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## Logarithmic dimension and bases in Whitney spaces

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**Abstract:** We give a formula for the logarithmic dimension of the generalized Cantor-type set  $K$ . In the case when the logarithmic dimension of  $K$  is smaller than 1, we construct a Faber basis in the space of Whitney functions  $\mathcal{E}(K)$ .

**Key words:** Topological bases, Whitney spaces, Hausdorff dimension, logarithmic capacity

### 1. Introduction

This paper is the extension of [2] and [12]. In [2], the logarithmic dimension  $\lambda_0$  was suggested as the Hausdorff dimension corresponding to the function  $\psi(r) = \frac{1}{\log \frac{1}{r}}$  that defines the logarithmic measure. Some applications of the logarithmic dimension to the isomorphic classification of Whitney spaces were presented. In [12], the first author constructed bases in the spaces  $\mathcal{E}(K_2^{(\alpha_n)})$ , where the set  $K_2^{(\alpha_n)}$  is obtained by the Cantor procedure with replacing each interval by two adjacent subintervals of equal length. Here, as in [2], we consider more general Cantor-type sets  $K_{(N_n)}^{(\alpha_n)}$ , see the definition below. In Section 2, we generalize Proposition 1 from [2], where the logarithmic dimension was calculated for regular  $K_{(N_n)}^{(\alpha_n)}$ . In Sections 3 we discuss applications of the logarithmic dimension to potential theory and to analysis of linear topological properties of Whitney spaces. Section 4 is devoted to construction of an interpolating Faber basis in  $\mathcal{E}(K_{(N_n)}^{(\alpha_n)})$  provided  $\lambda_0(K_{(N_n)}^{(\alpha_n)}) < 1$ .

### 2. Logarithmic dimension for the generalized Cantor-type sets

Recall that a function  $\varphi : (0, b] \rightarrow (0, \infty)$ , where  $b = b_\varphi > 0$ , is said to be a dimension function if it is nondecreasing, continuous and  $\varphi(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . Given  $A \subset \mathbb{R}$ ,  $\varepsilon > 0$ , let  $\mu_\varepsilon(A, \varphi) = \inf\{\sum \varphi(\delta_i) : A \subset \cup G_i \text{ with } \text{diam}(G_i) = \delta_i \leq \varepsilon\}$ . Here, the infimum can be taken over open coverings or closed coverings without changing the result. The value  $\mu_\varepsilon(A, \varphi)$  increases as  $\varepsilon \searrow 0$  and  $\mu(A, \varphi) = \lim_{\varepsilon \rightarrow 0} \mu_\varepsilon(A, \varphi)$  is called the Hausdorff  $\varphi$ -measure of  $A$ .

Logarithmic dimension is a special case of the Hausdorff dimension. Take the function  $\psi(r) = \frac{1}{\log \frac{1}{r}}$  corresponding to the logarithmic measure. Then, for any  $A \subset \mathbb{R}$  there exists a critical value  $\lambda_0 = \lambda_0(A) \in [0, \infty]$ , which we call the *logarithmic dimension of  $A$* , such that for  $\lambda < \lambda_0$  the Hausdorff  $\psi^\lambda$ -measure of  $A$  is infinite,

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and for  $\lambda > \lambda_0$  it is zero. As usual, the  $\psi^{\lambda_0}$ -measure of  $A$  can take any value from  $[0, +\infty]$ .

We follow [2] to define generalized Cantor-type sets. Let  $(N_n)_{n=1}^\infty$  be a sequence of integers with  $N_n \geq 2$  for all  $n$ . Let  $\ell_0 := 1$  and  $\ell_1$  be such that  $N_1 \ell_1 < \ell_0$ . We replace  $E_0 = I_{0,1} = [0, 1]$  by  $N_1$  closed intervals  $I_{n,1}$  of length  $\ell_1$  with  $N_1 - 1$  equal gaps of length  $h_0$ . We enumerate intervals in ascending order, so  $I_{1,1} = [0, \ell_1], I_{N_1,1} = [1 - \ell_1, 1]$ . Continuing in this way, we get  $E_n$  for  $n \geq 1$  as a union of  $N_1 N_2 \dots N_n$  disjoint closed intervals  $I_{k,n}$  of length  $\ell_n$ , and  $E_{n+1}$  is obtained by replacing each interval  $I_{k,n}$  by  $N_{n+1}$  disjoint subintervals  $I_{j,n+1}$  of length  $\ell_{n+1}$  with  $N_{n+1} - 1$  equal gaps of length  $h_n$ . The intervals  $I_{k,n}$  that make up the set  $E_n$  are called *basic intervals*. The set is well-defined if for all  $n$  we have  $N_n \ell_n < \ell_{n-1}$ . Then  $h_n = \frac{\ell_n - N_{n+1} \ell_{n+1}}{N_{n+1} - 1}$  is a gap between To simplify the calculation of the norms, assume that for each  $n$

$$h_n \geq \ell_{n+1}. \tag{2.1}$$

Thus, we get a sequence  $(\ell_n)_{n=0}^\infty$  of positive decreasing numbers. Let  $\alpha_1 = 1$ , and for  $n \geq 2$  let  $\alpha_n$  satisfy  $\ell_n = \ell_{n-1}^{\alpha_n}$ , so  $\alpha_n > 1$ . Thus,  $\ell_n = \ell_1^{\alpha_1 \dots \alpha_n}$ . Let  $K_{(N_n)}^{(\alpha_n)} := \bigcap_{n=0}^\infty E_n$ . We will denote by  $K_N^\alpha$  the case when  $N_n = N$  and  $\alpha_n = \alpha$ , for all indices.

**Lemma 2.1** For each  $K_{(N_n)}^{(\alpha_n)}$  we have  $\alpha_1 \dots \alpha_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

**Proof** The sequence  $(\alpha_1 \dots \alpha_n)_{n=1}^\infty$  increases. If it is bounded, then  $\alpha_n \rightarrow 1$  as  $n \rightarrow \infty$ . But  $N_n \ell_n < \ell_{n-1}$  implies  $N_n \ell_1^{\alpha_1 \dots \alpha_{n-1}(\alpha_n-1)} < 1$ , a contradiction.  $\square$

We say that the Cantor-type set  $K_{(N_n)}^{(\alpha_n)}$  is regular if there exists  $\lim_n \frac{\log N_n}{\log \alpha_n}$ . The logarithmic dimension of a regular Cantor-type set was given in [2] as follows:

**Proposition 2.2** Suppose that for  $K_{(N_n)}^{(\alpha_n)}$  the limit  $\lambda_0 = \lim_n \frac{\log N_n}{\log \alpha_n}$ , exists in the set of extended real numbers. Then  $\lambda_0$  is the logarithmic dimension of  $K$ . In particular,  $\lambda_0(K_N^\alpha) = \frac{\log N}{\log \alpha}$ .

