Turkish Journal of Mathematics

Volume 45 | Number 4 | Article 6

1-1-2021

Theory and numerical approaches of high order fractional Sturm-Liouville problems

TAHEREH HOULARI

MOHAMMAD DEHGHAN

JAFAR BIAZAR

ALIREZA NOURI

Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article
Turk J Math
(2021) 45: 1564 – 1579
© TÜBİTAK
doi:10.3906/mat-1710-103

Theory and numerical approaches of high order fractional Sturm–Liouville problems

Tahereh HOULARI[1], **Mohammad DEHGHAN**[2,*], **Jafar BIAZAR**[1], **Alireza NOURI**[3]
[1]Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran
[2]School of Mathematics and Statistics, Carleton University, Ottawa, Canada
[3]School of Mining and Petroleum Engineering Department Civil and Environmental Engineering, University of Alberta, Edmonton, Canada

Received: 30.10.2017 • **Accepted/Published Online:** 29.04.2021 • **Final Version:** 27.07.2021

Abstract: In this paper, fractional Sturm–Liouville problems of high-order are studied. A simple and efficient approach is presented to determine more eigenvalues and eigenfunctions than other approaches. Existence and uniqueness of solutions of a fractional high-order differential equation with initial conditions is addressed as well as the convergence of the proposed approach. This class of eigenvalue problems is important in finding solutions to linear fractional partial differential equations (LFPDE). This method is illustrated by three examples to signify the efficiency and reliability of the proposed numerical approach.

Key words: High-order fractional Sturm–Liouville problems, eigenvalues, fractional derivative, fixed point theory

1. Introduction

The Sturm-Liouville problems play an important role in applied mathematics, physics, and engineering. Although the second order Sturm–Liouville equations describe many physical phenomena mathematically such as classical and quantum mechanics (see [8, 25, 31, 42]), higher order Sturm–Liouville problems can be used in mathematical modeling of more important phenomena. For example, fourth order Sturm–Liouville problem is modeling the free vibration analysis of beam structures (see[14, 16, 34]), and a variety of fluid mechanics models are governed by high-order Sturm–Liouville problems. For instance, when the ordinary convection arises from heating the beneath layer of the fluid regarding the action of rotation, an eight order Sturm–Liouville problem can model the instability of moleculas. On the other hand, the marginal state is delineated by sixth order Sturm–Liouville problem (see [12, 13, 17, 34]). The governed differential equation would be ten and twelfth order Sturm–Liouville boundary value problems provided that a uniform magnetic field was applied across the fluid in some direction as gravity (see [12, 13, 18, 35]). However, not much work has been done for higher order Sturm–Liouville problems and it is still a challenging task for scientists.

Adomian decomposition method (ADM), variational iteration method (VIM), Chebyshev spectral collocation method (CSCM), and modified Adomian decomposition method are numerical or semi-analytic schemes available on this subject (see [6, 33, 34]).

*Correspondence: m-dehghan@tabrizu.ac.ir
2010 *AMS Mathematics Subject Classification:* 35G15 , 34L16 , 65L15, 26A33, 34A08

The non-singular high order Sturm–Liouville problem is in the following form.

$$\begin{aligned}&(-1)^m(p_m(x)y^{(m)}(x))^{(m)}+(-1)^{(m-1)}(p_{m-1}(x)y^{(m-1)}(x))^{(m-1)}+\cdots\\&\quad+(p_2(x)y''(x))''-(p_1(x)y'(x))'+p_0(x)y(x)\\&=\lambda w(x)y(x),\qquad a=0<x<b,\end{aligned}\tag{1.1}$$

subject to some $2m$ specified conditions at the boundary points $x\in\{a,b\}$ on

$$\begin{aligned}&u_k=y^{(k-1)},\qquad 1\leqslant k\leqslant m,\\&v_1=p_1y'-(p_2y'')'+(p_3y''')''+\cdots+(-1)^{m-1}(p_my^{(m)})^{(m-1)},\\&v_2=p_2y''-(p_3y''')'+(p_4y^{(4)})''+\cdots+(-1)^{m-2}(p_my^{(m)})^{(m-2)},\\&\vdots\\&v_k=p_ky^{(k)}-(p_{k+1}y^{(k+1)})'+(p_{k+2}y^{(k+2)})''+\cdots+(-1)^{m-k}(p_my^{(m)})^{(m-k)},\\&\vdots\\&v_m=p_my^{(m)},\end{aligned}\tag{1.2}$$

where all coefficients $p_i(x),\ i=0,\cdots,m$ are real valued functions. The interval (a,b) is finite; the coefficient functions $p_i(x)$ $(0\leq i\leq m)$, the weight function $w(x)$, and $\dfrac{1}{p_m(x)}$ are in $L^1(a,b)$; $p_m(x)$ and $w(x)$ are both positive.

The fractional-order SLP is a generalization of the integer-order SLP and this paper considers the high order fractional Sturm–Liouville equation where integer-order derivatives are replaced by fractional-order ones.

Without loss of gerality and for the sake of simplicity, this paper focuses on the following n^{th}-order fractional Sturm–Liouville eigenvalue problem,

$$D^{\alpha}_{0+}y(x)+\sum_{j=0}^{n-2}q_j(x)y^{(j)}(x)=\lambda y(x),\quad 0\leq x\leq b,\tag{1.3}$$

where $q_j(x)$, $j=0,1,2,\cdots,n-2$, are integrable functions over the closed interval $[0,b]$, D^{α}_{0+} is the Caputo fractional derivative, α is bounded by $n-1<\alpha\leqslant n$, and n should be an even number, that will be denoted by $2m$. The separated boundary conditions are as follows:

$$y^{(i)}(0)=0,\tag{1.4}$$

$$y^{(i)}(b)=0,\tag{1.5}$$

for $i\in S'\subset S:=\{0,1,2,\cdots,2m-1\}$, where S' has m elements.

The fractional Sturm–Liouville problem (FSLP) (1.3)-(1.4) often appears in the separable linear fractional partial differential equations (see [2, 3, 9, 26]). This equation has found a significant interest in engineering, geology, economy, biology, and acoustics (see [26, 30] and references therein).

