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## Generalized Stević–Sharma type operators from Hardy spaces into $n$ th weighted type spaces

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**Abstract:** In this paper, some characterizations for boundedness, essential norm and compactness of generalized Stević–Sharma type operators from Hardy spaces into  $n$ th weighted type spaces are given.

**Key words:** Essential norm, generalized Stević–Sharma type operators,  $n$ th weighted type spaces, Hardy spaces

### 1. Introduction

Let  $\mathbb{D}$  be the open unit disc in the complex plane  $\mathbb{C}$  and  $H(\mathbb{D})$  be the class of all analytic functions on  $\mathbb{D}$ . Every positive and continuous function on  $\mathbb{D}$  is called a weight. Suppose that  $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$  and  $\mu$  be a weight. The  $n$ th weighted type space  $\mathcal{W}_\mu^{(n)}(\mathbb{D}) = \mathcal{W}_\mu^{(n)}$  consists of all analytic functions on  $\mathbb{D}$  for which the following statement is finite

$$b_{\mathcal{W}_\mu^{(n)}}(f) := \sup_{z \in \mathbb{D}} \mu(z) |f^{(n)}(z)|.$$

The above statement is just a semi norm and  $\mathcal{W}_\mu^{(n)}$  is a Banach space equipped with the norm

$$\|f\|_{\mathcal{W}_\mu^{(n)}} = \sum_{i=0}^{n-1} |f^{(i)}(0)| + b_{\mathcal{W}_\mu^{(n)}}(f),$$

See for example [1, 9, 10]. Let  $\alpha > 0$ . Then  $\mathcal{W}_{(1-|z|^2)^\alpha}^{(1)} = \mathcal{B}^\alpha$  (Bloch type space),  $\mathcal{W}_{(1-|z|^2)^\alpha}^{(2)} = \mathcal{Z}^\alpha$  (Zygmund type space) and  $\mathcal{W}_{(1-|z|^2) \log \frac{2}{1-|z|^2}}^{(1)}$  coincides with the logarithmic Bloch space  $\mathcal{B}_{\log}$ . Also  $\mathcal{W}_\mu^{(0)} = H_\mu$  (weighted type space),  $\mathcal{W}_\mu^{(1)} = \mathcal{B}_\mu$  (weighted Bloch space) and  $\mathcal{W}_\mu^{(2)} = \mathcal{Z}_\mu$  (weighted Zygmund space). For more information about Bloch type spaces or Zygmund type spaces see [8, 15, 16].

For  $0 < p < \infty$  a function  $f \in H(\mathbb{D})$  belong to the Hardy space  $H^p$  if

$$\|f\|_{H^p} = \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty.$$

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where  $0 < p < \infty$ . If  $1 \leq p < \infty$ ,  $H^p$  is a Banach space and if  $0 < p < 1$ ,  $H^p$  is nonlocally convex topological vector space and in this case it is a complete metric space (see [4]).

For Banach spaces  $X$  and  $Y$  and a continuous linear operator  $T : X \rightarrow Y$ , the essential norm is the distance of  $T$  from the space of all compact operators, that is

$$\|T\|_e = \inf\{\|T - K\| : K : X \rightarrow Y \text{ is compact}\}.$$

$T$  is compact if and only if  $\|T\|_e = 0$ .

Let  $u, v \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$ , the set of all analytic self-maps of  $\mathbb{D}$ . The Stević-Sharma type operator is defined as follows

$$T_{u,v,\varphi}f(z) = u(z)f(\varphi(z)) + v(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

Indeed  $T_{u,v,\varphi} = uC_\varphi + vC_\varphi D$  where  $D$  is the differentiation operator and  $C_\varphi$  is composition operator. More information about this operator can be found in [7, 11, 12].

From the above definition we generalize the Stević-Sharma type operator. Let  $m \in \mathbb{N}$ ,  $u, v \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$ . We denote the generalized Stević-Sharma type operator with  $T_{u,v,\varphi}^m$  and define it as follows:

$$T_{u,v,\varphi}^m f(z) = (uC_\varphi f)(z) + (D_{\varphi,v}^m f)(z) \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D},$$

where  $D_{\varphi,u}^m$  is the generalized weighted composition operator. When  $v = 0$ , then  $T_{u,0,\varphi}^m = uC_\varphi$  is the well-known weighted composition operator. If  $u = 0$ , then,  $T_{0,v,\varphi}^m = D_{\varphi,v}^m$  and for  $m = 1$ ,  $T_{u,v,\varphi}^m$  is Stević-Sharma type operator.

For  $n, k \in \mathbb{N}_0$  and  $k \leq n$ , the partial Bell polynomials are triangulares

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{\prod_{t=1}^{n-k+1} j_t!} \prod_{t=1}^{n-k+1} \left(\frac{x_t}{t!}\right)^{j_t}.$$

In the above equation we take the sum over all sequences  $j_1, j_2, \dots, j_{n-k+1}$  of nonnegative integers with the following properties

$$\sum_{t=1}^{n-k+1} j_t = k \quad \text{and} \quad \sum_{t=1}^{n-k+1} t j_t = n.$$

See [3, pp 134].