We now extend this result to the general case. The proof is adapted from [2].

**Theorem 2.3** For the generalized Cantor-type set  $K_{(N_n)}^{(\alpha_n)}$ , we have

$$\lambda_0(K_{(N_n)}^{(\alpha_n)}) = \liminf_n \frac{\log(N_1 N_2 \dots N_n)}{\log(\alpha_1 \alpha_2 \dots \alpha_n)}.$$

**Proof** As above,  $\psi = \frac{1}{\log \frac{1}{r}}$  for  $0 < r < 1$  and, for a given  $\lambda > 0$ , let  $\mu(K, \psi^\lambda)$  be the Hausdorff  $\psi^\lambda$ -measure of  $K$ . For simplicity of notation and calculations, we write  $K$  instead of a fixed  $K_{(N_n)}^{(\alpha_n)}$  and set  $\ell_1 = 1/e$  in order to have  $\psi(\ell_1) = 1$ . Then  $\psi^\lambda(\ell_n) = (\alpha_1 \alpha_2 \dots \alpha_n)^{-\lambda}$ . Define  $\lambda_n = \frac{\log(N_1 N_2 \dots N_n)}{\log(\alpha_1 \alpha_2 \dots \alpha_n)}$  for  $n \geq 2$ . Then

$$(\alpha_1 \alpha_2 \dots \alpha_n)^{\lambda_n} = N_1 N_2 \dots N_n. \tag{2.2}$$

Let  $\lambda_0 = \liminf_n \lambda_n$ . We claim that  $\lambda_0 = \lambda_0(K)$ .

There are two cases to consider: finite and infinite  $\lambda_0$ . Suppose first that  $0 \leq \lambda_0 < \infty$  and  $\lambda > \lambda_0$ . Let  $\lambda = \lambda_0 + 2\sigma$ . We need to show that  $\mu(K, \psi^\lambda) = 0$ .

By definition, there exists  $n_k \rightarrow \infty$  such that  $\lambda_0 = \lim_k \lambda_{n_k}$  so  $\lambda > \lambda_{n_k} + \sigma$  for large enough  $k$ . Since  $E_n$  is a covering of  $K$  by  $N_1 \dots N_n$  intervals of length  $\ell_n$ , by (2.2), we have

$$\begin{aligned} \mu(K, \psi^\lambda) &\leq \liminf_n (N_1 \dots N_n) \psi^\lambda(\ell_n) = \liminf_n \frac{N_1 \dots N_n}{(\alpha_1 \dots \alpha_n)^\lambda} = \\ &= \liminf_n (\alpha_1 \dots \alpha_n)^{\lambda_n - \lambda} \leq \liminf_k (\alpha_1 \dots \alpha_{n_k})^{\lambda_{n_k} - \lambda} \leq \liminf_k (\alpha_1 \dots \alpha_{n_k})^{-\sigma}. \end{aligned}$$

By Lemma 2.1, the above limit is zero.

We now turn to the case  $0 < \lambda_0 < \infty$  and  $\lambda < \lambda_0$ . We aim to show  $\mu(K, \psi^\lambda) = \infty$ . Let  $\lambda_0 - \lambda = 2\sigma$ . There are only finitely many  $n$  with  $\lambda_n \leq \lambda_0 - \sigma$ . Let  $\tilde{n}$  be such that  $\lambda_n > \lambda_0 - \sigma$  for  $n \geq \tilde{n}$ . Then  $\lambda_n > \lambda + \sigma$ .

We fix  $\epsilon > 0$  and consider  $\mu_\epsilon(K, \psi^\lambda)$ . Here we use coverings of  $K$  by open intervals. Let us fix a finite covering  $\bigcup_{i=1}^M G_i$  of  $K$  by open intervals with lengths  $\delta_i < \epsilon$ , such that

$$\sum_{i=1}^M \psi^\lambda(\delta_i) \leq \mu_\epsilon(K, \psi^\lambda) + 1. \tag{2.3}$$

For each  $\delta_i$  fix  $n = n(i) \in \mathbb{N}$  with  $\ell_n \leq \delta_i < \ell_{n-1}$ . Let  $n_0 = \min_{i \leq M} n(i)$  and  $n_1 = \max_{i \leq M} n(i)$ . We can assume, by decreasing  $\epsilon$  if necessary, that  $n_0 \geq \tilde{n} + 1$ .

For  $1 \leq i \leq M$ , let  $k_i$  be the number of intervals from  $E_{n_1}$  that have non-empty intersection with  $G_i$ . We follow [10] and [2], where the main idea was to estimate  $k_i$  from above in terms of  $\psi^\lambda(\delta_i)$ .

For each  $i$  we have  $\psi^\lambda(\delta_i) \geq \psi^\lambda(\ell_n) = (\alpha_1 \alpha_2 \dots \alpha_n)^{-\lambda}$ . Since  $\lambda_n > \lambda$  for  $n \geq n_0$ , (2.2) implies

$$(\alpha_1 \dots \alpha_n)^\lambda < (\alpha_1 \dots \alpha_{n_0-1})^\lambda \cdot (\alpha_{n_0} \dots \alpha_n)^{\lambda_n} = (\alpha_1 \dots \alpha_{n_0-1})^{\lambda - \lambda_n} \cdot N_1 N_2 \dots N_n. \tag{2.4}$$

Therefore,

$$1 \leq (\alpha_1 \dots \alpha_{n_0-1})^{\lambda - \lambda_n} \cdot N_1 N_2 \dots N_n \cdot \psi^\lambda(\delta_i). \tag{2.5}$$

In what follows we will use (2.4) with another index,  $n - 1$  instead of  $n$ . The left hand side of (2.4) exceeds 1. Hence,

$$1 \leq (\alpha_1 \dots \alpha_{n_0-1})^{\lambda - \lambda_{n-1}} \cdot N_1 N_2 \dots N_{n-1}. \tag{2.6}$$

We decompose the sum  $\sum \psi^\lambda(\delta_i)$  into two parts. Let  $\sum'$  be the sum over all  $i$  such that  $\ell_n \leq \delta_i < \frac{\ell_{n-1}}{N_n}$ , and  $\sum''$  be the sum over the remaining  $i$ 's. Since  $\frac{\ell_{n-1}}{N_n} < \ell_n + h_{n-1}$ , for any  $i$  in the sum  $\sum'$ , the interval  $G_i$  can intersect at most two basic intervals of  $E_n$ . By construction, it can intersect at most  $2N_{n+1}$  basic intervals of  $E_{n+1}, \dots, 2N_{n+1} \dots N_{n_1}$  basic intervals of  $E_{n_1}$ .