Due to difficulties of applying analytic approaches for any FSLP, several numerical procedures have been implemented to find approximate solutions. Almost all of these numerical methods have been proposed for classical forms of Sturm–Liouville problems and are generalized to FSLP. Some of these approaches are as follows: homotopy analysis method (HAM) [1], adomian decomposition method (ADM) [3, 26], fractional differential transform method (FDTM) [15], iterative approximation method [28], and variational iteration method [24]. In order to find the eigenvalues and relevant eigenfunctions, discretizing of the fractional operator with the composition of the left and right Caputo derivative, subjected to the mixed boundary conditions, has been done in [11]. The combination of two techniques: the method of external excitation (MEE) and the backward substitution method (BSM), constructed a novel method for solving fractional eigenvalue problems of the second order [32]. Polyfractonomials as the eigenfunctions of fractional Jacobi problems have been introduced in [40] and extended to the case with the tempered fractional derivatives [41]. In [21], a fractional-order Legendre Tau method is devoted to approximate of the eigenvalues of a nonsingular fractional second-order Sturm–Liouville problem.

Since Klimek and Agrawal has introduced the new fractional Sturm–Liouville operator, which is the combination of Caputo and Riemann fractional derivative [23], many mathematicians put some tremendous effort to find the solution of this kind of problem numerically and analytically [7, 37, 38].

There exist many articles containing the existence and uniqueness theorems of fractional differential equations. Using fixed point theorem, the existence and approximation of solutions to initial value problems for nonlinear fractional differential equations of arbitrary order with Riemann–Liouville derivative have been considered in [36]. Authors in [39] examined the existence of solutions for a higher-order coupled system of fractional differential equations with Sturm–Liouville boundary value conditions at resonance by applying Mawhin continuation theorem.

It must be mentioned that most of the aforementioned methods have been applied only to the second-order FSLP. To the best of our knowledge, a few studies have been carried out to find the eigenvalues of high order FSLP and [19] is the first report in the field of eigenvalue problems of high order FSLP with variable coefficients. Authors, in [4] utilized the fractional series solution to find eigenfunctions of fourth order FSLP and then determined the eigenvalues by imposing the boundary conditions. We note that, despite extensive literature on the second order FSLP and their diverse physical and engineering applications, the research on the high order fractional problems has remained very scarce.

Therefore, we are motivated to propose an efficient approach in order to solve the high order fractional Sturm-Liouville problem, based on fractional differential transform method. The proposed method is flexible to solve any FSLP with an arbitrary order and we obtain more eigenvalues in our numerical results in comparison with the examples addressed in some mentioned literature. The convergence of the proposed method has been investigated in this paper. Moreover, we prove the existence and uniqueness of the solution of a Cauchy problem for the high order fractional initial value problem with respect to the Caputo derivative.

The paper is organized as follows. Preliminary definitions of fractional calculus are presented in Section 2. In Section 3, the existence and uniqueness of high-order fractional differential equation with initial conditions is considered. The fractional differential transform method and its implementation to approximate the solution of Eq. (1.3), with respect to the initial condition (1.4), is demonstrated in Section 4. The convergence of the proposed approach is studied in Section 5 followed by numerical examples in Section 6. Finally, the discussion of results is stated in Section 7.

2. Preliminary

In this section, we present some essential information about fractional calculus theory that will be used intensively in this paper.

Definition 2.1 Let us denote by $AC^n(\Omega)$, where $n = 1, 2, \cdots$ and Ω is an interval, the space of functions $f(x)$ which have continuous derivatives up order $n-1$ on Ω with $f^{(n-1)}(x) \in AC(\Omega)$.

Definition 2.2 The left-sided Riemann-Liouville fractional integral operator of order α is defined by

$$I^{\alpha}_{0+}y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t)dt, \tag{2.1}$$

where $y(x)$ belongs to the space $L^1[0,b]$, Lebesgue measurable functions on the finite interval $[0,b]$, $b > 0$, and $\alpha \in \mathbb{R}^+$.

Some useful properties of the operator I^{α}_{0+} are summarized in the following lemma [27, 29, 30].

Definition 2.3 The left-sided Caputo fractional derivative of order α, $n-1 < \alpha \leqslant n$ is defined by

$$D^{\alpha}_{0+}y(x) := (I^{n-\alpha}_{0+}D^n y)(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{y^{(n)}dt}{(x-t)^{\alpha-n+1}}, \tag{2.2}$$

whenever the right hand side exists.

Theorem 2.4 Let $n-1 < \alpha \leqslant n$. If $y(x)$ belongs to the space $AC^n[0,b]$, then the Caputo fractional derivative $D^{\alpha}_{0+}y(x)$ exists almost everywhere on $[0,b]$.

Proof See [22] for more details. □

Lemma 2.5 For $\alpha \in \mathbb{R}^+$, $n-1 < \alpha \leqslant n$ and $y \in L^1[0,b]$, we have

$$1. D^{\alpha}_{0+}I^{\alpha}_{0+}y(x) = y(x),$$

$$2. I^{\alpha}_{0+}D^{\alpha}_{0+}y(x) = y(x) - \sum_{i=0}^{n-1} y^{(m)}(0^+)\frac{x^m}{m!},$$

$$3. D^{\alpha}_{0+}x^r = \begin{cases} \frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)}x^{r-\alpha}, & for\ n \leqslant r, \\ 0 & for\ r < n. \end{cases}$$

3. Existence and uniqueness of solutions of High-order fractional initial value problems

Consider the following high-order fractional initial value problem

$$D^{\alpha}_{0+}y(x) = f(x, y, y', \cdots, y^{(n-1)}), \qquad x \in J := [0,b], \tag{3.1}$$

subject to Cauchy initial conditions

$$y^{(i)}(0) = y_0^i, \qquad i = 0, 1, \cdots, n-1, \tag{3.2}$$

where $n-1 < \alpha \leqslant n$, y_0^i are known constants.