In this paper, first we obtain some characterizations for boundedness of operator  $T_{u,v,\varphi}^m : H^p \rightarrow \mathcal{W}_\mu^{(n)}$ . Then estimations for the essential norm of these operators are given. Finally some equivalence conditions for compactness of generalized Stević-Sharma type operators from Hardy spaces into  $n$ th weighted type spaces are presented. As some applications, we get some characterizations for boundedness, essential norm and compactness of (generalized) weighted composition operators from the Hardy spaces into  $n$ th weighted type spaces.

By  $A \succeq B$  we mean there exists a constant  $C$  such that  $A \geq CB$  and  $A \approx B$  means that  $A \succeq B \succeq A$ .

## 2. Preliminaries

This section is devoted to giving some lemmas we use in the next sections.

**Lemma 2.1** ([16], Propositions 7 and 8) *Let  $\alpha > 0$  and  $H_\alpha^\infty = \mathcal{W}_{(1-|z|^2)^\alpha}^{(0)}$ . Then  $H_\alpha^\infty = \mathcal{B}^{\alpha+1}$ . Moreover, for any  $f \in \mathcal{B}^\alpha$  and  $n \in \mathbb{N}$ ,*

$$\|f\|_{\mathcal{B}^\alpha} \approx \sum_{i=0}^n |f^{(i)}(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha+n} |f^{(n+1)}(z)|.$$

**Lemma 2.2** ([5], Lemma 2.1) *Let  $\alpha > 0$ . The sequence  $\{j^{\alpha-1}z^j\}_1^\infty$  is bounded in  $\mathcal{B}_0^\alpha$  and*

$$\lim_{j \rightarrow \infty} j^{\alpha-1} \|z^j\|_{\mathcal{B}^\alpha} = \left(\frac{2\alpha}{e}\right)^\alpha.$$

**Lemma 2.3** ([4]) *Let  $0 < p < \infty$ ,  $n \in \mathbb{N}_0$  and  $f \in H^p$ . Then*

$$|f^{(n)}(z)| \leq \frac{\|f\|_{H^p}}{(1 - |z|^2)^{\frac{1}{p}+n}}, \quad z \in \mathbb{D}.$$

Let  $u \in H(\mathbb{D})$ ,  $i$  and  $n$  be integer numbers. For simplicity in calculation, we set

$$I_{i,\varphi}^{n,u}(z) = \begin{cases} \sum_{l=i}^n \binom{n}{l} u^{(n-l)}(z) B_{l,i}(\varphi'(z), \dots, \varphi^{(l-i+1)}(z)) & 0 \leq i \leq n \in \mathbb{N}_0 \\ 0 & \text{otherwise} \end{cases}$$

The proof of next lemma resembles to the proof of Lemma 4 [10], therefore it is omitted.

**Lemma 2.4** *Let  $f, u, v \in H(\mathbb{D})$ ,  $\varphi \in S(\mathbb{D})$  and  $m, n \in \mathbb{N}_0$ . If  $T_{u,v,\varphi}^m = uC_\varphi + D_{\varphi,v}^m$ , then*

$$\left(T_{u,v,\varphi}^m f\right)^{(n)}(z) = \sum_{i=0}^{m+n} f^{(i)}(\varphi(z)) (I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(z).$$

For any  $a \in \mathbb{D}$  and  $j \in \mathbb{N}$ , set

$$f_{j,a}(z) = \frac{(1 - |a|^2)^j}{(1 - \bar{a}z)^{\frac{1}{p}+j}}. \tag{2.1}$$

One can see that  $f_{j,a} \in H^p$ , for each  $j \in \mathbb{N}$ ,  $\sup_{a \in \mathbb{D}} \|f_{j,a}\|_{H^p} < \infty$  and  $f_{j,a}$  converges to 0 as  $|a| \rightarrow 1$ .

**Lemma 2.5** *Let  $m, n \in \mathbb{N}$  such that  $n \geq m$ . For any  $0 \neq a \in \mathbb{D}$  and  $i \in \{0, 1, \dots, m+n\}$ , there exists a function  $g_{i,a} \in H^p$  such that*

$$g_{i,a}^{(k)}(a) = \frac{\bar{a}^k \delta_{ik}}{(1 - |a|^2)^{\frac{1}{p}+k}},$$

where  $\delta_{ik}$  is Kronecker delta. If  $i \in \{0, 1, \dots, m-1\}$ , then  $g_{i,a} \in \text{span}\{f_{1,a}, \dots, f_{m,a}\}$  and for  $i \in \{m, \dots, n\}$ ,  $g_{i,a} \in \text{span}\{f_{m+1,a}, \dots, f_{n+1,a}\}$  also when  $i \in \{n+1, \dots, m+n\}$

$$g_{i,a} \in \text{span}\{f_{n+2,a}, \dots, f_{m+n+1,a}\}.$$

**Proof** For any fixed  $0 \neq a \in \mathbb{D}$  and coefficients  $c_1, \dots, c_{m+n+1}$  we set

$$\begin{aligned}
 e_{1,a,c_1,\dots,c_m}(z) &= \sum_{j=1}^m c_j f_{j,a}(z), \\
 e_{2,a,c_{m+1},\dots,c_{n+1}}(z) &= \sum_{j=1}^{n-m+1} \frac{c_{j+m}}{\prod_{t=0}^{m-1} (m+j+\frac{1}{p}+t)} f_{j+m,a}(z) \\
 e_{3,a,c_{n+2},\dots,c_{m+n+1}}(z) &= \sum_{j=1}^m \frac{c_{j+1+n}}{\prod_{t=0}^n (n+1+j+\frac{1}{p}+t)} f_{j+1+n,a}(z),
 \end{aligned}$$

