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On b-generalized skew derivations in Banach algebras

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Abstract: Let $\mathscr{A}$ be a Banach algebra over $\mathbb{R}$ or $\mathbb{C}$. In this paper, we describe the behavior of recently defined b-generalized skew derivations which satisfy certain differential identities on some specific subsets of $\mathscr{A}$.

Key words: Banach algebra, b- generalized skew derivation

1. Introduction

In this paper, unless otherwise mentioned, $\mathscr{A}$ always denotes an unital prime Banach algebra over $\mathbb{R}$ or $\mathbb{C}$. A linear map $\delta : \mathscr{A} \to \mathscr{A}$ is said to be a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathscr{A}$. A linear map $\mathscr{F} : \mathscr{A} \to \mathscr{A}$ is called a generalized derivation if there exists a derivation δ of $\mathscr{A}$ such that $\mathscr{F}(xy) = \mathscr{F}(x)y + x\delta(y)$ for all $x, y \in \mathscr{A}$. Let $\mathscr{A}$ be an associative ring and α be an automorphism of $\mathscr{A}$. A linear mapping $\delta : \mathscr{A} \to \mathscr{A}$ is said to be a skew derivation of $\mathscr{A}$ if $\delta(xy) = \delta(x)y + \alpha(x)\delta(y)$ for all $x, y \in \mathscr{A}$. The derivation δ is uniquely determined by $\mathscr{F}$, which is called an associated derivation of $\mathscr{F}$. The definition of generalized skew derivation is a unified notion of skew derivation and generalized derivation, which are considered as classical linear mappings of noncommutative algebras. Let $\mathscr{A}$ be an associative algebra and α be an automorphism of $\mathscr{A}$. A linear mapping $\mathscr{F} : \mathscr{A} \to \mathscr{A}$ is said to be a generalized skew derivation of $\mathscr{A}$ if $\mathscr{F}(xy) = \mathscr{F}(x)y + \alpha(x)\delta(y)$ for all $x, y \in \mathscr{A}$. In this case, δ is called an associated skew derivation of $\mathscr{F}$ and α is called an associated automorphism of $\mathscr{F}$. In a recent paper [10], Koşan and Lee proposed that an additive map $\mathscr{F} : \mathscr{R} \to \mathscr{Q}$ is called a left b-generalized derivation, with associated additive mapping δ from $\mathscr{R}$ to $\mathscr{Q}$, if $\mathscr{F}(xy) = \mathscr{F}(x)y + bx\delta(y)$ for all $x, y \in \mathscr{R}$ and $b \in \mathscr{Q}$, where $\mathscr{R}$ is a prime ring and $\mathscr{Q}$ is the right Martindale quotient ring of $\mathscr{R}$. In the same paper, it is proved that, if $\mathscr{R}$ is a prime ring, then δ is a derivation of $\mathscr{R}$. For simplicity of notation, this mapping $\mathscr{F}$ will be called a b-generalized derivation with associated pair (b, δ). Clearly, any generalized derivation with associated derivation δ is a b-generalized derivation with associated pair $(1, \delta)$. Similarly, the mapping $x \to ax + b[x, c]$, for $a, b, c \in \mathscr{Q}$, is a b-generalized derivation with associated pair $(b, ad(c))$, where $ad(c)(x) = [x, c]$ denotes the inner derivation of $\mathscr{R}$ induced by the element c. More generally, the mapping $x \to ax + qxc$, for $a, c, q \in \mathscr{Q}$, is a b-generalized derivation with associated pair $(q, ad(c))$. This mapping is called inner b-generalized derivation. Moreover, if $\alpha \in Aut(\mathscr{R})$, with $\alpha(x) = qxq^{-1}$ for q an invertible element of $\mathscr{Q}$, and $\mathscr{F}$ is the inner generalized skew derivation with associated automorphism α, then $\mathscr{F}$ is a b-generalized derivation with associated pair $(q, ad(q^{-1}b))$, for a suitable element $b \in \mathscr{Q}$.