We also know $f : J \times AC^n \times AC^{n-1} \times \cdots \times AC \to L^1$ as a continuous function. The space $AC^n[0,b]$ with the norm $\|y\|_{AC^n} = \|y\|_{L^1} + \|y'\|_{L^1} + \cdots + \|y^{(n)}\|_{L^1}$ where $\|y\|_{L^1} = \int_0^b |y(x)|dx$, can be readily shown that it is a Banach space.

(LC) Assume that $f(x, y, y', \ldots, y^{(n-1)})$ satisfies the following Lipschitz condition

$$\|f(x, y, y', \cdots, y^{(n-1)}) - f(x, z, y', \cdots, y^{(n-1)})\|_{AC^n} \leqslant K\|y - z\|_{AC^n},$$

where $K > 0$.

From Lemma (2.5), it can be seen that Eq. (3.1) is equivalent to the integral equation

$$y(x) = \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} (f(x, y, y', \cdots, y^{(n-1)}) ds.$$

For brevity, let us take $\gamma = \frac{b^\alpha}{\Gamma(\alpha+1)}$ and $N := \|f(x, 0, 0, \cdots, 0)\|_{AC^n}$.

Theorem 3.1 If the hypothesis (LC) is satisfied and if $2\gamma K < 1$, then the high-order fractional initial value problem (3.1)-(3.2) has a unique solution.

Proof Let $Z = C(J; AC^n)$. Define a mapping $\Phi : Z \to Z$ by

$$\Phi y(x) = \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} x^k - \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} (f(x, y, y', \cdots, y^{(n-1)})) ds.$$

We are going to show that Φ has a fixed point. Choose $r \geqslant 2(\sum_{k=0}^{n-1} \frac{\|y^{(k)}(0)\|_{AC^n}}{k!} + N\gamma)$. Then, we can show that $\Phi B_r \subseteq B_r$, where $B_r := \{y \in Z : \|y\|_{AC^n} \leqslant r\}$. From the assumption, we have

$$\begin{aligned}
&\|\Phi y\|_{AC^n} \leqslant \sum_{k=0}^{n-1} \frac{\|y^{(k)}(0)\|_{AC^n}}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} (\|f(x,y,y',\ldots,y^{(n-1)})\|_{AC^n}) ds \\
&\leqslant \sum_{k=0}^{n-1} \frac{\|y^{(k)}(0)\|_{AC^n}}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} (\|f(x,y,\ldots,y^{(n-1)}) - f(x,0,\ldots,0)\|_{AC^n} \\
&+\|f(x,0,\ldots,0)\|_{AC^n}) ds \\
&\leqslant \sum_{k=0}^{n-1} \frac{\|y^{(k)}(0)\|_{AC^n}}{k!} + \frac{Nb^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} K \|y\|_{AC^n} ds \\
&\leqslant \sum_{k=0}^{n-1} \frac{\|y^{(k)}(0)\|_{AC^n}}{k!} + N\gamma + K\|y\|_{AC^n} \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} ds \\
&\leqslant \sum_{k=0}^{n-1} \frac{\|y^{(k)}(0)\|_{AC^n}}{k!} + N\gamma + K\gamma r \\
&\leqslant \frac{r}{2} + \frac{r}{2} \\
&\leqslant r.
\end{aligned}$$

Thus, Φ maps B_r into itself. Now, for $y_1, y_2 \in AC^n$, we have

$$\begin{aligned}
&\|\Phi y_1 - \Phi y_2\|_{AC^n} \leqslant \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} (\|f(x,y_1,\ldots,y_1^{(n-1)}) - f(x,y_2,\ldots,y_2^{(n-1)})\|_{AC^n}) ds \\
&\leqslant \frac{1}{\Gamma(\alpha)} K \|y_1 - y_2\|_{AC^n} \int_0^x (x-s)^{\alpha-1} ds \\
&\leqslant K\gamma \|y_1 - y_2\|_{AC^n}.
\end{aligned}$$

Since $\gamma K < \frac{1}{2}$, the mapping Φ is a contraction, therefore there exists a unique fixed point $y \in B_r$ such that $\Phi y(x) = y(x)$. Any fixed point of Φ is the solution of Eq. (3.1). □

4. Fractional differential transform method

The basic definition of differential transformation is introduced in this section. Further details can be found in Arikoglu and Ozkol [5].

By expanding the analytic function $y(x)$ in terms of fractional power series, the following presentation is obtained:

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x-a)^{\frac{k}{\beta}}, \tag{4.1}$$

where β is the order of fraction and $Y(k)$ is the fractional differential transform of $y(x)$. We consider the first

N terms of series (4.1), as an approximation of the solution of Eq. (1.3), as follows

$$y(x;\lambda) \approx \sum_{k=0}^{N} Y(k;\lambda)(x-a)^{\frac{k}{\beta}}. \tag{4.2}$$

Some essential theorems of DTM, which can be proved easily (see [5]), are listed as follows:

Theorem 4.1 *If* $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$.

Theorem 4.2 *If* $f(x) = g(x)h(x)$, then $F(k) = \sum_{l=0}^{k} G(l)H(k-l)$.

Theorem 4.3 *If* $f(x) = (x-a)^p$, then $F(k) = \delta(k-p)$, *where*

$$\delta(k) = \begin{matrix} 1 & if & k = 0, \\ 0 & if & k \neq 0. \end{matrix}$$

Theorem 4.4 *If* $f(x) = D_a^{\alpha}[g(x)]$, then $F(k) = \dfrac{\Gamma(\alpha+1+\frac{k}{\beta})}{\Gamma(1+\frac{k}{\beta})} G(k+\beta\alpha)$.