where  $f_{j,a}$  are defined in (2.1). For each  $i \in \{0, 1, \dots, m+n\}$  the system of linear equations

$$\begin{aligned}
 e_{1,a,c_1,\dots,c_m}(a) &= \frac{1}{(1-|a|^2)^{\frac{1}{p}}} \sum_{j=1}^m c_j = \frac{\delta_{i0}}{(1-|a|^2)^{\frac{1}{p}}} \\
 &\vdots \\
 e_{1,a,c_1,\dots,c_m}^{(m-1)}(a) &= \frac{\bar{a}^{m-1}}{(1-|a|^2)^{m-1+\frac{1}{p}}} \sum_{j=1}^m c_j \prod_{t=0}^{m-2} (j+\frac{1}{p}+t) = \frac{\bar{a}^{m-1} \delta_{i(m-1)}}{(1-|a|^2)^{m-1+\frac{1}{p}}} \\
 e_{2,a,c_{m+1},\dots,c_{n+1}}^{(m)}(a) &= \frac{\bar{a}^m}{(1-|a|^2)^{m+\frac{1}{p}}} \sum_{j=1}^{n-m+1} c_{j+m} = \frac{\bar{a}^m \delta_{im}}{(1-|a|^2)^{m+\frac{1}{p}}} \\
 &\vdots \\
 e_{2,a,c_{m+1},\dots,c_{n+1}}^{(n)}(a) &= \frac{\bar{a}^n}{(1-|a|^2)^{n+\frac{1}{p}}} \sum_{j=1}^{n-m+1} c_{j+m} \prod_{t=m}^{n-1} (m+j+\frac{1}{p}+t) = \frac{\bar{a}^n \delta_{in}}{(1-|a|^2)^{n+\frac{1}{p}}} \\
 e_{3,a,c_{n+2},\dots,c_{m+n+1}}^{(n+1)}(a) &= \frac{\bar{a}^{n+1}}{(1-|a|^2)^{n+1+\frac{1}{p}}} \sum_{j=1}^m c_{j+1+m} = \frac{\bar{a}^{n+1} \delta_{i(n+1)}}{(1-|a|^2)^{n+1+\frac{1}{p}}} \\
 &\vdots \\
 e_{3,a,c_{n+2},\dots,c_{m+n+1}}^{(m+n)}(a) &= \dots = \frac{\bar{a}^{m+n} \delta_{i(m+n)}}{(1-|a|^2)^{m+n+\frac{1}{p}}}
 \end{aligned}$$

has a unique solution [9, Lemma 2.3] which is independent of the choice of  $a$  and therefore it can be shown by  $(c_1^i, c_2^i, \dots, c_{m+n+1}^i)$ . Now we set

$$g_{i,a}(z) = e_{1,a,c_1^i,\dots,c_m^i}(z) + e_{2,a,c_{m+1}^i,\dots,c_{n+1}^i}(z) + e_{3,a,c_{n+2}^i,\dots,c_{m+n+1}^i}(z).$$

□

The proof of the following lemma is similar to the proof of the previous lemma so it is omitted.

**Lemma 2.6** *Let  $m, n \in \mathbb{N}$  such that  $n < m$ . For any  $0 \neq a \in \mathbb{D}$  and  $i \in \{0, \dots, n\} \cup \{m, \dots, m+n\}$ , there exists a function  $g_{i,a} \in H^p$  such that*

$$g_{i,a}^{(k)}(a) = \frac{\bar{a}^k \delta_{ik}}{(1-|a|^2)^{\frac{1}{p}+k}}.$$

Also for  $i \in \{0, 1, \dots, n\}$  then  $g_{i,a} \in \text{span}\{f_{1,a}, \dots, f_{n+1,a}\}$  and when  $i \in \{m, m + 1, \dots, m + n\}$

$$g_{i,a} \in \text{span}\{f_{m+1,a}, \dots, f_{m+n+1,a}\}.$$

In Sections 3 and 4,  $m, n \in \mathbb{N}$ ,  $0 < p < \infty$ ,  $u, v \in H(\mathbb{D})$ ,  $\mu$  is a weight and  $\varphi \in S(\mathbb{D})$ .

### 3. Boundedness

In this section, we give some necessary and sufficient conditions for the generalized Stević–Sharma type operators to be bounded.