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Motivated by the aforementioned definitions, recently the author and his team proved the following result in [9] for skew generalized derivations:

Theorem 1.1 *Let $\mathscr{A}$ be a unital prime Banach algebra with centre $\mathscr{Z}(\mathscr{A})$ and Θ_1, Θ_2 be open subsets of $\mathscr{A}$, $\mathscr{F}: \mathscr{A} \to \mathscr{A}$ be a continuous linear generalized skew derivation, and $\mathscr{G}: \mathscr{A} \to \mathscr{A}$ be a continuous linear map. If, for each $x \in \Theta_1, y \in \Theta_2$, there exists an integer $m = m(x,y) > 1$ such that either $\mathscr{F}((xy)^m) - \mathscr{G}(x^m y^m) \in \mathscr{Z}(\mathscr{A})$ or $\mathscr{F}((xy)^m) - \mathscr{G}(y^m x^m) \in \mathscr{Z}(\mathscr{A})$, then $\mathscr{A}$ is commutative.*

Theorem 1.2 *Let $\mathscr{A}$ be a unital prime Banach algebra with centre $\mathscr{Z}(\mathscr{A})$ and Θ_1, Θ_2 be open subsets of $\mathscr{A}$, $\mathscr{F}: \mathscr{A} \to \mathscr{A}$ be a continuous linear generalized skew derivation, and $\mathscr{G}: \mathscr{A} \to \mathscr{A}$ be a continuous linear map. If, for each $x \in \Theta_1, y \in \Theta_2$, there exists an integer $m = m(x,y) > 1$ such that either $\mathscr{F}((xy)^m) + \mathscr{G}(x^m y^m) \in \mathscr{Z}(\mathscr{A})$ or $\mathscr{F}((xy)^m) - \mathscr{G}(y^m x^m) \in \mathscr{Z}(\mathscr{A})$, then $\mathscr{A}$ is commutative.*

In 2018, De Filippis and Wei [5] extended the notion of b-generalized derivation to b-generalized skew derivation as follows. Let $\mathscr{R}$ be a prime ring, $b \in \mathscr{Q}$, the right Martindale quotient ring of $\mathscr{R}$, $\delta : \mathscr{R} \to \mathscr{R}$ a linear mapping and α be an automorphism of $\mathscr{R}$. An additive mapping $\mathscr{F}: \mathscr{R} \to \mathscr{R}$ is called a b-generalized skew derivation of $\mathscr{R}$, with an associated term (b, α, d) if $\mathscr{F}(xy) = \mathscr{F}(x)y + b\alpha(x)\delta(y)$ for all $x, y \in \mathscr{R}$. Moreover, they proved that the linear map δ is a skew derivation with associated automorphism α.

According to the definition of b-generalized skew derivation, we can conclude that general results about b-generalized skew derivations may give useful and powerful corollaries about derivations, generalized derivations, skew derivations, and generalized skew derivations. The definition of b-generalized skew derivations is a unified notion of skew derivation and b-generalized derivation, which are considered as classical linear mappings of associative algebras. Interestingly, every b-generalized skew derivation neither b-generalized derivation nor generalized skew derivation (for example see [4, Section 4]). In light of these interesting facts, we shall establish the following results:

Theorem 1.3 *Let $\mathscr{A}$ be a unital noncommutative prime Banach algebra with centre $\mathscr{Z}(\mathscr{A})$ and Θ_1, Θ_2 be open subsets of $\mathscr{A}$, $\mathscr{F}: \mathscr{A} \to \mathscr{A}$ be a continuous linear b-generalized skew derivation, and $\mathscr{G}: \mathscr{A} \to \mathscr{A}$ be a continuous linear map. If, for each $x \in \Theta_1, y \in \Theta_2$, there exists an integer $m = m(x,y) > 1$ such that either $\mathscr{F}((xy)^m) - \mathscr{G}(x^m y^m) \in \mathscr{Z}(\mathscr{A})$ or $\mathscr{F}((xy)^m) - \mathscr{G}(y^m x^m) \in \mathscr{Z}(\mathscr{A})$, then $\mathscr{A}$ satisfies $s_4(x_1, ..., x_4)$, the standard identity of degree four.*