Subsequently, we apply fractional differential transform method to Eq. (1.3), which results in the following equation. Taking differential transformation of Eq. (1.3) and using the aforementioned theorems, we get

$$\begin{aligned} &\frac{\Gamma(1+\alpha+\frac{k}{\beta})}{\Gamma(1+\frac{k}{\beta})} Y(k+\alpha\beta;\lambda) + \sum_{j=0}^{n-2}\sum_{l=0}^{k} \frac{\Gamma(1+j+\frac{k-l}{\beta})}{\Gamma(1+\frac{k-l}{\beta})} Q_j(l) Y(k-l+j\beta;\lambda) \\ &= \lambda\, Y(k;\lambda), \qquad k = 0,\ldots,N-\alpha\beta, \end{aligned} \tag{4.3}$$

where $Y(i)$, $i \in I := \{0,1,2,\ldots,\alpha\beta-1\}$ are starting values and can be determined by

$$Y(k) = \begin{cases} \dfrac{1}{\left(\frac{k}{\beta}\right)!}\left[\dfrac{d^{\frac{k}{\beta}} y(x)}{dx^{\frac{k}{\beta}}}\right]_{x=a} & \dfrac{k}{\beta} \in \mathbb{Z}^+, \\ 0 & \dfrac{k}{\beta} \notin \mathbb{Z}^+, \end{cases} \tag{4.4}$$

for $k = 0,1,2,\ldots,(\alpha\beta-1)$, where α is the order of fractional differential equation; see [5] for more details.

Simplifying Eq. (4.3) results in the following equation

$$\begin{aligned} Y(k+\alpha\beta;\lambda) = \frac{\Gamma(1+\frac{k}{\beta})}{\Gamma(1+\alpha+\frac{k}{\beta})}(\lambda Y(k;\lambda) - \sum_{j=0}^{n-2}\sum_{l=0}^{k} \frac{\Gamma(1+j+\frac{k-l}{\beta})}{\Gamma(1+\frac{k-l}{\beta})} \\ \times Q_j(l) Y(k-l+j\beta;\lambda)), \end{aligned} \tag{4.5}$$

which is a recursive relation to find coefficients of FDTM, i.e., $Y(k;\lambda)$. Let's partition the index set I into three subsets:

- (1) $J_1 = \{k \in I | \ \dfrac{k}{\beta} \notin \mathbb{Z}^+\}$,

- (2) $J_2 = \{k \in I| \ \dfrac{k}{\beta} \in \mathbb{Z}^+, \ \ \dfrac{k}{\beta} \in S \setminus S^{'} \}$,

- (3) $J_3 = \{k \in I| \ \dfrac{k}{\beta} \in \mathbb{Z}^+, \ \ \dfrac{k}{\beta} \in S' \}$.

Both subsets J_2 and J_3 have m elements. Starting values Y(k) for $k \in J_1 \cup J_3$ can be easily obtained. Let's take m other starting values $Y(k)$, $k \in J_2$, identical to unknown constants c_k. According to the recursive relation (4.5), the approximate solution can be written as follows:

$$y(x;\lambda) \approx \sum_{k=0}^{N} Y(k;\lambda, c_1 \dots, c_m)(x-a)^{\frac{k}{\beta}}. \tag{4.6}$$

By imposing m boundary conditions (1.5), $y^{(i)}(b) = 0$, into (4.6), one obtains a system of m nonlinear equations with $m+1$ unknown coefficients $\lambda, c_1, \dots,$ and c_m.

$$\begin{array}{ll} \sum_{k=0}^{N} Y(k;\lambda, c_1 \dots, c_m)(b-a)^{\frac{k}{\beta}} = 0 & if \ \ i = 0, \\ \sum_{k=1}^{N} Y(k;\lambda, c_1 \dots, c_m)(\frac{k}{\beta})(\frac{k}{\beta}-1)\dots(\frac{k}{\beta}-(i-1))(b-a)^{\frac{k}{\beta}-i} = 0 & if \ \ i \in S' - \{0\}. \end{array} \tag{4.7}$$

Regarding the normalization of these coefficients, c_1 can be equal to 1. Hereby, with respect to the parameter λ, there will be m unknown coefficients that can be calculated by solving the nonlinear system (4.7).

In this paper, we use Mathematica to solve the system of nonlinear equations (4.7), numerically by applying NSolve command.

5. Convergence of FDTM on FSLP

This section presents a convergence criterion for the fractional differential transform method in the context of the fractional Sturm–Liouville problem.

In the following theorem, $(AC^n[a,b], \|.\|_{AC^n})$ denotes a Banach space consisting of all continuous functions on $[a,b]$, with the formation norm.

Theorem 5.1 Let

$$\Phi_k(x;\lambda, c_1, \cdots, c_m) = Y(k;\lambda, c_1, \cdots, c_m)(x-a)^{k/\beta},$$

in series (4.6), then the series solution $\sum_{k=0}^{\infty} \Phi_k(x;\lambda, c_1, \cdots, c_m)$ converges if there exist a γ , $0 < \gamma < 1$, such that

$$\|\Phi_{k+1}(x;\lambda, c_1, \cdots, c_m)\|_{AC^n} \leqslant \gamma \|\Phi_k(x;\lambda, c_1, \cdots, c_m)\|_{AC^n}, \tag{5.1}$$

for all $k \geqslant k_0$, and some $k_0 \in \mathbb{N}$, in which $\lambda, c_1, \cdots,$and c_m have already been obtained from the nonlinear system of Equations (4.7). Condition (5.1) should be ignored when $\Phi_k(x;\lambda, c_1, \cdots, c_m)$ is zero, that happens frequently in FDTM procedure.