**Theorem 3.1** *Let  $u \in \mathcal{W}_\mu^{(n)}$ . If  $n \geq m$ , then the following statements are equivalent.*

- (i) *The operator  $T_{u,v,\varphi}^m : H^p \rightarrow \mathcal{W}_\mu^{(n)}$  is bounded.*
- (ii) *If  $p_j(z) = z^j$ , then  $\sup_{j \geq 1} j^{\frac{1}{p}} \|T_{u,v,\varphi}^m p_j\|_{\mathcal{W}_\mu^{(n)}} < \infty$ .*
- (iii) *For each  $i \in \{0, 1, \dots, m + n\}$ ,  $\sup_{a \in \mathbb{D}} \|T_{u,v,\varphi}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} < \infty$  and  $\sup_{z \in \mathbb{D}} \mu(z) |(I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(z)| < \infty$ .*
- (iv) *For each  $i \in \{0, 1, \dots, m + n\}$ ,  $\sup_{z \in \mathbb{D}} \frac{\mu(z) |(I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p} + i}} < \infty$ .*

**Proof** (i)  $\Rightarrow$  (iii) For  $i \in \{0, 1, \dots, m + n\}$ ,  $\sup_{a \in \mathbb{D}} \|f_{i+1,a}\|_{H^p} < \infty$ , so

$$\sup_{a \in \mathbb{D}} \|T_{u,v,\varphi}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} \leq \|T_{u,v,\varphi}^m\|_{H^p \rightarrow \mathcal{W}_\mu^{(n)}} \sup_{a \in \mathbb{D}} \|f_{i+1,a}\|_{H^p} < \infty.$$

Applying the operator  $T_{u,v,\varphi}^m$  to  $p_j(z) = z^j$  for  $j = 0, 1, \dots, m + n$  respectively and using Lemma 2.4, we obtain the other part of (iii).

(iii)  $\Rightarrow$  (iv) For any  $i \in \{0, 1, \dots, m + n\}$  and  $\varphi(a) \neq 0$ , by using Lemmas 2.4 and 2.5, we obtain

$$\frac{\mu(a) |\varphi(a)|^i |(I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(a)|}{(1 - |\varphi(a)|^2)^{i + \frac{1}{p}}} \leq \sup_{a \in \mathbb{D}} \|T_{u,v,\varphi}^m g_{i,\varphi(a)}\|_{\mathcal{W}_\mu^{(n)}} \leq \sum_{j=1}^{m+n} |c_j^i| \sup_{a \in \mathbb{D}} \|T_{u,v,\varphi}^m f_{j,a}\|_{\mathcal{W}_\mu^{(n)}} < \infty.$$

From the previous inequality,

$$\sup_{|\varphi(a)| > \frac{1}{2}} \frac{\mu(a) |(I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(a)|}{(1 - |\varphi(a)|^2)^{i + \frac{1}{p}}} \leq \sum_{j=1}^{m+n} |c_j^i| \sup_{a \in \mathbb{D}} \|T_{u,v,\varphi}^m f_{j,a}\|_{\mathcal{W}_\mu^{(n)}} < \infty,$$

and from (iii), we get

$$\sup_{|\varphi(a)| \leq \frac{1}{2}} \frac{\mu(a) |(I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(a)|}{(1 - |\varphi(a)|^2)^{i + \frac{1}{p}}} \leq \sup_{|\varphi(a)| \leq \frac{1}{2}} \mu(a) |(I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(a)| < \infty.$$

Hence from last inequalities, we get (iv).

(iv)  $\Rightarrow$  (i) For any  $f \in H^p$ , by using Lemmas 2.4 and 2.3, we have

$$\begin{aligned} \mu(z) \left| (T_{u,v,\varphi}^m f)^{(n)}(z) \right| &\leq \mu(z) \left| \sum_{i=0}^{m+n} f^{(i)}(\varphi(z)) (I_{i,\varphi}^{n,u}(z) + I_{i-m,\varphi}^{n,v}(z)) \right| \\ &\leq \|f\|_{H^p} \sum_{i=0}^{m+n} \sup_{z \in \mathbb{D}} \frac{\mu(z) |I_{i,\varphi}^{n,u}(z) + I_{i-m,\varphi}^{n,v}(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p}+i}}. \end{aligned} \tag{3.1}$$

Also for each  $k < n$

$$\left| (T_{u,v,\varphi}^m f)^{(k)}(0) \right| \leq \|f\|_{H^p} \sum_{i=0}^{m+k} |(I_{i,\varphi}^{k,u} + I_{i-m,\varphi}^{k,v})(0)|. \tag{3.2}$$

Hence, from (3.1) and (3.2), the operator  $T_{u,v,\varphi}^m : H^p \rightarrow \mathcal{W}_\mu^{(n)}$  is bounded.

(ii)  $\Rightarrow$  (iii) For each  $i \in \{0, \dots, m+n\}$  and  $a \in \mathbb{D}$

$$f_{i+1,a}(z) = (1 - |a|^2)^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{p} + i + 1 + j)}{j! \Gamma(\frac{1}{p} + i + 1)} \bar{a}^j z^j.$$

So,

$$\|T_{u,v,\varphi}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} \leq (1 - |a|^2)^{i+1} \sum_{j=0}^{\infty} j^{i+\frac{1}{p}} |\bar{a}|^j \|T_{u,v,\varphi}^m p_j\|_{\mathcal{W}_\mu^{(n)}} \leq 2^{i+1} \max\{\|u\|_{\mathcal{W}_\mu^{(n)}}, \sup_{j \geq 1} j^{\frac{1}{p}} \|T_{u,v,\varphi}^m p_j\|_{\mathcal{W}_\mu^{(n)}}\}.$$

Therefore,  $\sup_{a \in \mathbb{D}} \|T_{u,v,\varphi}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} < \infty$ . The proof of other part is similar to the proof (i)  $\Rightarrow$  (iii).