Theorem 1.4 *Let $\mathscr{A}$ be a unital noncommutative prime Banach algebra with centre $\mathscr{Z}(\mathscr{A})$ and Θ_1, Θ_2 be open subsets of $\mathscr{A}$, $\mathscr{F}: \mathscr{A} \to \mathscr{A}$ be a continuous linear b-generalized skew derivation, and $\mathscr{G}: \mathscr{A} \to \mathscr{A}$ be a continuous linear map. If, for each $x \in \Theta_1, y \in \Theta_2$, there exists an integer $m = m(x,y) > 1$ such that either $\mathscr{F}((xy)^m) + \mathscr{G}(x^m y^m) \in \mathscr{Z}(\mathscr{A})$ or $\mathscr{F}((xy)^m) - \mathscr{G}(y^m x^m) \in \mathscr{Z}(\mathscr{A})$, then $\mathscr{A}$ satisfies $s_4(x_1, ..., x_4)$, the standard identity of degree four.*

2. Preliminaries results

En route to establishing the above mentioned theorems, we recall some well known facts without proof:

Fact 1 ([1]) Let $p(t)=\sum_{r=0}^{n} b_r t^r$ be a polynomial in real variable t for infinite values of t and each $b_r \in \mathscr{A}$. If $p(t) \in \mathscr{M}$ for infinite real values t, then each b_r lies in $\mathscr{M}$.

Fact 2 ([2]) It is well known that automorphisms, derivations and skew derivations of a prime ring $\mathscr{A}$ can be extended to the right Martindale quotient ring of $\mathscr{A}$, $\mathscr{Q}_r$.

Fact 3 ([5], Remark 1.9) *Every b-generalized skew derivation $\mathscr{F}$ with associated term (b,α,δ) can be extended to $\mathscr{Q}$ and assumes of the form $\mathscr{F}(x)=ax+b\delta(x)$, where $a,b\in\mathscr{Q}$.*

We begin our discussion with the following key result, which have been proved in [9]. Nevertheless, for the sake of completeness, here, we would like to provide the proof:

Proposition 2.1 *Let $\mathscr{A}$ be a unital prime Banach algebra and Θ_1, Θ_2 be open subsets of $\mathscr{A}$, $\mathscr{F}:\mathscr{A}\to\mathscr{A}$ and $\mathscr{G}:\mathscr{A}\to\mathscr{A}$ be two continuous linear maps. If, for each $x\in\Theta_1, y\in\Theta_2$, there exists an integer $m=m(x,y)>1$ such that either $\mathscr{F}((xy)^m)-\mathscr{G}(x^m y^m)\in\mathscr{M}$ or $\mathscr{F}((xy)^m)-\mathscr{G}(y^m x^m)\in\mathscr{M}$, then the following facts hold simultaneously:*

1. *$\mathscr{F}(x)-\mathscr{G}(x)\in\mathscr{M}$ for all $x\in\mathscr{A}$;*
2. *$\mathscr{F}([x,y])\in\mathscr{M}$ and $\mathscr{G}([x,y])\in\mathscr{M}$ for all $x,y\in\mathscr{A}$;*
3. *there exists a fixed integer $m>1$ such that for all $x,y\in\mathscr{A}$, both $\mathscr{F}\big((xy)^m-x^m y^m\big)\in\mathscr{M}$ and $\mathscr{F}\big((xy)^m-y^m x^m\big)\in\mathscr{M}$.*

In the proof of above proposition, we adapt the arguments from the proof of [[12], Theorem 1]. However, we omit the details of arguments for brevity.

Proof Fix any $x\in\Theta_1$. For each $n>1$ we define the set $U_n=\{y\in\mathscr{A} \mid \mathscr{F}((xy)^n)-\mathscr{G}(x^n y^n)\notin\mathscr{M}$ and $\mathscr{F}((xy)^n)-\mathscr{G}(y^n x^n)\notin\mathscr{M}\}$. It is easy to prove that U_n is open. Thus, by the Baire category theorem, if every U_n is dense then their intersection is also dense, which contradicts the existence of Θ_2. Hence, there exists a positive integer r such that U_r is not dense. Therefore, there exists a nonempty open set Θ_3 in the complement of U_r such that for all $y\in\Theta_3$ either $\mathscr{F}((xy)^r)-\mathscr{G}(x^r y^r)\in\mathscr{M}$ or $\mathscr{F}((xy)^r)-\mathscr{G}(y^r x^r)\in\mathscr{M}$. Let $e_0\in\Theta_3$ and $z\in\mathscr{A}$. Then $e_0+tz\in\Theta_3$ for all sufficiently small real t. Thus, for each such t, we have

