_x + \theta v_x \tag{2.16}$$

Hence, Eq. (2.13) becomes

$$\tilde{T} = -\frac{1}{2}up + 2\theta u_x + \theta v_x \tag{2.17}$$

Furthermore, one obtains the following system of 3rd-order NC HS-coupled KdV equations:

$$-\frac{\dot{u}}{2\theta} = 2\theta^2 u_{xxx} + 3uu_x - 6vv_x \tag{2.18}$$

$$-\frac{\dot{v}}{2\theta} = -4\theta^2 v_{xxx} - 3uv_x \tag{2.19}$$

This coincides with the standard HS-coupled KdV equations given in Eq. (1.1) once the limit $\theta \rightarrow \frac{1}{2}$ and the scaling transformation $-\frac{1}{2\theta} \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t}$ are performed. Consequently, the NC Lax pair associated to the θ -deformed 3rd-order HS-coupled KdV equations is explicitly given by

$$\begin{cases} L = p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \\ T = p^3 + \frac{3}{2}up - 3\theta v_x \end{cases} \tag{2.20}$$

Let us note that the same analysis used for $n = 0$ and $n = 1$ is actually extended to build the 5th-order, 7th-order and the 9th-order NC HS-coupled KdV equations and their associated Lax pair operators.

- Now, for $n = 2$, the NC Lax differential equation given in Eq. (2.1) yields the 5th-order NC HS-coupled KdV equations.

The NC Lax pair is given by

$$\begin{aligned}
 L &= (p^2 + u + v) \star (p^2 + u - v) \\
 &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2
 \end{aligned} \tag{2.21}$$

and

$$T = p \star L + \tilde{T} \tag{2.22}$$

From Eq. (2.1), the \tilde{T} - operator is explicitly given by

$$\tilde{T} = \frac{1}{2}up^3 - \theta(2u_x + v_x)p^2 + (4\theta^2 v_{xx} + 3\theta^2 u_{xx} + \frac{7}{8}u^2 - \frac{1}{4}v^2)p - \frac{5}{2}\theta uv_x - 2\theta uu_x - 2\theta^3 u_{xxx} + 2\theta vv_x - 5\theta^3 v_{xxx} \tag{2.23}$$

After calculation, we find the following system of 5th-order NC HS-coupled KdV equations:

$$-\frac{\dot{u}}{2\theta} = 2\theta^4 u_{5x} - 10\theta^2 v v_{xxx} + 5\theta^2 u u_{xxx} + 10\theta^2 u_x u_{xx} - 10\theta^2 v_x v_{xx} + \frac{15}{4} u_x u^2 - 5u v v_x - \frac{5}{2} u_x v^2 \quad (2.24)$$

$$-\frac{\dot{v}}{2\theta} = -8\theta^4 v_{5x} - 10\theta^2 u v_{xxx} - 10\theta^2 u_x v_{xx} - 5\theta^2 v_x u_{xx} - \frac{5}{4} v_x u^2 - \frac{5}{2} v_x v^2 \quad (2.25)$$

Hence, the NC Lax pair for the 5th-order NC HS-coupled KdV equations is explicitly given by

$$\begin{cases} L = p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \\ T = p^5 + \frac{5}{2} up^3 - 5\theta v_x p^2 + (5\theta^2 u_{xx} + \frac{15}{8} u^2 - \frac{5}{4} v^2) p - 5\theta^3 v_{xxx} - \frac{5}{2} \theta u v_x. \end{cases} \quad (2.26)$$

- Similarly, for $n = 3$, the NC Lax differential equation given in Eq. (2.1) becomes the 7th-order NC HS-coupled KdV equations.

The NC Lax pair is given by

$$\begin{aligned} L &= (p^2 + u + v) \star (p^2 + u - v) \\ &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \end{aligned} \quad (2.27)$$