Then by (2.5) we obtain for each  $i$  corresponding to  $\sum'$

$$k_i \leq 2N_{n+1} \dots N_{n_1} \leq 2N_1 \dots N_{n_1} \cdot (\alpha_1 \dots \alpha_{n_0-1})^{\lambda - \lambda_n} \cdot \psi^\lambda(\delta_i). \tag{2.7}$$

For  $i$  corresponding to  $\sum''$  we fix  $j \in \{1, 2, \dots, N_n - 1\}$  such that  $\frac{j}{N_n} \ell_{n-1} \leq \delta_i < \frac{j+1}{N_n} \ell_{n-1}$ . It is easy to check that the interval  $G_i$  can intersect at most  $j + 2$  basic intervals of  $E_n$  and hence  $(j + 2)N_{n+1} \dots N_{n_1}$  basic intervals of  $E_{n_1}$ .

Here,

$$\psi^\lambda(\delta_i) \geq \psi^\lambda\left(\frac{j}{N_n} \ell_{n-1}\right) \geq \left(\alpha_1 \dots \alpha_{n-1} + \log \frac{N_n}{j}\right)^{-\lambda}.$$

If  $\log \frac{N_n}{j} \geq \alpha_1 \dots \alpha_{n-1}$ , then  $\psi^\lambda(\delta_i) \geq (2 \log \frac{N_n}{j})^{-\lambda}$ . Recall that  $1 < \frac{N_n}{j} \leq N_n$ . Take a constant  $A_\lambda$  such that  $\log^\lambda t \leq A_\lambda t$  for  $t \geq 1$ . Then  $1 \leq 2^\lambda A_\lambda \frac{N_n}{j} \psi^\lambda(\delta_i)$  and

$$k_i \leq (j + 2)N_{n+1} \dots N_{n_1} \leq 2^\lambda A_\lambda \frac{j + 2}{j} N_n N_{n+1} \dots N_{n_1} \psi^\lambda(\delta_i).$$

Here,  $\frac{j+2}{j} \leq 3$ . Let  $C'_\lambda = 3 \cdot 2^\lambda A_\lambda$ . By (2.6),

$$k_i \leq C'_\lambda (\alpha_1 \dots \alpha_{n_0-1})^{\lambda-\lambda_{n-1}} \cdot N_1 \dots N_{n_1} \psi^\lambda(\delta_i). \tag{2.8}$$

Suppose now that  $\log \frac{N_n}{j} < \alpha_1 \dots \alpha_{n-1}$ . Then  $\psi^\lambda(\delta_i) \geq (2 \alpha_1 \dots \alpha_{n-1})^{-\lambda}$ . Since  $j + 2 \leq N_n + 1 < 2 N_n$ , we have

$$k_i \leq 2N_n \dots N_{n_1} \leq 2^{\lambda+1} (\alpha_1 \dots \alpha_{n-1})^\lambda N_n \dots N_{n_1} \psi^\lambda(\delta_i).$$

By (2.4),

$$k_i \leq 2N_n \dots N_{n_1} \leq 2^{\lambda+1} (\alpha_1 \dots \alpha_{n_0-1})^{\lambda-\lambda_n} N_1 \dots N_{n_1} \psi^\lambda(\delta_i).$$

Combining this with (2.7) and (2.8), we see that for each  $i$ , the inequality

$$k_i \leq C_\lambda (\alpha_1 \dots \alpha_{n_0-1})^{-\sigma} N_1 \dots N_{n_1} \psi^\lambda(\delta_i)$$

is valid with  $C_\lambda = \max\{C'_\lambda, 2^{\lambda+1}\}$ . Here we use the conditions  $\lambda_k - \lambda > \sigma$  for  $k \in \{n - 1, n\}$ .

The covering  $\bigcup_{i=1}^M G_i$  intersects all basic intervals of  $E_{n_1}$ , so  $\sum_{i=1}^M k_i \geq N_1 \dots N_{n_1}$ . This gives

$$C_\lambda^{-1} (\alpha_1 \dots \alpha_{n_0-1})^\sigma \leq \sum_{i=1}^M \psi^\lambda(\delta_i). \tag{2.9}$$

By Lemma 2.1, the left hand side here is as big as we want for small enough  $\epsilon$ . By (2.3),  $\mu_\epsilon(K, \psi^\lambda) \rightarrow \infty$  as  $\epsilon \rightarrow 0$ , which is our claim.

It remains to consider the case of infinite  $\lambda_0$ . Fix any  $\lambda$ . We repeat the previous arguments with minor modifications. Here,  $\tilde{n}$  is given by the condition  $\lambda_n \geq 2\lambda$  for  $n \geq \tilde{n}$ . In the same manner we get (2.9) with  $\lambda$  instead of  $\sigma$  and  $\mu(K, \psi^\lambda) = \infty$ . □

**Remarks.** 1. A set  $K$  is called *dimensional* if there is at least one dimension function  $\varphi$  such that  $0 < \mu(K, \varphi) < \infty$ . Best in [4] presented an example of a dimensionless Cantor set. The theorem above does not mean that each sets  $K = K_{(N_n)}^{(\alpha_n)}$  is dimensional, because the value  $\mu(K, \psi^{\lambda_0})$  may be 0 or  $\infty$ . Nevertheless, we think that for every  $K$  of the given type, there is a function  $\varphi$  (possibly more complex in structure than  $\psi^\lambda$ ) with a proper value of  $\mu(K, \varphi)$ . See for instance [1] for the construction of such function for a more complicated Cantor-type set that is not geometrically symmetric.

2. In the proof we did not use the condition (2.1).

**3. Relation to potential theory and the extension property**

The value  $\lambda_0 = 1$  is critical in potential theory: by Theorem III.19 and Theorem III.20 in [16], we have the following simple observation.

**Proposition 3.1** *Assume  $\lambda_0 = \lambda_0(K_2^{(\alpha_n)}) \neq 1$ . Then  $K_2^{(\alpha_n)}$  is polar if and only if  $\lambda_0 < 1$ .*

In the case of  $\lambda_0(K) = 1$ , the finiteness of the logarithmic measure is sufficient for polarity.