$$\mathscr{F}((x(e_0+tz))^r)-\mathscr{G}(x^r(e_0+tz)^r)\in\mathscr{M} \tag{2.1}$$

or

$$\mathscr{F}((x(e_0+tz))^r)-\mathscr{G}((e_0+tz)^r x^r)\in\mathscr{M}. \tag{2.2}$$

Thus at least one of (2.1) and (2.2) is valid for infinitely many t. Suppose (2.1) holds for these t. Then the expression $\mathscr{F}((x(e_0+tz))^r)-\mathscr{G}(x^r(e_0+tz)^r)$ can be written as

$$\begin{aligned}
&\mathscr{F}\big(\Upsilon_{r,0}(x,e_0,z)\big)-\mathscr{G}\big(x^r\Upsilon'_{r,0}(e_0,z)\big)\\
&+\mathscr{F}\big(\Upsilon_{r-1,1}(x,e_0,z)\big)t-\mathscr{G}\big(x^r\Upsilon'_{r-1,1}(e_0,z)\big)t\\
&+\dots\\
&+\mathscr{F}\big(\Upsilon_{1,r-1}(x,e_0,z)\big)t^{r-1}-\mathscr{G}\big(x^r\Upsilon'_{1,r-1}(e_0,z)\big)t^{r-1}\\
&+\mathscr{F}\big(\Upsilon_{0,r}(x,e_0,z)\big)t^{r}-\mathscr{G}\big(x^r\Upsilon'_{0,r}(e_0,z)\big)t^{r},
\end{aligned}$$

where $\Upsilon_{i,j}(x,e_0,z)$ denotes the sum of all terms in which xe_0 appears exactly i times and xz appears exactly j times in the expansion of $(x(e_0+tz))^r$ where i and j are nonnegative integers such that $i+j=r$. Similarly, $\Upsilon'_{i,j}(e_0,z)$ is sum of all terms in which e_0 appears exactly i times and z appears exactly j times in the expansion of $(e_0+tz)^r$, where i and j are nonnegative integers such that $i+j=r$. The above expression is a polynomial in t and the coefficient of t^r in this polynomial is $\mathscr{F}((xz)^r)-\mathscr{G}(x^rz^r)$. Therefore in view of Fact 1, we have $\mathscr{F}((xz)^r)-\mathscr{G}(x^rz^r)\in\mathscr{M}$. On the other hand, if (2.2) holds for infinitely many t, then $\mathscr{F}((xz)^r)-\mathscr{G}(z^rx^r)\in\mathscr{M}$. Thus, given $x\in\Theta_1$ and for all $z\in\mathscr{A}$, there is a positive integer $r>1$ depending on x such that either $\mathscr{F}((xz)^r)-\mathscr{G}(x^rz^r)\in\mathscr{M}$ or $\mathscr{F}((xz)^r)-\mathscr{G}(z^rx^r)\in\mathscr{M}$. Next, fix $y\in\mathscr{A}$ and for each positive integer k, set $V_k=\{e\in\mathscr{A}\mid\mathscr{F}((ey)^k)-\mathscr{G}(e^ky^k)\notin\mathscr{M}$ and $\mathscr{F}((ey)^k)-\mathscr{G}(y^ke^k)\notin\mathscr{M}\}$. Each V_k is open (X). If each V_k is dense, then by the Baire category theorem, their intersection is also dense; which contradicts the existence of the open set Θ_1. Thus, there is an integer $m=m(y)>1$ and a nonempty open subset Θ_4 in the complement of V_m. If $x_0\in\Theta_4$ and $u\in\mathscr{A}$, then $x_0+tu\in\Theta_4$ for all sufficiently small real t. Hence for positive integer $m>1$ either