(iv)  $\Rightarrow$  (ii) Let  $p_j(z) = z^j (j \geq n)$ . By using Lemmas 2.1, 2.2 and 2.4, we get

$$\begin{aligned} j^{\frac{1}{p}} \mu(z) \left| (T_{u,v,\varphi}^m p_j)^{(n)}(z) \right| &\leq \mu(z) \sum_{i=0}^{m+n} j^{\frac{1}{p}} (1 - |\varphi(z)|^2)^{\frac{1}{p}+i} \frac{j!}{(j-i)!} \times \frac{|\varphi(z)|^{j-i} |I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v}(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p}+i}} \\ &\leq \sup_j j^{\frac{1}{p}} \|z^j\|_{\mathcal{B}^{\frac{1}{p}+1}} \sum_{i=0}^{m+n} \frac{|(I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p}+i}} \\ &\leq \left(\frac{2(\frac{1}{p} + 1)}{e}\right)^{\frac{1}{p}+1} \sum_{i=0}^{m+n} \frac{|(I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p}+i}}. \end{aligned} \tag{3.3}$$

For any  $k < n$ , we have

$$j^{\frac{1}{p}} \left| (T_{u,v,\varphi}^m p_j)^{(k)}(0) \right| \leq \left(\frac{2(\frac{1}{p} + 1)}{e}\right)^{\frac{1}{p}+1} \sum_{i=0}^{m+k} j^{\frac{1}{p}} \frac{|(I_{i,\varphi}^{k,u} + I_{i-m,\varphi}^{k,v})(0)|}{(1 - |\varphi(0)|^2)^{\frac{1}{p}+i}}. \tag{3.4}$$

From (3.3) and (3.4), we obtain (ii). The proof is completed.  $\square$

In the same way as in the proof of Theorem 3.1 we can prove the following theorem, just use Lemma 2.6 instead of Lemma 2.5.

**Theorem 3.2** Let  $u \in \mathcal{W}_\mu^{(n)}$ . If  $n < m$ , then the following statements are equivalent.

(i) The operator  $T_{u,v,\varphi}^m : H^p \rightarrow \mathcal{W}_\mu^{(n)}$  is bounded.

(ii) If  $p_j(z) = z^j$ , then  $\sup_{j \geq 1} j^{\frac{1}{p}} \|T_{u,v,\varphi,u}^m p_j\|_{\mathcal{W}_\mu^{(n)}} < \infty$ .

(iii) For each  $i \in \{0, 1, \dots, n\} \cup \{m, \dots, m+n\}$ ,  $\sup_{a \in \mathbb{D}} \|T_{u,v,\varphi}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} < \infty$  and

$$\sup_{z \in \mathbb{D}} \mu(z) \left| (I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(z) \right| < \infty.$$

(iv) For each  $i \in \{0, 1, \dots, n\} \cup \{m, \dots, m+n\}$ ,  $\sup_{z \in \mathbb{D}} \frac{\mu(z) \left| (I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(z) \right|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i}} < \infty$ .

#### 4. Essential norm

In this section, we obtain some estimates for the essential norm of generalized Stević-Sharma type operators from Hardy spaces into  $n$ th weighted type spaces. Then we give some equivalence conditions for compactness of such operators.

**Theorem 4.1** Let  $T_{u,v,\varphi}^m : H^p \rightarrow \mathcal{W}_\mu^{(n)}$  is bounded. If  $n \geq m$ , then

$$\|T_{u,v,\varphi}^m\|_e \approx \max\{A_i\}_{i=0}^{m+n} \approx \max\{B_i\}_{i=0}^{m+n},$$

where

$$A_i = \limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}}, \quad B_i = \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) \left| I_{i,\varphi}^{n,u}(z) + I_{i-m,\varphi}^{n,v}(z) \right|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i}}.$$

**Proof** For all  $i \in \{0, \dots, m+n\}$ ,  $\sup_{a \in \mathbb{D}} \|f_{i+1,a}\|_{H^p} < \infty$  and  $f_{i+1,a}$  converge to 0 uniformly on compact subsets of  $\mathbb{D}$  as  $|a| \rightarrow 1$ . Using Lemma 2.10 [13], for any compact operator  $K$  from  $H^p$  into  $\mathcal{W}_\mu^{(n)}$ , we get

$$\lim_{|a| \rightarrow 1} \|K f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} = 0.$$

Thus, for any  $i \in \{0, \dots, m+n\}$

$$\begin{aligned} \|T_{u,v,\varphi}^m - K\|_{H^p \rightarrow \mathcal{W}_\mu^{(n)}} &\geq \limsup_{|a| \rightarrow 1} \|(T_{u,v,\varphi}^m - K) f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} \\ &\geq \limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} - \limsup_{|a| \rightarrow 1} \|K f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} = A_i. \end{aligned}$$

So,

$$\|T_{u,v,\varphi}^m\|_e = \inf_K \|T_{u,v,\varphi}^m - K\|_{H^p \rightarrow \mathcal{W}_\mu^{(n)}} \geq \max\{A_i\}_{i=0}^{m+n}.$$