**Proposition 3.2** ([9]) *If  $\mu(K, \psi) < \infty$  then  $Cap(K) = 0$ .*

By Carleson [5] (see also [6]), we have

**Proposition 3.3** *The set  $K_2^{(\alpha_n)}$  is polar if and only if  $\sum_{n=1}^{\infty} \frac{A_n}{2^n} = \infty$ , where  $A_n = \alpha_1 \alpha_2 \dots \alpha_n$ .*

It is easy to give examples of both polar and non-polar Cantor sets of logarithmic dimension 1. Let  $K_1 := K_2^{(\alpha_n)}$  with  $A_n = 2^n/n^2$  for large  $n$  and  $K_2 := K_2^2$ . Then  $Cap(K_1) > 0$ ,  $Cap(K_2) = 0$ ,  $\lambda_0(K_1) = \lambda_0(K_2) = 1$ .

Also, the example  $K_2^{(\alpha_n)}$  with  $\alpha_2 = 2$  and  $\alpha_n = 2^{\frac{n-1}{n}}$ ,  $n \geq 3$  (here,  $A_n = 2^n/n$ ) shows that the inverse implication in Propositions 3.2 is not valid.

Let  $K \subset \mathbb{R}$  be a perfect compact set and  $I$  be a closed interval containing  $K$ . By  $\mathcal{F}(K, I) = \{F \in C^\infty(I) : F^{(p)}|_K = 0, \forall p\}$  we denote the ideal of flat on  $K$  functions. The Whitney space  $\mathcal{E}(K)$  of extendable functions consists of traces on  $K$  of  $C^\infty$ -functions defined on  $I$ , so it is a factor space of  $C^\infty(I)$  and the restriction operator  $R : C^\infty(I) \rightarrow \mathcal{E}(K)$  is surjective. This means that the sequence  $0 \rightarrow \mathcal{F}(K, I) \xrightarrow{J} C^\infty(I) \xrightarrow{R} \mathcal{E}(K) \rightarrow 0$  is exact. If it splits, then the right inverse to  $R$  is the linear continuous extension operator  $W : \mathcal{E}(K) \rightarrow C^\infty(I)$ . In this case we say that  $K$  has the extension property.

By the celebrated Whitney theorem ([18]), the quotient topology of  $\mathcal{E}(K)$  can be given by the norms

$$\|f\|_q = |f|_q + \sup\{|(R_y^q f)^{(i)}(x)| \cdot |x - y|^{i-q} : x, y \in K, x \neq y, i = 0, 1, \dots, q\},$$

where  $q = 0, 1, \dots$ ,  $|f|_q = \sup\{|f^{(i)}(x)| : x \in K, i \leq q\}$  and  $R_y^q f(\cdot) = f(\cdot) - \sum_{k=0}^q \frac{f^{(k)}(y)}{k!} (\cdot - y)^k$  is the  $q$ -th Taylor remainder of  $f$  at  $y$ .

The following result was proved for the considered Cantor-type sets with  $N_n = N$ .

**Proposition 3.4** ([2]) *If  $\liminf \alpha_n > N$ , then  $K_N^{(\alpha_n)}$  does not have the extension property. If  $\limsup \alpha_n < N$ , then  $K_N^{(\alpha_n)}$  has the extension property.*

**Corollary 3.5** *For a compact set  $K_N^{(\alpha_n)}$ , let the limit  $\alpha = \lim \alpha_n$  exist and be not equal to  $N$ . Then  $K_N^{(\alpha_n)}$  has the extension property if and only if  $\lambda_0(K_N^{(\alpha_n)}) > 1$ .*

In general, the logarithmic dimension cannot be used for characterization of the extension property. What is more, recently it was shown in [13] that there is no such characterization in terms of Hausdorff measures, Hausdorff contents, their densities or related characteristics.

On the other hand, the logarithmic dimension is quite suitable to describe the diametral dimension of the space  $\mathcal{E}(K)$ , see Section 4 in [2] for more details. In particular,

**Corollary 3.6** ([2]) *If spaces of the type  $\mathcal{E}(K_N^\alpha)$  are isomorphic, then the corresponding compact sets have the same logarithmic dimension.*

#### 4. Polynomial bases for small Cantor-type sets

The Grothendieck problem of the existence of a basis in a nuclear Fréchet (NF) space was open for a long time. In 1974 the first example of a NF space without basis was found in [15]. After this many other examples of nuclear spaces without basis were presented, but all of them are either artificial as in [3], [17] or non-metrizable [8]. Therefore, no natural NF space of functions without basis has been found so far. This explains the interest to basis problem in concrete functional spaces.

Any Schauder basis in a NF space is absolute, therefore in order to construct a basis in such a space, it is enough to present a biorthogonal system satisfying the following Dynin-Mityagin criterion ([14]).

Let  $E$  be a nuclear Fréchet space with topology given by an increasing sequence of norms  $(\|\cdot\|_p)_{p=1}^\infty$ . Let  $E'$  be the topological dual space and  $|\cdot|_{-q}$  denote the dual norm, that is, for  $\xi \in E$ ,  $|\xi|_{-q} := \sup\{|\xi(f)|, \|f\|_q \leq 1\}$ . Suppose  $\{e_n \in E, \xi_n \in E', n \in \mathbb{N}\}$  is a biorthogonal system such that the set of functionals  $(\xi_n)_{n=1}^\infty$  is total over  $E$ . The last means that  $f = 0$  if  $\xi_n(f) = 0$  for all  $n$ . Assume that for every  $p$  there exist a  $q$  and a  $C$  such that for all  $n$

$$\|e_n\|_p \cdot |\xi_n|_{-q} \leq C. \tag{4.1}$$

Then the system  $(e_n, \xi_n)_{n=1}^\infty$  is an absolute basis in  $E$ .

Given a perfect compact set  $K \subset \mathbb{R}$  and a sequence of distinct points  $(x_k)_1^\infty \subset K$ , let  $e_0 = 1$  and  $e_n(x) = \prod_1^n (x - x_k)$  for  $n \in \mathbb{N}$ . By  $\xi_n(f)$  we denote the  $n$ -th divided difference  $[x_1, x_2, \dots, x_{n+1}]f$  of a function  $f$ . By the properties of divided differences, see for instance [7], the system  $(e_n, \xi_n)_{n=1}^\infty$  is biorthogonal. If, in addition, the sequence  $(x_k)_1^\infty$  is dense in  $K$ , then the functionals  $\xi_n, n = 0, 1, \dots$ , are total over  $\mathcal{E}(K)$ .