$$\mathscr{F}\Big(((x_0+tu)y)^m\Big)-\mathscr{G}\Big((x_0+tu)^my^m\Big)\in\mathscr{M}$$

or

$$\mathscr{F}\Big(((x_0+tu)y)^m\Big)-\mathscr{G}\Big(y^m(x_0+tu)^m\Big)\in\mathscr{M}$$

for each $u\in\mathscr{A}$ and $x_0\in\Theta_4$. Arguing as above we see that, for any $y\in\mathscr{A}$ there exists $m=m(y)>1$ such that, for any $u\in\mathscr{A}$, either

$$\mathscr{F}((uy)^m)-\mathscr{G}(u^my^m)\in\mathscr{M} \tag{2.3}$$

or

$$\mathscr{F}((uy)^m)-\mathscr{G}(y^mu^m)\in\mathscr{M}. \tag{2.4}$$

Let S_k, $k>1$ be the set of $y\in\mathscr{A}$ such that for each $z\in\mathscr{A}$ either $\mathscr{F}((zy)^k)-\mathscr{G}(z^ky^k)\in\mathscr{M}$ or $\mathscr{F}((zy)^k)-\mathscr{G}(y^kz^k)\in\mathscr{M}$, then the union of S_k will be $\mathscr{A}$. It can be easily proved that each S_k is closed. Hence again by Baire category theorem some S_n, $n>1$ must have a nonempty open subset Θ_5. Let $y_0\in\Theta_5$, for all sufficiently small real t and each $v,z\in\mathscr{A}$ either

$$\mathscr{F}\Big((z(y_0+tv))^n\Big)-\mathscr{G}\Big(z^n(y_0+tv)^n\Big)\in\mathscr{M}$$

or

$$\mathscr{F}\Big((z(y_0+tv))^n\Big)-\mathscr{G}\Big((y_0+tv)^nz^n\Big)\in\mathscr{M}.$$

By the same above argument, we have for each $v,z\in\mathscr{A}$ either

$$\mathscr{F}((zv)^n)-\mathscr{G}(z^nv^n)\in\mathscr{M} \tag{2.5}$$

or

$$\mathscr{F}((zv)^n)-\mathscr{G}(v^nz^n)\in\mathscr{M}. \tag{2.6}$$

Let e be the unity of $\mathscr{A}$. Hence, for all real t and for any $x, y \in \mathscr{A}$, either

$$\mathscr{F}\Big(((e+tx)y)^n\Big) - \mathscr{G}\Big((e+tx)^n y^n\Big) \in \mathscr{M}$$

or

$$\mathscr{F}\Big(((e+tx)y)^n\Big) - \mathscr{G}\Big(y^n(e+tx)^n\Big) \in \mathscr{M}.$$

By the computation of the coefficient of t in the expansion of the above equations and using Fact 1, it follows that, for all $x, y \in \mathscr{A}$, either

$$\mathscr{F}(xy^n + \sum_{k=1}^{n-1} y^k x y^{n-k}) - n\mathscr{G}(xy^n) \in \mathscr{M} \tag{2.7}$$

or

$$\mathscr{F}(xy^n + \sum_{k=1}^{n-1} y^k x y^{n-k}) - n\mathscr{G}(y^n x) \in \mathscr{M}. \tag{2.8}$$

Now, taking $\mathscr{F}([(y(e+tx))^n])$ in place of $\mathscr{G}([((e+tx)y)^n])$, we have that, for all $x, y \in \mathscr{A}$, either

$$\mathscr{F}(y^n x + \sum_{k=1}^{n-1} y^k x y^{n-k}) - n\mathscr{G}(xy^n) \in \mathscr{M} \tag{2.9}$$

or

$$\mathscr{F}(y^n x + \sum_{k=1}^{n-1} y^k x y^{n-k}) - n\mathscr{G}(y^n x) \in \mathscr{M}. \tag{2.10}$$

Then, at least one of pairs of equations $\{(2.7), (2.9)\}$, $\{(2.7), (2.10)\}$, $\{(2.8), (2.9)\}$ and $\{(2.8), (2.10)\}$ must hold.