Proof The sequence $\{u_n\}_{n=0}^{\infty}$ is defined as follows

$$u_n = \sum_{i=0}^{n} \Phi_i(x; \lambda, c_1, \cdots, c_m).$$

To prove the convergence, it is enough to show that $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space. To this end, we have

$$\begin{aligned}\|u_{n+1} - u_n\|_{AC^n} = \|\Phi_{n+1}(x; \lambda, c_1, \cdots, c_m)\|_{AC^n} &\leqslant \gamma \|\Phi_n(x; \lambda, c_1, \cdots, c_m)\|_{AC^n} \\ &\leqslant \gamma^{n-p+1} \|\Phi_p(x; \lambda, c_1, \cdots, c_m)\|_{AC^n},\end{aligned}$$

where $p = k_0, k_0 + 1, \cdots$. These inequalities are obtained from Eq.(5.1). For every $n, l \in \mathbb{N}, n \geqslant l > k_0$. We can write

$$\begin{aligned}\|u_n - u_l\|_{AC^n} = \|\sum_{i=l}^{n-1}(u_{i+1} - u_i)\|_{AC^n} &\leqslant \sum_{i=l}^{n-1} \|u_{i+1} - u_i\|_{AC^n} \\ &\leqslant \sum_{i=l}^{n-1} \gamma^{i-k_0+1} \|\Phi_{k_0}(x; \lambda, c_1, \cdots, c_m)\|_{AC^n} \\ &= \frac{1 - \gamma^{n-l}}{1 - \gamma} \gamma^{l-k_0+1} \|\Phi_{k_0}(x; \lambda, c_1, \cdots, c_m)\|_{AC^n},\end{aligned} \tag{5.2}$$

which results in

$$lim_{n,l \to \infty} \|u_n - u_l\|_{AC^n} = 0,$$

regarding $0 < \gamma < 1$. □

Theorem 5.2 If the series solution $u_n(x) = \sum_{k=0}^{n} Y(k; \lambda, c_1, \cdots, c_m)(x - a)^{\frac{k}{\beta}}$ convergence to the solution of problem (1.3), then the truncated error can be estimated as follows

$$\|y(x) - \sum_{k=0}^{N} Y(k; \lambda, c_1, \cdots, c_m)(x - a)^{\frac{k}{\beta}}\|_{AC^n} \leqslant \frac{1}{1 - \gamma} \gamma^{N-k_0+1} \|Y(k_0; \lambda, c_1, \cdots, c_m)(x - a)^{\frac{k_0}{\beta}}\|_{AC^n},$$

for some $k_0 \geqslant 0$, where $Y(k_0; \lambda, c_1, \cdots, c_m) \neq 0$.

Proof For $0 < \gamma < 1$, and $n \geqslant N > k_0$, we have $(1 - \gamma^{n-N}) < 1$. Hence, from (5.2) we can write

$$\|u_n - u_N\|_{AC^n} \leqslant \frac{1}{1 - \gamma} \gamma^{N-k_0+1} \|Y(k_0; \lambda, c_1, \cdots, c_m)(x - a)^{\frac{k_0}{\beta}}\|_{AC^n}.$$

The proof will be completed by the assumption of the Theorem (3.3), i.e., $lim_{n \to \infty} u_n = y(x)$. □

6. Numerical example

Some illustrative examples will be presented in this section.

Example 6.1 Consider the following α-order FSLP

$$D^{\alpha}[y(x)] + y'(x) + \lambda y(x) = 0, \quad x \in (0,1)$$

subject to the following boundary conditions

$$y'(0) = 0, \quad y(1) = 0,$$

where $1 < \alpha \leq 2$. Applying FDTM on this example, regarding (4.5) we have

$$Y(k+\beta\alpha) = \frac{-\Gamma(1+\frac{k}{\beta})\left[\lambda Y(k) + \frac{\Gamma(2+\frac{k}{\beta})}{\Gamma(1+\frac{k}{\beta})} Y(k+\beta)\right]}{\Gamma(1+\alpha+\frac{k}{\beta})}.$$

For instance, when $\alpha = 1.9$ we have $\beta = 10$ with starting values

$$Y(0) = 1, \quad Y(1) = Y(2) = \cdots = Y(18) = 0,$$

by means of (4.4).

The first five eigenvalues are determined by the proposed approach. These eigenvalues and those reported in [19] and [3] for different values of α, are presented in Table 1. Three eigenvalues are just reported in [3], while we have found the same number and the same eigenvalues as reported in [19], by an easier approach.

Table 1. First five eigenvalues in example 6.1.

	$\alpha = 1.75$			$\alpha = 1.9$		
λ_k	FDTM	[3]	[19]	FDTM	[3]	[19]
λ_1	3.7921553	3.7921553	3.792145	3.648055	3.648062	3.648054
λ_2	19.5210356	19.5210356	19.521342	21.215101	21.215101	21.215116
λ_3	42.8785420	42.8785424	42.876382	52.999382	52.999382	52.999304
λ_4	76.767607	–	76.767929	98.902684	–	98.902945
λ_5	114.004	–	114.004005	157.898040	–	157.892919

	$\alpha = 2$		
λ_k	FDTM	[3]	[19]
λ_1	3.623089	3.623089	3.623089
λ_2	2344237	2344237	2344337
λ_3	62.929723	62.929723	62.929723
λ_4	122.149923	–	122.149923
λ_5	201.107831	–	201.107831

Example 6.2 Consider the following α-order FSLP

$$D^{\alpha}[y(x)] = \lambda y(x), \quad x \in (0,1),$$

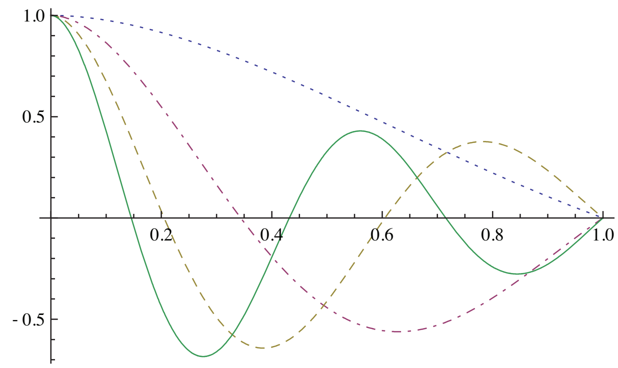

Figure 1. The first four eigenfunctions of Example 6.1 for $\alpha = 1.9$ are plotted by the dotted line, dot-dashed line, dashed line, and thick line, respectively.

where $3 < \alpha \leqslant 4$, subject to the boundary conditions

$$y(0) = y''(0) = 0, \quad y(1) = y''(1) = 0.$$

Applying FDTM on this example, regarding (4.5) we have

$$Y(k+\beta\alpha) = \frac{\Gamma(1+\frac{k}{\beta})\left[\lambda Y(k)\right]}{\Gamma(1+\alpha+\frac{k}{\beta})}.$$

For instance, when $\alpha = 3.7$ we have $\beta = 10$ with starting values

$$Y(0) = \cdots = Y(9) = 0, Y(10) = 1, Y(11) = \cdots = Y(29) = 0,$$

$$Y(30) = c_2, Y(31) = \cdots = Y(36) = 0$$

by means of (4.4).