Our claim is that the space  $\mathcal{E}(K_{(N_n)}^{(\alpha_n)})$  possesses an interpolating Faber basis provided  $\lambda_0(K_{(N_n)}^{(\alpha_n)}) < 1$ . Recall that a polynomial basis  $(P_n)_{n=0}^\infty$  in a function space  $X$  is called a Faber basis if  $\deg P_n = n$  for all  $n$ . The task is to find a sequence  $(x_k)_1^\infty \subset K_{(N_n)}^{(\alpha_n)}$  such that the corresponding system  $(e_n, \xi_n)_{n=0}^\infty$  satisfies (4.1). When the sequence will be determined, set  $Z_M := (x_k)_1^M$ . As in Theorem 2.3, we write  $K$  instead of  $K_{(N_n)}^{(\alpha_n)}$ .

Let us first consider the representation of numbers in mixed numerical bases. Let  $A_n$  denote the number of intervals in  $E_n$ , so  $A_0 = 1$  and  $A_n = N_1 \cdot \dots \cdot N_n$ .

**Lemma 4.1** *Suppose that  $A_n \leq M < A_{n+1}$ . Then  $M$  has a unique representation in the form  $M = \sum_{j=0}^n k_j A_j$  with  $1 \leq k_n \leq N_{n+1} - 1$  and  $0 \leq k_j \leq N_{j+1} - 1$  for  $0 \leq j \leq n - 1$ .*

**Proof** Indeed, let us subtract from  $M$  the value  $A_n$  several times in succession while the result is nonnegative. We can do this  $k_n$  times with  $k_n \leq N_{n+1} - 1$ . For the remainder we have  $0 \leq M - k_n A_n < A_n$  and the same reasoning applies to  $k_j$  for  $j = n - 1, n - 2, \dots, 0$ . □

We compose the desired sequence  $(x_k)_1^\infty$  from all left endpoints of basic intervals. Write each basic interval as  $I_{j,n} = [a_{j,n}, b_{j,n}]$ . Let  $x$  be a left endpoint of some basic interval. Then there exists a minimal number  $s$  (the *type* of  $x$ ) such that  $x$  is the endpoint of some  $I_{j,m}$  for every  $m \geq s$ . By  $X_n$  we denote all points of the type  $n$ . Hence,  $X_0 := \{0\}$ ,  $X_1$  contains  $N_1 - 1$  points  $a_{i,1} = (i - 1)(\ell_1 + h_0)$  for  $2 \leq i \leq N_1$ . Continuing in this manner, we obtain

$$X_2 = \{(i - 1)(\ell_1 + h_0) + (j - 1)(\ell_2 + h_1) \text{ with } 1 \leq i \leq N_1, 2 \leq j \leq N_2\} \tag{4.2}$$

and, in general,  $X_n = \{(i_1 - 1)(\ell_1 + h_0) + (i_2 - 1)(\ell_2 + h_1) + \dots + (i_n - 1)(\ell_n + h_{n-1})\}$ , where  $1 \leq i_j \leq N_j$  for  $1 \leq j \leq n - 1$  and  $2 \leq i_n \leq N_n$ . We see that  $X_n$  contains  $A_n - A_{n-1}$  points  $a_{j,n}$  with  $j \neq kN_n + 1$  for  $0 \leq k \leq A_{n-1} - 1$ . Set  $Y_n = \cup_{k=0}^n X_k$ . Then  $\#(Y_n) = A_n$ . Here and below,  $\#(Z)$  denotes the cardinality of a finite set  $Z$ . If  $Z$  is fixed then for brevity  $\nu_{j,s} := \#(I_{j,s} \cap Z)$ . Also, for each  $x \in \mathbb{R}$ , by  $d_k(x, Z), k = 1, 2, \dots, \#(Z)$ , we denote the distances  $|x - z_{j_k}|$  from  $x$  to points of  $Z$  arranged in the nondecreasing order.

Let us arrange points from  $\cup_{k=0}^\infty X_k$  in order, including successively points of all types in ascending order. For points of the same type, the following procedure is used to ensure a uniform distribution of points on  $K$ . First  $x_1 = 0$ . The points from  $X_1$  we arrange in their natural order:  $x_k = (k - 1)(\ell_1 + h_0)$  for  $2 \leq k \leq N_1$ . Now each  $I_{j,1}$  contains exactly one point from  $Z_{A_1}$ . To enumerate points from  $X_2$ , we fix the value  $j = 2$  in (4.2) and consider  $i = 1, 2, \dots, N_1$ . Then the same we do for  $j = 3, 4, \dots, N_2$ . This gives  $x_{N_1+1} = \ell_2 + h_1 = a_{2,2}, x_{N_1+2} = \ell_1 + h_0 + \ell_2 + h_1 = a_{N_1+2,2}$ , so  $x_{N_1+k}$  is the left endpoint of the second subinterval  $I_{j,2}$  of  $I_{k,1}$  for  $1 \leq k \leq N_1$ . Next,  $x_{2N_1+k} = (k - 1)(\ell_1 + h_0) + 2(\ell_2 + h_1)$  is  $a_{j,2}$  of the third  $I_{j,2}$  subinterval of  $I_{k,1}$  for  $1 \leq k \leq N_1$ , etc. Maximal possible values  $i = N_1, j = N_2$  give the point  $x_k = 1 - \ell_2$  with the index  $k = N_1 + (N_2 - 1)N_1 = A_2$ . We note that, if  $A_1 \leq M < A_2$ , then for the set  $Z_M$  the condition  $\nu_{j,2} \in \{0, 1\}$  is valid for each  $j$  with  $1 \leq j \leq A_2$ , whereas  $\nu_{i,1} \in \{1, \dots, N_2\}$  for  $1 \leq i \leq A_1$ .