Now, we prove that

$$\max\{B_i\}_{i=0}^{m+n} \leq \|T_{u,v,\varphi}^m\|_e. \tag{4.1}$$



Let  $\{z_j\}_{j \in \mathbb{N}}$  be a sequence in  $\mathbb{D}$  such that  $|\varphi(z_j)| \rightarrow 1$  as  $j \rightarrow \infty$ . Since  $T_{u,v,\varphi}^m : H^p \rightarrow \mathcal{W}_\mu^{(n)}$  is bounded, using Lemmas 2.4 and 2.5 for any compact operator  $K : H^p \rightarrow \mathcal{W}_\mu^{(n)}$  and  $i \in \{0, \dots, m+n\}$ , we obtain

$$\begin{aligned} \|T_{u,v,\varphi}^m - K\|_{H^p \rightarrow \mathcal{W}_\mu^{(n)}} &\succeq \limsup_{j \rightarrow \infty} \|T_{u,v,\varphi}^m(g_{i,\varphi(z_j)})\|_{\mathcal{W}_\mu^{(n)}} - \limsup_{j \rightarrow \infty} \|K(g_{i,\varphi(z_j)})\|_{\mathcal{W}_\mu^{(n)}} \\ &\succeq \limsup_{j \rightarrow \infty} \frac{\mu(z_j) |\varphi(z_j)|^i |I_{i,\varphi}^{n,u}(z_j) + I_{i-m,\varphi}^{n,v}(z_j)|}{(1 - |\varphi(z_j)|^2)^{i + \frac{1}{p}}} = B_i. \end{aligned}$$

From the last inequality, we get (4.1).

For each  $0 < r < 1$  we consider the compact operator  $K_r$  on  $H^p$  given by  $K_r f(z) = f_r(z) = f(rz)$ . Let  $\{r_j\} \subset (0, 1)$  be a sequence such that  $r_j \rightarrow 1$  as  $j \rightarrow \infty$ . Since  $f_r \rightarrow f$  uniformly on compact subsets of  $\mathbb{D}$  as  $r \rightarrow 1$  then for any positive integer  $j$ , the operator  $T_{u,v,\varphi}^m K_{r_j} : H^p \rightarrow \mathcal{W}_\mu^{(n)}$  is compact. So

$$\|T_{u,v,\varphi}^m\|_e \leq \limsup_{j \rightarrow \infty} \|T_{u,v,\varphi}^m - T_{u,v,\varphi}^m K_{r_j}\|. \tag{4.2}$$

Hence, it is sufficient to prove that

$$\limsup_{j \rightarrow \infty} \|T_{u,v,\varphi}^m - T_{u,v,\varphi}^m K_{r_j}\| \preceq \min\{\max\{A_i\}_{i=0}^{m+n}, \max\{B_i\}_{i=0}^{m+n}\}.$$

For any  $f \in H^p$  such that  $\|f\|_{H^p} \leq 1$ ,

$$\begin{aligned} &\|(T_{u,v,\varphi}^m - T_{u,v,\varphi}^m K_{r_j})f\|_{\mathcal{W}_\mu^{(n)}} \leq \\ &\sum_{t=0}^{n-1} \underbrace{\left| \left( T_{u,v,\varphi}^m (f - f_{r_j}) \right)^{(t)}(0) \right|}_{S_t} + \underbrace{\sup_{|\varphi(z)| \leq r_N} \mu(z) \left| \sum_{k=0}^{m+n} (f - f_{r_j})^{(k)}(\varphi(z)) \left( I_{k,\varphi}^{n,u} + I_{k-m,\varphi}^{n,v} \right)(z) \right|}_{H_1} \\ &+ \underbrace{\sup_{|\varphi(z)| > r_N} \mu(z) \left| \sum_{k=0}^{m+n} (f - f_{r_j})^{(k)}(\varphi(z)) \left( I_{k,\varphi}^{n,u} + I_{k-m,\varphi}^{n,v} \right)(z) \right|}_{H_2}, \end{aligned} \tag{4.3}$$

where  $N \in \mathbb{N}$  such that  $r_j \geq \frac{2}{3}$  for all  $j \geq N$ . Since  $(f - f_{r_j})^{(s)} \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$  as  $j \rightarrow \infty$ , for any nonnegative integer  $s$ , then Theorem 3.1 implies that

$$\limsup_{j \rightarrow \infty} H_1 = \limsup_{j \rightarrow \infty} S_t = 0 \quad (t = 0, \dots, n-1). \tag{4.4}$$

Also

$$\begin{aligned} H_2 &\leq \underbrace{\sum_{k=0}^{m+n} \sup_{|\varphi(z)| > r_N} \mu(z) |f^{(k)}(\varphi(z))| | (I_{k,\varphi}^{n,u} + I_{k-m,\varphi}^{n,v})(z) |}_{M_{2,k}} \\ &+ \underbrace{\sum_{k=0}^{m+n} \sup_{|\varphi(z)| > r_N} \mu(z) |r_j^k f^{(k)}(r_j \varphi(z))| | (I_{k,\varphi}^{n,u} + I_{k-m,\varphi}^{n,v})(z) |}_{N_{2,k}}. \end{aligned} \tag{4.5}$$