The exact eigenvalues of this equation, for $\alpha = 4$, are known as $\lambda_k = (k\pi)^4$, $k \geq 1$ (see [20]). Table 2 shows the results of applying FDTM and those reported in [19]. The novelty is the number of eigenvalues, which are more than those reported in [1], for the values $\alpha = 3.7, 3.9$, and 4, and the accuracy of the results are notable.

Example 6.3 Consider the following α-order FSLP

$$D^{\alpha}[y(x)] + \sum_{j=0}^{4} q_j(x) y^{(j)}(x) + \lambda y(x) = 0, \quad x \in (0,5),$$

subject to the following boundary conditions,

$$y(0) = y''(0) = y^{(4)}(0) = 0,$$
$$y(5) = y''(5) = y^{(4)}(5) = 0,$$

Table 2. Eigenvalue results for Example 6.2.

	$\alpha = 3.5$		$\alpha = 3.7$		$\alpha = 3.9$	
λ_k	FDTM	[19]	FDTM	[19]	FDTM	[19]
λ_1	102.504591	102.504591	91.412292	91.412293	93.533242	93.533230
λ_2	610.167030	610.167042	944.796195	944.795695	1324.156313	1324.156357
λ_3	4489.429015	4489.426817	4544.318706	4544.336950	6456.132847	6456.132485
λ_4	5312.582021	5312.585078	12012.662956	12012.483491	19613.917656	19613.888783
λ_5	–	–	29458.060993	–	42828645577	–
λ_6	–	–	52916.459013	–	95054.064096	–
λ_7	–	–	102692.701627	–	173711.520853	–
λ_8	–	–	149959.089170	–	291370.684659	–
λ_9	–	–	265355.521567	–	465226.862289	–

	$\alpha = 4$	
FDTM	[19]	$\lambda^{(exact)}$
97.409091	97.409091	97.409091
1558.545456	1558.545456	1558.545456
7890.136374	7890.136374	7890.136374
24936.727305	24936.727305	24936.727305
60880.681896	–	60880.681896
126242.181980	–	126242.181980
233879.227572	–	233879.227572
398987.636875	–	398987.636875
639101.046277	–	639101.046274

Table 3. c_2 results for Example 6.2.

	$\alpha = 3.5$	$\alpha = 3.7$	$\alpha = 3.9$	$\alpha = 4$
λ_k	c_2	c_2	c_2	c_2
λ_1	-2.39681	-1.95182	-1.72334	-1.64493
λ_2	-6.51175	-6.76237	-6.6493	-6.57974
λ_3	-20.3619	-15.8076	-14.986	-14.8044
λ_4	-22.418	-26.7337	-26.495	-26.3189
λ_5	–	-43.4145	-41.4436	-41.1234
λ_6	–	-59.5856	-59.5188	-59.2176
λ_7	–	-85.2686	-81.085	-80.6018
λ_8	–	-104.634	-105.713	-105.276
λ_9	–	-142.445	-134.383	-133.24

where $5 < \alpha \leqslant 6$, $q_j(x)$, $0 \le j \le 5$ are given as the following

$$q_0(x) = -r_3(x), \quad q_1(x) = r_2'(x), \quad q_2(x) = r_2(x) - r_1''(x), \quad q_3(x) = -2r_1'(x),$$

$$q_4(x) = -r_1(x), \quad r_1(x) = 0.03x^2, \quad r_2(x) = 0.0003x^4 - 0.08, \quad r_3(x) = 10^{-6}x^6 - 0.0014x^2.$$

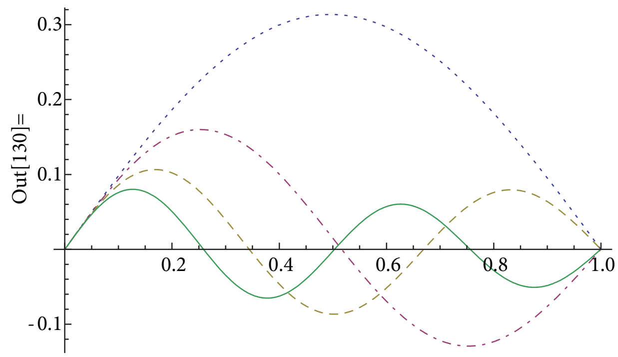

Figure 2. The first four eigenfunctions of Example 6.2 for $\alpha = 3.9$ are plotted by the dotted line, dot-dashed line, dashed line, and thick line, respectively.

Applying FDTM on this example, regarding (4.5) we have

$$Y(k+\beta\alpha) = \frac{\Gamma(1+\frac{k}{\beta})}{\Gamma(1+\alpha+\frac{k}{\beta})}\Big[-\lambda Y(k) + 0.03\sum_{l=0}^{k}\delta(l-2)\frac{\Gamma(5+\frac{k-l}{\beta})}{\Gamma(1+\frac{k-l}{\beta})}Y(k-l+4\beta)$$

$$+0.12\frac{\Gamma(4+\frac{k}{\beta})}{\Gamma(1+\frac{k}{\beta})}Y(k+3\beta) + 0.14\frac{\Gamma(3+\frac{k}{\beta})}{\Gamma(1+\frac{k}{\beta})}Y(k+2\beta)$$

$$-0.0003\sum_{l=0}^{k}\delta(l-4)\frac{\Gamma(3+\frac{k-l}{\beta})}{\Gamma(1+\frac{k-l}{\beta})}Y(k-l+2\beta) - 0.0012\sum_{l=0}^{k}\delta(l-3)\frac{\Gamma(2+\frac{k-l}{\beta})}{\Gamma(1+\frac{k-l}{\beta})}Y(k-l+\beta)$$

$$-0.0014\sum_{l=0}^{k}\delta(l-2)Y(k-l) + 10^{-6}\sum_{l=0}^{k}\delta(l-6)Y(k-l)\Big].$$

For instance, when $\alpha = 5.5$ we have $\beta = 10$ with starting values

$$Y(0) = \cdots = Y(9) = 0, Y(10) = 1, Y(11) = \cdots = Y(29) = 0,$$

$$Y(30) = c_1, Y(31) = \cdots = Y(49) = 0, Y(50) = c_2, Y(51) = \cdots = Y(54) = 0,$$

by means of (4.4).