We use the same lexicographic order to list points from  $X_n$  for  $n \geq 3$ : first fix the values  $i_n = 2, i_{n-1} = \dots = i_2 = 1$ , and consider  $i_1 = 1, 2, \dots, N_1$ , after this enlarge  $i_2$  by 1, take again  $i_1 = 1, 2, \dots, N_1$ , etc. Maximal  $x_k$  in  $X_n$  is  $1 - \ell_n$  with  $k = A_n$ . Clearly,  $(x_n)_1^\infty$  is dense in  $K$ . We warn the reader that in [12] a different, more symmetric distribution of points  $x_k$  was used. Nevertheless, as in [12] and [13], the points  $Z_M$  are distributed uniformly on  $K$  in the following sense: for each  $s \in \mathbb{N}$  and  $i, j \in \{1, 2, \dots, A_s\}$  we have

$$|\nu_{j,s} - \nu_{i,s}| \leq 1, \tag{4.3}$$

so any two intervals of the same level contain the same number of points from  $Z_M$  or, perhaps, one of the intervals contains one extra point  $x_k$ , compared to another interval.

Suppose  $A_n \leq M < A_{n+1}$ . Then  $M = k_n A_n + r_n$  with  $1 \leq k_n \leq N_{n+1} - 1$  and  $0 \leq r_n < A_n$ . There are  $A_n$  intervals of  $n$ -th level. Hence, for each  $j$  we have  $k_n \leq \nu_{j,n} \leq k_n + 1$ . Lemma 4.1 yields the representation  $M = (k_n N_n + k_{n-1})A_{n-1} + r_{n-1}$  with  $0 \leq r_{n-1} < A_{n-1}$ . Therefore,  $k_n N_n + k_{n-1} \leq \nu_{j,n-1} \leq k_n N_n + k_{n-1} + 1$ . Similarly, for  $0 \leq s \leq n - 1$  and  $1 \leq j \leq A_s$  we have

$$k_n N_n \dots N_{s+1} + k_{n-1} N_{n-1} \dots N_{s+1} + \dots + k_s \leq \nu_{j,s} \leq k_n N_n \dots N_{s+1} + \dots + k_s + 1. \tag{4.4}$$

In the case of bounded sequence, let  $N_k \leq N$  for all  $k$ , we have  $1 \leq \nu_{j,n} \leq N$  and, for  $s < n$ ,

$$N_n \dots N_{s+1} \leq \nu_{j,s} \leq N^{n-s+1}.$$



Our next objective is to associate with a given  $M$  a set  $(m_k)_{k=0}^n$  of natural numbers which will be used in estimations of  $\|e_M\|_p$  and  $|\xi_M|_{-q}$ . For each  $x \in K$  we have the chain of basic intervals containing  $x$ :  $x \in I_{j,n} \subset I_{j_1,n-1} \subset \dots \subset I_{j_n,0} = [0, 1]$ .

Let  $m_n(x) = \nu_{j,n} = \#(Z_M \cap I_{j,n})$  and  $m_k(x) = \nu_{j_{n-k},k} - \nu_{j_{n-k-1},k+1}$  for  $0 \leq k \leq n-1$ , so  $m_k(x)$  is the number of zeros of  $e_M$  in  $I_{j_{n-k},k}$  which do not belong to  $I_{j_{n-k-1},k+1}$ . Then  $|e_M(x)| = \prod_{i=1}^M d_i(x, Z_M)$  with  $d_i(x, Z_M) \leq \ell_n$  for  $1 \leq i \leq m_n(x)$ ,  $d_i(x, Z_M) \leq \ell_{n-1}$  for the next  $m_{n-1}(x)$  values of  $i$ , etc. This gives

$$|e_M(x)| \leq \ell_n^{m_n(x)} \dots \ell_0^{m_0(x)}. \tag{4.5}$$

Let us find minimal possible values of  $(m_k)_{k=0}^n$  for which (4.5) is valid for all  $x \in K$ . Since  $k_n A_n \leq M < (k_n + 1)A_n$ , at least one  $I_{j,n}$  contains exactly  $k_n$  points from  $Z_M$ . Hence we must take  $m_n = k_n$ . Since  $(k_n N_n + k_{n-1}) A_{n-1} \leq M < (k_n N_n + k_{n-1} + 1) A_{n-1}$ , there is  $I_{j,n-1}$  containing exactly  $k_n N_n + k_{n-1}$  points from  $Z_M$ . For at least of one of its subintervals  $I_{j,n}$  we have  $\#(Z_M \cap I_{j,n}) = k_n$ . It follows that  $m_{n-1} = k_n(N_n - 1) + k_{n-1}$ . Continuing in this manner, we obtain for  $0 \leq s \leq n-1$  the representation

$$m_s = k_n N_n \dots N_{s+2}(N_{s+1} - 1) + \dots + k_{s+1}(N_{s+1} - 1) + k_s. \tag{4.6}$$

Then for each  $x \in K$  we have

$$|e_M(x)| = \prod_{i=1}^M d_i(x, Z_M) \leq \ell_n^{m_n} \dots \ell_0^{m_0}, \tag{4.7}$$

where the set  $(m_k)_{k=0}^n$  does not depend on  $x$ . It is easy to check that  $m_n + \dots + m_0 = M$ , so  $\ell_n^{m_n} \dots \ell_0^{m_0}$  is a product of  $M$  nondecreasing terms:

$$\ell_n^{m_n} \dots \ell_0^{m_0} = \prod_{k=1}^M \rho_k \quad \text{where } \rho_1 \leq \rho_2 \leq \dots \leq \rho_M. \tag{4.8}$$

**Lemma 4.2** *Suppose  $N_n \leq N$  for all  $n$ . Let  $M$  be as in Lemma 4.1,  $m_n = k_n$  and  $(m_s)_{s=0}^{n-1}$  be given by (4.6). Then for any natural numbers  $r, s$  with  $2 \leq r \leq r + s \leq n$  we have*

$$\sum_{j=r}^{r+s} m_{n-j} \leq N^{s+3} m_{n-r+1}.$$

**Proof** By Lemma 4.1,  $k_n \geq 1$  and  $k_j \geq 0$  for  $0 \leq j \leq n-1$ . This gives

$$N_n \cdot N_{n-1} \dots N_{n-r+2}(N_{n-r+1} - 1) \leq m_{n-r}. \tag{4.9}$$

Substituting the maximal possible values  $k_j = N_{j+1} - 1$  into (4.6) yields

$$m_{n-r} \leq N_{n+1} \cdot N_n \dots N_{n-r+2}(N_{n-r+1} - 1). \tag{4.10}$$

We note that (4.10) is valid for  $r = 1$  as well. By (4.10),

$$\sum_{j=r}^{r+s} m_{n-j} \leq N_{n+1} \cdot N_n \dots N_{n-r+3}[N_{n-r+2}(N_{n-r+1} - 1) + \dots + N_{n-r+2} \dots N_{n-r-s+2}(N_{n-r-s+1} - 1)].$$