and

$$T = p^3 \star L + \tilde{T}. \quad (2.28)$$

The \tilde{T} - operator is explicitly given by

$$\begin{aligned} \tilde{T} &= \frac{3}{2} up^5 - 3\theta(v_x + 2u_x)p^4 + (\frac{19}{2}\theta^2 u_{xx} + 12\theta^2 v_{xx} + \frac{27}{8}u^2 - \frac{3}{4}v^2)p^3 + (6\theta v v_x - 8\theta^3 u_{xxx} - 16\theta^3 v_{xxx} \\ &\quad - \frac{21}{2}\theta u v_x - 6\theta u u_x)p^2 + (4\theta^4 v_{4x} + \frac{9}{2}\theta^4 u_{4x} + \frac{5}{2}\theta^2 v_x^2 - 8\theta^2 v v_{xx} + \frac{23}{2}\theta^2 u u_{xx} - \frac{13}{8}\theta^2 u_x^2 \\ &\quad + \frac{35}{16}u^3 - \frac{21}{8}u v^2)p + 7\theta^5 v_{5x} - 2\theta^5 u_{5x} + 2\theta^3 v v_{xxx} + 6\theta^3 v_x v_{xx} + \frac{21}{4}\theta v_x v^2 - 2\theta^3 u u_{xxx} \\ &\quad - 6\theta^3 u_x u_{xx} + 21\theta^3 u_x v_{xx} - \frac{21}{8}\theta v_x u^2 + \frac{7}{2}\theta^3 u v_{xxx}. \end{aligned} \quad (2.29)$$

Finally, we get the following system of 7th-order NC HS-coupled KdV equations:

$$\begin{aligned} -\frac{\dot{u}}{2\theta} &= 2\theta^6 u_{7x} + 14\theta^4 v v_{5x} + 7\theta^4 u u_{5x} + 21\theta^4 u_x u_{4x} - 14\theta^4 v_x v_{4x} + 35\theta^4 u_{xx} u_{xxx} - \frac{21}{4} u u_x v^2 - \frac{21}{4} v v_x u^2 \\ &\quad - \frac{7}{2}\theta^2 u_{xxx} v^2 - 56\theta^4 v_{xx} v_{xxx} + \frac{35}{4}\theta^2 u_{xxx} u^2 + 14\theta^2 v u_x v_{xx} - 21\theta^2 u v_x v_{xx} + 35\theta^2 u u_x u_{xx} \\ &\quad + 7\theta^2 u v v_{xxx} - 7\theta^2 u_x v_x^2 + \frac{21}{2} v_x v^3 + \frac{35}{4}\theta^2 u_x^3 + \frac{35}{8} u_x u^3 \end{aligned} \quad (2.30)$$

$$\begin{aligned} -\frac{\dot{v}}{2\theta} &= 16\theta^6 v_{7x} + 28\theta^4 u v_{5x} + 56\theta^4 u_x v_{4x} + 7\theta^4 v_x u_{4x} + 42\theta^4 v_{xx} u_{xxx} + 70\theta^4 u_{xx} v_{xxx} + \frac{21}{2}\theta^2 v_{xxx} u^2 \\ &\quad + 21\theta^2 u u_x v_{xx} + \frac{7}{2}\theta^2 u v_x u_{xx} - \frac{7}{4}\theta^2 v_x u_x^2 + 14\theta^2 v_x^3 + 35\theta^2 v v_x v_{xx} + 7\theta^2 v_{xxx} v^2 \\ &\quad - \frac{7}{8} v_x u^3 + \frac{21}{4} u v_x v^2. \end{aligned} \quad (2.31)$$

Consequently, the NC Lax pair for the 7th-order NC HS-coupled KdV equations is explicitly given by

$$\left\{ \begin{aligned} L &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \\ T &= p^7 + \frac{7}{2}up^5 - 7\theta v_x p^4 + \left(\frac{35}{2}\theta^2 u_{xx} + \frac{35}{8}u^2 - \frac{7}{4}v^2\right)p^3 - (28\theta^3 v_{xxx} + \frac{21}{2}\theta uv_x)p^2 \\ &\quad + \left(\frac{21}{2}\theta^4 u_{4x} - \frac{7}{2}\theta^2 v_x^2 + \frac{35}{8}\theta^2 u_x^2 + \frac{35}{2}\theta^2 uu_{xx} - 14\theta^2 vv_{xx} - \frac{21}{8}uv^2 + \frac{35}{16}u^3\right)p \\ &\quad + \frac{7}{2}\theta^3 uv_{xxx} + 21\theta^3 u_x v_{xx} + 7\theta^5 v_{5x} + \frac{21}{4}\theta v_x v^2 - \frac{21}{8}\theta v_x u^2 \end{aligned} \right. \quad (2.32)$$

- For $n = 4$, the NC Lax differential equation given in Eq. (2.1) is the 9th-order NC HS-coupled KdV equations.