For  $M_{2,k}$ ,  $k \in \{0, \dots, m+n\}$ , from Lemmas 2.3, 2.4 and 2.5, we get

$$M_{2,k} = \sup_{|\varphi(z)| > r_N} \mu(z) \frac{(1 - |\varphi(z)|^2)^{k + \frac{1}{p}} |f^{(k)}(\varphi(z))|}{|\varphi(z)|^k} \times \frac{|\varphi(z)|^k |(I_{k,\varphi}^{n,u} + I_{k-m,\varphi}^{n,v})(z)|}{(1 - |\varphi(z)|^2)^{k + \frac{1}{p}}}$$

$$\preceq \|f\|_{H^p} \sup_{|\varphi(z)| > r_N} \|T_{u,v,\varphi}^m g_{k,\varphi}(z)\|_{\mathcal{W}_\mu^{(n)}} \preceq \sum_{j=0}^{m+n} |c_{j+1}^k| \sup_{|a| > r_N} \|T_{u,v,\varphi}^m f_{j+1,a}\|_{\mathcal{W}_\mu^{(n)}}$$

As  $N \rightarrow \infty$ , we obtain

$$\limsup_{j \rightarrow \infty} M_{2,k} \preceq \underbrace{\sum_{i=0}^{m+n} \limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}}}_{A_i} \preceq \max\{A_i\}_{i=0}^{m+n} \quad \text{and} \quad \limsup_{j \rightarrow \infty} M_{2,k} \preceq B_k. \tag{4.6}$$

Similarly, we get

$$\limsup_{j \rightarrow \infty} N_{2,k} \preceq \underbrace{\sum_{i=0}^{m+n} \limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}}}_{A_i} \preceq \max\{A_i\}_{i=0}^{m+n} \quad \text{and} \quad \limsup_{j \rightarrow \infty} N_{2,k} \preceq B_k. \tag{4.7}$$

Thus, by using (4.3), (4.4), (4.5), (4.6) and (4.7), we obtain

$$\limsup_{j \rightarrow \infty} \|T_{u,v,\varphi}^m - T_{u,v,\varphi}^m K_{r_j}\|_{H^p \rightarrow \mathcal{W}_\mu^{(n)}} = \limsup_{j \rightarrow \infty} \sup_{\|f\|_{H^p} \leq 1} \|(T_{u,v,\varphi}^m - T_{u,v,\varphi}^m K_{r_j})f\|_{\mathcal{W}_\mu^{(n)}} \preceq \max\{A_i\}_{i=0}^{m+n}$$

and

$$\limsup_{j \rightarrow \infty} \|T_{u,v,\varphi}^m - T_{u,v,\varphi}^m K_{r_j}\|_{H^p \rightarrow \mathcal{W}_\mu^{(n)}} \preceq \max\{B_i\}_{i=0}^{m+n}.$$

Hence, from (4.2),

$$\|T_{u,v,\varphi}^m\|_e \preceq \min\{\max\{A_i\}_{i=0}^{m+n}, \max\{B_i\}_{i=0}^{m+n}\}.$$

The proof is completed. □

The proof of the next theorem is similar to the proof of Theorem 4.1, except that we use Lemma 2.6 instead of Lemma 2.5.

**Theorem 4.2** *Let  $T_{u,v,\varphi}^m : H^p \rightarrow \mathcal{W}_\mu^{(n)}$  is bounded. If  $n < m$ , then*

$$\|T_{u,v,\varphi}^m\|_e \approx \max\{\{A_i\}_{i=0}^n \cup \{A_i\}_{i=m}^{m+n}\} \approx \max\{\{B_i\}_{i=0}^n \cup \{B_i\}_{i=m}^{m+n}\},$$

where

$$A_i = \limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}}, \quad B_i = \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |(I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p} + i}}$$

and  $i \in \{0, 1, \dots, n\} \cup \{m, \dots, m+n\}$ .

**Theorem 4.3** Let  $T_{u,v,\varphi}^m : H^p \rightarrow \mathcal{W}_\mu^{(n)}$  is bounded. If  $n \geq m$ , then the following statements are equivalent.

(i) The operator  $T_{u,v,\varphi}^m : H^p \rightarrow \mathcal{W}_\mu^{(n)}$  is compact.

(ii) If  $p_j(z) = z^j$  then,  $\lim_{j \rightarrow \infty} j^{\frac{1}{p}} \|T_{u,v,\varphi}^m p_j\|_{\mathcal{W}_\mu^{(n)}} = 0$ .

(iii) For each  $i \in \{0, 1, \dots, m + n\}$ ,  $\limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} = 0$ .

(iv) For each  $i \in \{0, 1, \dots, m + n\}$   $\limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |(I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p} + i}} = 0$ .

**Proof** By using Theorem 4.1, (i), (iii) and (iv) are equivalent.

(ii)  $\Rightarrow$  (iii) For any given  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for  $k \geq N$ ,

$$k^{\frac{1}{p}} \|T_{u,v,\varphi}^m p_k\|_{\mathcal{W}_\mu^{(n)}} < \epsilon.$$

For any  $j \in \{0, 1, \dots, m + n\}$

$$f_{j+1,a}(z) = (1 - |a|^2)^{j+1} \left( \sum_{k=0}^{N-1} + \sum_{k=N}^{\infty} \right) \frac{\Gamma(\frac{1}{p} + j + 1 + k)}{k! \Gamma(\frac{1}{p} + j + 1)} \bar{a}^k z^k.$$

So,

$$\|T_{u,v,\varphi}^m f_{j+1,a}\|_{\mathcal{W}_\mu^{(n)}} \leq 2 \max\{\|u\varphi^k\|_{\mathcal{W}_\mu^{(n)}}\}_{k=0}^{N-1} (1 - |a|^2)^j (1 - |a|^N)^{\frac{\Gamma(\frac{1}{p} + N + j)}{N! \Gamma(\frac{1}{p} + j)}} + 2^{j+1} \epsilon.$$