Here, the sum in square brackets does not exceed  $N^{s+2}$ , as is easy to check. Hence,

$$\sum_{j=r}^{r+s} m_{n-j} \leq N^{s+3} N_n \cdots N_{n-r+3}.$$

On the other hand, by (4.9),  $m_{n-r+1} \geq N_n \cdots N_{n-r+3}$  as  $N_{n-r+2} \geq 2$ . □

**Lemma 4.3** *Let  $A_n \leq M < A_{n+1}$  and  $p < M$ . Then  $\|e_M\|_p \leq C_p M^p \prod_{k=p+1}^M \rho_k$ , where  $C_p$  does not depend on  $M$ .*

**Proof** The  $i$ -th derivative of  $e_M$  at  $x$  is a sum of  $M!/(M-i)!$  products, where each product contains  $M-i$  terms of the type  $x-x_j$ . Hence,  $|e_M^{(i)}(x)| \leq M^i \prod_{j=i+1}^M d_j(x, Z_M) \leq M^i \prod_{k=i+1}^M \rho_k$ , by (4.7) and (4.8). Taking supremum over all  $i \leq p$  and  $x \in K$  we get  $|e_M|_p \leq M^p \prod_{k=p+1}^M \rho_k$ .

As for the norms  $\|e_k\|_p$ , by (2.1), we can repeat the reasoning from the proof of Theorem 1 in [12], see page 354. □

We proceed to estimate the dual norms. For each  $x_r \in Z_M$  we have  $x_r \in I_{i,n} \subset I_{i_1,n-1} \subset \cdots \subset I_{i_n,0} = [0, 1]$  and

$$|e'_M(x_r)| = \prod_{j \neq r} |x_r - x_j| = \prod_{j=2}^M d_j(x_r, Z_M) \geq h_n^{m'_n(x_r)} \cdots h_0^{m'_0(x_r)}. \tag{4.11}$$

where  $m'_k(x_r)$  is the number of zeros of  $e_M$  (except the point  $x_r$ ) in  $I_{i_{n-k},k}$  which do not belong to  $I_{i_{n-k-1},k+1}$ . Thus,  $(m'_k(x_r))_{k=0}^n$  are natural numbers except perhaps  $m'_n(x_r)$  which is 0 if  $I_{i,n} \cap Z_M = \{x_r\}$ .

We search for maximal possible values of  $(m'_k)_{k=0}^n$  for which (4.11) is valid for all  $x_r \in Z_M$ . Since  $k_n \leq \nu_{j,n} \leq k_n + 1$  for all  $j$  and we remove  $x_r$  from consideration,  $m'_n = \max \nu_{j,n} - 1 \leq k_n = m_n$ . In the next step,  $m'_{n-1} = (\nu_{i_1,n-1} - 1) - m'_n$  with  $\nu_{i_1,n-1} \leq k_n N_n + k_{n-1} + 1$ . Hence,  $m'_{n-1} \leq (k_n - 1) N_n + k_{n-1} = m_{n-1}$ . Reapplying this argument yields  $m'_k \leq m_k$  for  $0 \leq k \leq n$  and the following uniform with respect to  $x_r$  bound

$$|e'_M(x_r)| \geq h_n^{m_n} \cdots h_0^{m_0}. \tag{4.12}$$

Given any product  $\prod_{j=1}^N \lambda_j$  with  $\lambda_j \geq 0$  and  $q < N$ , by  $(\prod_{j=1}^N \lambda_j)_q$  we denote this product without  $q$  smallest terms.

**Lemma 4.4** *Suppose  $A_n \leq M < A_{n+1}, 1 \leq q < M$ . Then  $|\xi_M|_{-q} \leq C_q 2^M ((h_n \cdot h_n^{m_n} \cdots h_0^{m_0})_q)^{-1}$ , where  $C_q$  does not depend on  $M$ .*

**Proof** To estimate the dual  $q$ -th norm of  $\xi_M$  we enumerate the points  $(x_k)_1^{M+1}$  in increasing order and denote the rearranged set by  $(y_k)_1^{M+1}$ . Then  $\xi_M(f) = [y_1, \dots, y_{M+1}]f$ . By (1) in [11], see also (2) in [12],

$$|\xi_M|_{-q} \leq C_q 2^M \left( \min_{k=q+1}^M \prod_{k=q+1}^M |y_{a(k)} - y_{b(k)}| \right)^{-1}, \tag{4.13}$$

where minimum is taken over all  $j$  with  $1 \leq j \leq M+1-q$  and all possible chains of strict embeddings  $[y_j, \dots, y_{j+q}] \subset \cdots \subset [y_1, \dots, y_{M+1}]$ . Here,  $[y_j, \dots, y_{j+q}] = [y_{a(q+1)}, \dots, y_{b(q+1)}] \subset [y_{a(q+2)}, \dots, y_{b(q+2)}] \subset \cdots \subset$

$[y_{a(M)}, \dots, y_{b(M)}] = [y_1, \dots, y_{M+1}]$  with  $a(k+1) = a(k)$ ,  $b(k+1) = b(k) + 1$ , or  $a(k+1) = a(k) - 1$ ,  $b(k+1) = b(k)$ . Let the minimal product  $\Pi$  in (4.13) be realized by  $[y_{j_0}, \dots, y_{j_0+q}]$ . We note that at least one point from the pair  $y_{j_0}, y_{j_0+q}$  belongs to  $Z_M$ . Without loss of generality let  $y_{j_0} \in Z_M$ . In each embedding of  $[y_{j_0}, \dots, y_{j_0+q}]$  into larger interval  $[y_a, \dots, y_b]$  some new endpoint, let for instance  $y_a$ , appears. Since  $y_b - y_a \geq |y_{j_0} - y_a|$ , we obtain  $\Pi = \prod_{k=q+1}^M |y_{a(k)} - y_{b(k)}| \geq (\prod_{k=1, k \neq j_0}^{M+1} |y_{j_0} - y_k|)^q$ . The last product represent largest  $M - q$  terms of  $|e'_{M+1}(y_{j_0})|$ . Here,  $|e'_{M+1}(y_{j_0})| = |e'_M(y_{j_0})| \cdot |y_{j_0} - x_{M+1}|$  with  $|y_{j_0} - x_{M+1}| \geq \ell_{n+1} + h_n > h_n$ , since  $M + 1 \leq A_{n+1}$ . Applying (4.12) yields the desired result.  $\square$

From now on, we assume that the sequence  $(N_n)_{n=1}^\infty$  is bounded. We present a Faber basis in the space  $\mathcal{E}(K_{(N_n)}^{(\alpha_n)})$  for two cases:

- 1)  $\alpha_n \geq N_n$  for all  $n$ . The corresponding result is a direct generalization of Theorem 1 from [12]. Here,  $\lambda_0(K_{(N_n)}^{(\alpha_n)}) \leq 1$  but perhaps  $\lim_n \lambda_n$  does not exist.
- 2) There exists  $\lim_n \lambda_n$  which is smaller than 1.