The NC Lax pair is given by

$$\begin{aligned} L &= (p^2 + u + v) \star (p^2 + u - v) \\ &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \end{aligned} \quad (2.33)$$

and

$$T = p^5 \star L + \tilde{T}. \quad (2.34)$$

The 9th-order NC HS-coupled KdV equations are explicitly given by

$$\begin{aligned} -\frac{\dot{u}}{2\theta} &= 2\theta^8 u_{9x} + 45\theta^2 vv_x^3 - \frac{135}{16}v^2 u_x u^2 + \frac{45}{4}uv_x v^3 - \frac{45}{8}vv_x u^3 + 9\theta^6 uu_{7x} - 18\theta^6 v_x v_{6x} \\ &\quad + \frac{105}{8}\theta^2 u_{xxx} u^3 + 84\theta^6 u_{xx} u_{5x} + 126\theta^6 u_{xxx} u_{4x} + \frac{63}{4}\theta^4 u_{5x} u^2 - 102\theta^6 v_{xx} v_{5x} \\ &\quad + \frac{483}{4}\theta^4 u_{xxx} u_x^2 - 30\theta^4 u_{xxx} v_x^2 + 36\theta^6 u_x u_{6x} - \frac{9}{2}\theta^4 u_{5x} v^2 + \frac{651}{4}\theta^4 u_x u_{xx}^2 \\ &\quad - 150\theta^6 v_{xxx} v_{4x} - 93\theta^4 u_x v_{xx}^2 + 30\theta^6 vv_{7x} + \frac{45}{2}\theta^2 v_{xxx} v^3 + \frac{45}{16}u_x v^4 \\ &\quad + \frac{315}{64}u_x u^4 - \frac{15}{4}\theta^2 vv_{3x} u^2 - \frac{45}{4}\theta^2 uu_{xxx} v^2 - \frac{135}{4}\theta^2 v_x v_{xx} u^2 + \frac{315}{2}\theta^4 uu_{xx} u_{xxx} \\ &\quad - 120\theta^4 uv_{xx} v_{xxx} + \frac{225}{2}\theta^2 v_x v_{xx} v^2 - 45\theta^4 uv_x v_{4x} + \frac{189}{2}\theta^4 uu_x u_{4x} - 84\theta^4 u_x v_x v_{xxx} \\ &\quad + 33\theta^4 uvv_{5x} - 3\theta^4 vv_x u_{4x} + \frac{315}{4}\theta^2 u_x u_{xx} u^2 + 105\theta^4 vv_{xxx} u_{xx} + 45\theta^4 vv_{xx} u_{xxx} \\ &\quad - 105\theta^4 v_x v_{xx} u_{xx} + 99\theta^4 vv_x v_{4x} - \frac{45}{2}\theta^2 u_x u_{xx} v^2 - \frac{75}{2}\theta^2 uu_x v_x^2 - \frac{75}{4}\theta^2 vv_x u_x^2 \\ &\quad - 30\theta^2 uu_{xx} vv_x - 15\theta^2 uu_x vv_{xx} + \frac{315}{8}\theta^2 uu_x^3 \end{aligned} \quad (2.35)$$

$$\begin{aligned}
 -\frac{\dot{v}}{2\theta} = & 32\theta^8 v_{9x} + \frac{195}{2}\theta^2 uvv_x v_{xx} + 105\theta^4 uu_{xxx} v_{xx} + 240\theta^4 vv_{xx} v_{xxx} + 120\theta^4 u_x u_{xx} v_{xx} \\
 & + 180\theta^4 uu_x v_{4x} + \frac{33}{2}\theta^4 uu_{4x} v_x + 12\theta^4 v_x u_x u_{xxx} + \frac{45}{2}\theta^2 uv_{xxx} v^2 + \frac{45}{2}\theta^2 u_x v_{xx} v^2 \\
 & + \frac{45}{4}\theta^2 u_x v_{xx} u^2 + \frac{45}{4}\theta^2 u_{xx} v_x v^2 - \frac{15}{8}\theta^2 u_{xx} v_x u^2 + 195\theta^4 uu_{xx} v_{xxx} + 129\theta^4 vv_x v_{4x} \\
 & - \frac{45}{64}v_x u^4 + \frac{45}{16}v_x v^4 + 114\theta^6 v_{xx} u_{5x} + 480\theta^6 u_{xxx} v_{4x} + 216\theta^6 u_x v_{6x} + 15\theta^6 v_x u_{6x} \\
 & + 72\theta^6 uv_{7x} + \frac{21}{4}\theta^4 v_x u_{xx}^2 + 18\theta^4 v_{5x} v^2 + \frac{195}{2}\theta^4 v_{xxx} u_x^2 + 318\theta^6 v_{xxx} u_{4x} + 246\theta^4 v_{xxx} v_x^2 \\
 & + 357\theta^4 v_x v_{xx}^2 + 420\theta^6 u_{xx} v_{5x} - \frac{45}{8}\theta^2 uvv_x u_x^2 + \frac{75}{2}\theta^2 vv_x v_x^2 + 30\theta^2 uv_x^3 + \frac{45}{16}v_x u^2 v^2 \\
 & + 45\theta^4 v_{5x} u^2 + \frac{15}{4}\theta^2 v_{xxx} u^3
 \end{aligned} \tag{2.36}$$