Six eigenvalues are determined by the proposed approach. These eigenvalues and those reported in [19], for different values of α, are presented in Table 3. Just four eigenvalues are reported in [19].

In all examples, the n^{th} eigenfunction has exactly, $n-1$ zeros in the domain of interest.

7. Conclusion

In this paper, the existence and uniqueness of high-order fractional initial value problem have been investigated. An efficient method is implemented to calculate eigenvalues and related eigenfunctions for this kind of problems. The proposed method is more convenient than the existing methods in the literature and is easy to use for complex high-order fractional Sturm–Liouville problems. The proposed approach leads to a nonlinear system of equation for which there are many mathematical packages to solve such systems.

Table 4. Eigenvalues for Example 6.3.

	$\alpha = 5.5$		$\alpha = 5.7$		$\alpha = 5.9$	
λ_k	FDTM	[19]	FDTM	[19]	FDTM	[19]
λ_1	0.214799	0.239520	0.128826	0.144551	0.080281	0.091268
λ_2	4.335007	4.332677	4.558590	4.572395	4.449796	4.512594
λ_3	47.988457	48.049021	44.686082	44.712872	46.215498	46.230890
λ_4	164.800230	164.785334	208.321324	0.699270	242.867316	1.923295
λ_5	768.065438	–	759.242968	–	894.119252	–
λ_6	1054.834367	–	1928.516381	–	2636.35167	–

Table 5. c_2 and c_3 results for Example 6.3.

	$\alpha = 5.5$		$\alpha = 5.7$		$\alpha = 5.9$	
λ_k	c_2	c_3	c_2	c_3	c_2	c_3
λ_1	-0.0815304	0.00316322	-0.154054	0.00499523	-0.0741744	0.0021073
λ_2	-0.2346	0.0246503	-0.511215	0.0493419	-0.268707	0.0236192
λ_3	-0.565208	0.139746	-1.14109	0.24092	-0.592188	0.112587
λ_4	-0.885778	0.341979	-1.95928	0.70757	-1.03927	0.345591
λ_5	-1.55067	1.04618	-3.08474	1.75181	-1.61689	0.835696
λ_6	-1.73494	1.30934	-4.27845	3.36855	-2.33279	1.73893

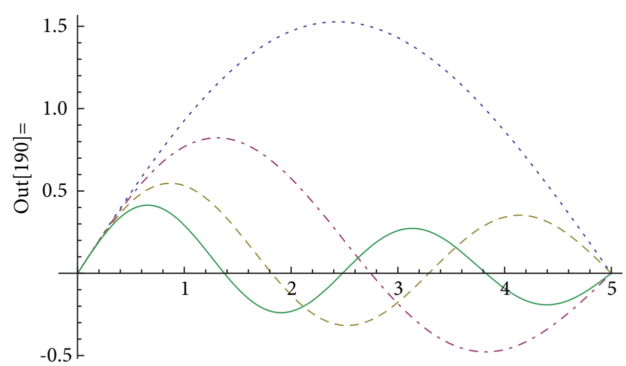

Figure 3. The first four eigenfunctions of Example 6.3 for $\alpha = 5.9$ are plotted by the dotted line, dot-dashed line, dashed line, and thick line, respectively.

As the Tables show, the number of eigenvalues gained in our method is more than those reported in [19] and [3]. Results in this paper demonstrate the high accuracy of our method to deal with high order fractional Sturm–Liouville problems in comparison with latter references.

Acknowledgement

I would like to express my very great appreciation to Prof. ANGELO MINGARELLI for his valuable and constructive suggestions during the revision process of this research work. His willingness to give his time so generously has been very much appreciated.

References

[1] Abbasbandy S, Shirzadi A. Homotopy analysis method for multiple solutions of the fractional Sturm–Liouville problems. Numerical Algorithms 2010; 54: 521-532. doi: 10.1007/s11075-009-9351-7

[2] Al-Mdallal QM. An efficient method for solving fractional Sturm–Liouville problems. Chaos Solitons Fractals 2009; 40 (1): 183-189.

[3] Al-Mdallal QM. On the numerical solution of fractional Sturm–Liouville problems. International Journal of Computer Mathematics 2010; 87 (12): 2837-2845.

[4] Al-Mdallal QM, Al-Refai M, Syam M, Sirhin M. Theoretical and computational perspectives on the eigenvalues of fourth order fractional Sturm-Liouville problem. International Journal of Computer Mathematics 2018; 95 (8): 1548-1564. doi: 10.1080/00207160.2017.1322690

[5] Arikoglu A, Ozkol I. Solution of fractional differential equations by using differential transform method. Chaos, Solitons and Fractals 2007; 34: 1473-1481.

[6] Attili BS, Lesnic D. An efficient method for computing eigenelements of Sturm-Liouville fourth order boundary value problems. Applied Mathematics and Computation 2006; 182 (2): 1247-1254.

[7] Bas E, Ozarslan R, Baleanu D, Ercan A. Comparative simulations for solutions of fractional Sturm–Liouville problems with non-singular operators. Advances in Difference Equations 2018; 350 (2018).

[8] Bas E, Ozarslan R, Baleanu D. Sturm-Liouville difference equations having Bessel and hydrogen atom potential type. Open Physics 2018; 16 (1): 801-809.