Hence,  $\limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi}^m f_{j+1,a}\|_{\mathcal{W}_\mu^{(n)}} \leq 2^{j+1} \epsilon$ . Since  $\epsilon$  is arbitrary, we obtain

$$\limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi}^m f_{j+1,a}\|_{\mathcal{W}_\mu^{(n)}} = 0.$$

(iv)  $\Rightarrow$  (ii) For any given  $\epsilon > 0$  and  $k \in \{0, 1, \dots, m + n\}$  there exists a positive constant  $\delta$  such that  $\delta < |\varphi(z)| < 1$ ,

$$\limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |(I_{k,\varphi}^{n,u} + I_{k-m,\varphi}^{n,v})(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p} + k}} < \epsilon. \tag{4.8}$$

Let  $p_j(z) = z^j (j \geq n)$ . By using Lemma 2.4, we have

$$\begin{aligned} j^{\frac{1}{p}} \|T_{u,v,\varphi}^m p_j\|_{\mathcal{W}_\mu^{(n)}} &\leq \\ &\underbrace{\sum_{t=0}^{n-1} j^{\frac{1}{p}} \left| (T_{u,v,\varphi}^m p_j)^{(t)}(0) \right|}_{H_1} + \underbrace{\sup_{|\varphi(z)| \leq r_N} \mu(z) \sum_{k=0}^{m+n} j^{\frac{1}{p}} \frac{j!}{(j-k)!} |\varphi(z)|^{j-k} |(I_{k,\varphi}^{n,u} + I_{k-m,\varphi}^{n,v})(z)|}_{H_1} \\ &+ \underbrace{\sup_{|\varphi(z)| > r_N} \mu(z) \sum_{k=0}^{m+n} j^{\frac{1}{p}} (1 - |\varphi(z)|^2)^{\frac{1}{p} + k} \frac{j!}{(j-k)!} |\varphi(z)|^{j-k} \frac{|(I_{k,\varphi}^{n,u} + I_{k-m,\varphi}^{n,v})(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p} + k}}}_{H_2}. \end{aligned} \tag{4.9}$$

From Theorem 3.1, it is obvious that

$$\limsup_{j \rightarrow \infty} H_1 = \limsup_{j \rightarrow \infty} S_t = 0 \quad (t = 0, \dots, n - 1). \tag{4.10}$$

By using Lemma 2.1, 2.2 and (4.8), we obtain

$$H_2 \leq \left( \frac{2(\frac{1}{p} + 1)}{e} \right)^{\frac{1}{p} + 1} \sum_{k=0}^{m-1} \sup_{|\varphi(z)| > \delta} \frac{\mu(z) |(I_{k,\varphi}^{n,u} + I_{k-m,\varphi}^{n,v})(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p} + k}} \leq \epsilon$$

which implies that

$$\limsup_{j \rightarrow \infty} H_2 = 0. \tag{4.11}$$

From (4.9), (4.10) and (4.11), we get (ii). The proof is completed. □

Using the same method as in the proof of Theorem 4.3 we can get the following theorem.

**Theorem 4.4** *Let  $T_{u,v,\varphi}^m : H^p \rightarrow \mathcal{W}_\mu^{(n)}$  is bounded. If  $n < m$ , then the following statements are equivalent.*

- (i) *The operator  $T_{u,v,\varphi}^m : H^p \rightarrow \mathcal{W}_\mu^{(n)}$  is compact.*
- (ii) *If  $p_j(z) = z^j$  then  $\lim_{j \rightarrow \infty} j^{\frac{1}{p}} \|T_{u,v,\varphi}^m p_j\|_{\mathcal{W}_\mu^{(n)}} = 0$ .*
- (iii) *For each  $i \in \{0, 1, \dots, n\} \cup \{m, \dots, m + n\}$ ,  $\limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi}^m f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} = 0$ .*
- (iv) *For each  $i \in \{0, 1, \dots, n\} \cup \{m, \dots, m + n\}$ ,  $\limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |(I_{i,\varphi}^{n,u} + I_{i-m,\varphi}^{n,v})(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p} + i}} = 0$ .*

**Remark 4.5** *By putting  $v \equiv 0$  in Theorems 3.1, 3.2, 4.1, 4.2, 4.3 and 4.4, we obtain some characterizations for boundedness, the essential norm and compactness of operator  $uC_\varphi : H^p \rightarrow \mathcal{W}_\mu^{(n)}$  (see [2]).*

**Remark 4.6** *By setting  $u \equiv 0$  in Theorems 3.1, 3.2, 4.1, 4.2, 4.3 and 4.4, we get some characterizations for boundedness, the essential norm and compactness of generalized weighted composition operator  $D_{\varphi,u}^m : H^p \rightarrow \mathcal{W}_\mu^{(n)}$  (see [6]).*

**Remark 4.7** *By taking  $m = 1$  in Theorems 3.1, 3.2, 4.1, 4.2, 4.3 and 4.4, we find some characterizations for boundedness, the essential norm and compactness of Stević-Sharma type operators from Hardy space into  $n$ th weighted type spaces (see [14]).*

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