Finally, the NC Lax pair for the 9th-order NC HS-coupled KdV equations is explicitly given by

$$\left\{ \begin{aligned}
 L &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \\
 T &= p^9 + \frac{9}{2}up^7 - 9\theta v_x p^6 + (42\theta^2 u_{xx} + \frac{63}{8}u^2 - \frac{9}{4}v^2)p^5 - (\frac{45}{2}\theta uv_x + 75\theta^3 v_{xxx})p^4 + (\frac{105}{16}u^3 \\
 &+ \frac{105}{8}\theta^2 u_x^2 + \frac{315}{4}\theta^2 uu_{xx} + 63\theta^4 u_{4x} - \frac{45}{8}uv^2 - 15\theta^2 v_x^2 - \frac{75}{2}\theta^2 vv_{xx})p^3 + (\frac{45}{4}\theta v_x v^2 \\
 &- 51\theta^5 v_{5x} - \frac{105}{2}\theta^3 v_x u_{xx} + 45\theta^3 u_x v_{xx} - 60\theta^3 uv_{xxx} - \frac{135}{8}\theta v_x u^2)p^2 + (\frac{315}{16}\theta^2 uu_x^2 \\
 &+ \frac{315}{128}u^4 + \frac{45}{32}v^4 - \frac{75}{4}\theta^2 uv_x^2 - 30\theta^2 uvv_{xx} - \frac{15}{4}\theta^2 u_x vv_x - \frac{45}{4}\theta^2 u_{xx} v^2 + \frac{315}{8}\theta^2 u_{xx} u^2 \\
 &+ \frac{189}{4}\theta^4 u_x u_{xxx} + \frac{189}{4}\theta^4 uu_{4x} - 42\theta^4 v_x v_{xxx} - \frac{51}{2}\theta^4 vv_{4x} - \frac{135}{32}u^2 v^2 + \frac{651}{8}\theta^4 u_x^2 \\
 &+ 18\theta^6 u_{6x} - \frac{183}{2}\theta^4 v_x^2)p - 15\theta^3 uu_{xx} v_x + \frac{45}{2}\theta^3 uu_x v_{xx} + 45\theta^3 vv_x v_{xx} + \frac{45}{8}\theta uv_x v^2 \\
 &- \frac{45}{8}\theta^3 v_x u_x^2 + \frac{105}{2}\theta^5 u_{xx} v_{xxx} - \frac{3}{2}\theta^5 v_x u_{4x} + \frac{33}{2}\theta^5 uv_{5x} + 75\theta^5 u_x v_{4x} + 60\theta^5 v_{xx} u_{xxx} \\
 &- \frac{15}{8}\theta^3 v_{xxx} u^2 + \frac{45}{4}\theta^3 v_{xxx} v^2 - \frac{45}{16}\theta v_x u^3 + 15\theta^7 v_{7x} + \frac{45}{2}\theta^3 v_x^3
 \end{aligned} \right. \tag{2.37}$$

3. Conclusion

In commutative case, it is well known that the existence of Lax pair operators (L, T) for nonlinear differential equations is a strong indication of integrability, that is, the existence of a complete set of conservation laws, of multisoliton solutions and so on. In this context, many efforts have been devoted to extend the Lax pair operators for several $(1 + 1)$ and $(2 + 1)$ -dimensional integrable systems on noncommutative spaces [24–28]. The noncommutative extension of Lax pair generating technique is one of the most powerful methods to do so.

In this paper, we have presented a systematic study of HS-coupled KdV hierarchy on noncommutative phase-space by using the noncommutative Lax pair generating technique. We have obtained a set of higher-order noncommutative HS-coupled KdV equations which are also integrable equations. This integrability is due to

the existence of a noncommutative Lax pair operators (L, T) which is explicitly derived. Also, we have shown that the obtained results reduce to the commutative case once the limit $\theta \rightarrow \frac{1}{2}$ and the scaling transformation $-\frac{1}{2\theta} \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t}$ are considered.

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