[9] Bouchaud JP, Georges A. Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications. Physics Reports 1990; 195 (45): 127-293.

[10] Caputo M. Linear models of dissipation whose Q is almost frequency independent. Geophysical Journal of the Royal Astronomical Society 1967; 13 (5): 529-539.

[11] Ciesielski M, Klimek M, Blaszczyk T. The fractional Sturm–Liouville problem—Numerical approximation and application in fractional diffusion. Journal of Computational and Applied Mathematics 2017; 317: 573-588.

[12] Chandrasekhar S. On characteristic value problems in high order differential equations which arise in studies on hydrodynamic and hydromagnetic stability. The American Mathematical Monthly 1954; 61 (7P2): 32-45.

[13] Chandrasekhar S. Hydrodynamic and Hydromagnetic Stability. Oxford: Clarenden Press, 1961. Reprinted by New York, NY, USA: Dover Books, 1981.

[14] Civalek O, Ulker M, Free vibration analysis of elastic beams using harmonic differential quadrature (HDQ). Mathematical and Computational Applications 2004; 9 (2): 257-264.

[15] Erturk VS, Computing eigenelements of Sturm–Liouville Problems of fractional order via fractional differential transform method. Mathematical and Computational Applications 2011; 16 (3): 712-720.

[16] Greenberg L, Marletta M. Oscillation theory and numerical solution of fourth order Sturm-Liouville problems. IMA Journal of Numerical Analysis 1995; 15 (3): 319-356.

[17] Greenberg L, Marletta M. Oscillation theory and numerical solution of sixth order Sturm-Liouville problems. SIAM Journal on Numerical Analysis 1998; 35 (5): 2070-2098.

[18] Greenberg L, Marletta M. Numerical methods for higher order Sturm-Liouville problems. Journal of Computational and Applied Mathematics 2000; 125 (1-2): 367-383.

[19] Hajji MA, Al-Mdallal QM, Allan FM. An efficient algorithm for solving higher-order fractional Sturm-Liouville eigenvalue problems. Journal of Computational Physics 2014; 272: 550-558.

[20] Hang Y, Chen J, Luo QZ. A simple approach for determining the eigenvalues of the fourth-order Sturm-Liouville problem with variable coefficients. Applied Mathematics Letter 2013; 26 (7): 729-734.

[21] Kashkari BSH, Syam MI. An accurate method for solving a class of fractional Sturm-Liouville eigenvalue problems. Results in Physics 2018; 9: 560-569.

[22] Kilbas AA, Srevastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. Volume 204 (North-Holand Mathematics Studies). New York, NY, USA: Elsevier Science Inc., 2006.

[23] Klimek M, Agrawal OP. Fractional Sturm–Liouvile problem. Computers and Mathematics with Applications 2013; 66 (5): 795-812.

[24] Klimek M, Odzijewicz T, Malinowska AB. Variational methods for the fractional Sturm–Liouville problem. Journal of Mathematical Analysis and Applications 2014; 416 (1): 402-426.

[25] Ledoux V, Daele MV. Solution of Sturm-Liouville problems using modified Neumann schemes. SIAM Journal on Scientific Computing 2010; 32 (2): 563-584.

[26] Luchko Y. Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation. Fractional Calculus and Applied Analysis 2012; 15 (1): 141-160.

[27] Miller KS, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations. New York, NY, USA: John Wiley and Sons Inc., 1993.

[28] Neamaty A, Darzi R, Dabbaghian A, Golipoor J. Introducing an iterative method for solving a special FDE. International Mathematical Forum 2009; 4 (30): 1449-1456.

[29] Oldham KB, Spanier J. The Fractional Calculus. New York, NY USA: Academic Press, 1974.

[30] Podlubny I. Fractional Differential Equations. New York, NY USA: Academic Press, 1999.

[31] Pryce JD. Numerical Solution of Sturm-Liouville Problems. New York, NY USA: Oxford University Press, 1993.

[32] Reutskiy SY. A novel method for solving second order fractional eigenvalue problems, Journal of Computational and Applied Mathematics 2016; 306: 133-153.

[33] Taher AHS, Malek A, Thabet ASA. Semi-analytical approximation for solving high-order Sturm-Liouville Problems. British Journal of Mathematics and Computer Science 2014; 23: 3345-3357.

[34] Taher AHS, Malek A, Momeni-Masuleh SH. Chebyshev differentiation matrices for efficient computation of the eigenvalues of fourth-order Sturm-Liouville problems. Applied Mathematical Modelling 2013; 37 (7): 4634-4642.

[35] Taher AHS, Malek A. An efficient algorithm for solving high-order Sturm-Liouville problems using variational iteration method. Fixed Point Theory 2013; 14: 193-210.

[36] Tisdell C, Ford N. On the application of sequential and fixed-point methods to fractional differential equations of arbitrary order. Journal of Integral Equations and Applications 2012; 24 (2): 283-319.

[37] Ugurlu E, Baleanu D, Tas T. On square integrable solutions of a fractional differential equation. Applied Mathematics and Computation 2018; 337: 153-157.

[38] Ugurlu E, Baleanu D, Tas T. Regular fractional differential equations in the Sobolev space. Fractional Calculus and Applied Analysis 2017; 20 (3): 810-817.

[39] Xue T, Liu W, Zhang W. Existence of solutions for Sturm-Liouville boundary value problems of higher-order coupled fractional differential equations at resonance. Advances in Difference Equations 2017; 301 (2017). doi: 10.1186/s13662-017-1345-5

[40] Zayernouri M, Karniadakis GE. Fractional Sturm-Liouville eigen-problems: Theory and numerical approximation. Journal of Computational Physics 2013; 252: 495-517.

[41] Zayernouri M, Ainsworth M, Karniadakis GE. Tempered fractional Sturm–Liouville eigenproblems. SIAM Journal on Scientific Computing; 37 (4): A1777-A1800.

[42] Zettl A. Sturm-Liouville Theory. Providence, RI, USA: American Mathematical Society, 2005.