**Theorem 4.5** *Let  $N_n \leq N$  for all  $n$ . Suppose that for a set  $K_{(N_n)}^{(\alpha_n)}$  either  $\alpha_n \geq N_n$  for all  $n$  or there exists  $\lim_n \lambda_n < 1$ . Then the sequence  $(e_M)_{M=0}^\infty$  is a Schauder basis in the space  $\mathcal{E}(K_{(N_n)}^{(\alpha_n)})$ .*

**Proof** Given  $p$ , we need to find  $q$  and  $C$  such that for all  $M$

$$\|e_M\|_p \cdot |\xi_M|_{-q} \leq C. \tag{4.14}$$

Let us fix any  $p \in \mathbb{N}$  and take  $q = p(1 + N^{w+3})$ , where  $w = w(N)$  will be specified later. We can consider only large enough  $M$  since otherwise (4.14) is valid with an appropriate choice of  $C$ . Hence, we can assume that  $M$  is so large that we can use above lemmas. Fix  $M$ . Let  $A_n \leq M < A_{n+1}$ .

Let us first apply Lemma 4.4 to the case of bounded sequence  $(N_n)_{n=0}^\infty$ . By (2.1), we have  $\ell_k \leq (2N_{k+1} - 1)h_k$ . It follows that  $h_k > (2N)^{-1}\ell_k$  for all  $k$  and  $|\xi_M|_{-q} \leq C_q(4N)^M (\prod_{k=q}^M \rho_k)^{-1}$ , by (4.8). Thus there is a constant  $C_0$  such that

$$\|e_M\|_p \cdot |\xi_M|_{-q} \leq C_0 \cdot M^p (4N)^M \prod_{k=p+1}^{q-1} \rho_k. \tag{4.15}$$

Given  $p$ , take  $u$  such that  $m_n + \dots + m_{n-u+2} < p \leq m_n + \dots + m_{n-u+1}$ . Consider the product from (4.8) in more detail:

$$\prod_{k=1}^M \rho_k = \underbrace{\ell_n \cdots \ell_n}_{m_n} \cdots \underbrace{\ell_{n-u+1} \cdots \ell_{n-u+1}}_{m_{n-u+1}} \underbrace{\ell_{n-u} \cdots \ell_{n-u}}_{m_{n-u}} \cdots \underbrace{\ell_{n-u-w+1} \cdots \ell_{n-u-w+1}}_{m_{n-u-w+1}} \cdots \underbrace{\ell_0 \cdots \ell_0}_{m_0}.$$

Here,  $m_n + m_{n-1} + \dots + m_{n-u-w+1} < p + \sum_{j=u-1}^{u+w-1} m_{n-j} \leq p + N^{w+3} m_{n-u+2}$ , by Lemma 4.2. But  $m_{n-u+2} < p$ . Hence, the sum above does not exceed  $q - 1$  and interval  $[\rho_{p+1}, \dots, \rho_{q-1}]$  covers

$$\underbrace{\ell_{n-u} \cdots \ell_{n-u}}_{m_{n-u}} \cdots \underbrace{\ell_{n-u-w+1} \cdots \ell_{n-u-w+1}}_{m_{n-u-w+1}}.$$

Therefore,  $\prod_{k=p+1}^{q-1} \rho_k \leq \ell_{n-u}^{m_{n-u}} \cdots \ell_{n-u-w+1}^{m_{n-u-w+1}}$ . The last product is  $\ell_1^\kappa$  with  $\kappa = m_{n-u}\alpha_1 \cdots \alpha_{n-u} + \cdots + m_{n-u-w+1}\alpha_1 \cdots \alpha_{n-u-w+1}$ . It remains to find a constant  $C$  such that for all  $M$

$$M^p (4N)^M \ell_1^\kappa \leq C.$$

Recall that  $M < A_{n+1} \leq N A_n$ ; therefore, the desired inequality reduces to

$$p \log(N A_n) + A_n N \log(4N) \leq C + \kappa \cdot \log(1/\ell_1). \tag{4.16}$$

By (4.9),  $m_{n-r} \geq N_n \cdot N_{n-1} \cdots N_{n-r+2} \geq (N)^{-1} N_n \cdot N_{n-1} \cdots N_{n-r+1}$ . For this reason,

$$m_{n-r}\alpha_1 \cdots \alpha_{n-r} \geq N^{-1} N_1 \cdots N_n \cdot \frac{\alpha_1 \cdots \alpha_{n-r}}{N_1 \cdots N_{n-r}}.$$

Hence,

$$\kappa \geq N^{-1} A_n \sum_{j=u}^{u+w-1} \frac{\alpha_1 \cdots \alpha_{n-j}}{N_1 \cdots N_{n-j}}.$$

In the first case, when  $\alpha_n \geq N_n$  for all  $n$ , we have  $\kappa \geq N^{-1} A_n w$ . We see that the choice  $w = N^3$  provides (4.16).

In the second case, when  $\lim_n \lambda_n = \lambda_0 < 1$ , let us take  $\tilde{n}$  such that  $\lambda_n \leq 1$  for  $n \geq \tilde{n}$ . By (2.2),  $\frac{\alpha_1 \cdots \alpha_{n-j}}{N_1 \cdots N_{n-j}} = (N_1 \cdots N_{n-j})^{\frac{1-\lambda_{n-j}}{\lambda_{n-j}}} \geq 1$  for large enough  $n$  and bounded  $j$ . Here, as above,  $\kappa \geq N^{-1} A_n w$  and we can take the same  $w$ . This gives (4.16) and (4.14). □

- Remarks.** 1. The same reasoning applies to the case when  $\lambda_n \searrow 1$  so fast that the sequence  $(\lambda_n - 1) \log A_n$  is bounded.  
 2. We think that for the general case, the method of local interpolations, see [12] and [13], can be used to construct topological (in general, not Faber) bases in  $\mathcal{E}(K_{(N_n)}^{(\alpha_n)})$ , see question on page 237 in [2].

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