Firstly, we notice that, for $y = e$ in any one of the equations (2.7) to (2.10), one has that $n\mathscr{F}(x) - n\mathscr{G}(x) \in \mathscr{M}$, that is $\mathscr{F}(x) - \mathscr{G}(x) \in \mathscr{M}$, for any $x \in \mathscr{A}$.

On the other hand, by combining the equations in the pairs $\{(2.7), (2.9)\}$, $\{(2.7), (2.10)\}$, $\{(2.8), (2.9)\}$ and $\{(2.8), (2.10)\}$, we have that, for any $x, y \in \mathscr{A}$, one of the following relation holds:

$$\mathscr{F}([x, y^n]) \in \mathscr{M} \tag{2.11}$$

$$\mathscr{F}([x, y^n]) + n\mathscr{G}([x, y^n]) \in \mathscr{M} \tag{2.12}$$

$$\mathscr{F}([x, y^n]) - n\mathscr{G}([x, y^n]) \in \mathscr{M}. \tag{2.13}$$

Replacing y by $e + ty$ in (2.11, 2.12 and 2.13), and using same above arguments, it follows that, for any $x, y \in \mathscr{A}$, one of the following holds:

$$\mathscr{F}([x, y]) \in \mathscr{M} \tag{2.14}$$

$$\mathscr{F}([x,y]) + n\mathscr{G}([x,y]) \in \mathscr{M} \tag{2.15}$$

$$\mathscr{F}([x,y]) - n\mathscr{G}([x,y]) \in \mathscr{M}. \tag{2.16}$$

In any case, since $n > 1$ and $\mathscr{F}(x) - \mathscr{G}(x) \in \mathscr{M}$, for all $x \in \mathscr{A}$, it follows that both $\mathscr{F}([x,y]) \in \mathscr{M}$ and $\mathscr{G}([x,y]) \in \mathscr{M}$, for any $x, y \in \mathscr{A}$. Since $\mathscr{F}(xy) - \mathscr{G}(yx) \in \mathscr{M}$, for any $x, y \in \mathscr{A}$, the relations (2.5) and (2.6) are equivalent. Moreover, since (2.5) and (2.6) hold simultaneously and $\mathscr{F}(x) - \mathscr{G}(x) \in \mathscr{M}$, for all $x \in \mathscr{A}$, then there exists a fixed integer $n > 1$ such that,

$$\mathscr{F}\Big((xy)^n - y^n x^n\Big) \in \mathscr{M} \text{ for all } x \in \mathscr{A},$$

and also

$$\mathscr{F}\Big((xy)^n - x^n y^n\Big) \in \mathscr{M} \text{ for all } x \in \mathscr{A},$$

as required. This completes the proof. □

Proof of Theorem 1.3. Since the centre $\mathscr{Z}(\mathscr{A})$ of $\mathscr{A}$, is also a closed linear subspace of $\mathscr{A}$, so we can replace $\mathscr{Z}(\mathscr{A})$ by $\mathscr{M}$. Thus, by Proposition 2.1, we have the following relations:

$$\mathscr{F}(x) - \mathscr{G}(x) \in \mathscr{Z}(\mathscr{A}), \text{ for all } x \in \mathscr{A}, \tag{2.17}$$

$$\mathscr{F}([x,y]) \in \mathscr{Z}(\mathscr{A}) \text{ for all } x \in \mathscr{A}, \tag{2.18}$$

$$\mathscr{G}([x,y]) \in \mathscr{Z}(\mathscr{A}), \text{ for all } x, y \in \mathscr{A}. \tag{2.19}$$

Moreover there exists a fixed integer $m > 1$, such that, for any $x, y \in \mathscr{A}$

$$\mathscr{F}((xy)^m - y^m x^m) \in \mathscr{Z}(\mathscr{A}). \tag{2.20}$$

Suppose, for that sake of a contradiction $[x,y] \notin \mathscr{Z}(\mathscr{A})$, for some $x, y \in \mathscr{A}$. In particular, since $\mathscr{A}$ is prime, by [7, Theorem 1.1.8] there is a commutator which does not commute with $[x,y]$. Consequently, $\Lambda = [\mathscr{A}, \mathscr{A}]$ is a noncommutative Lie ideal of $\mathscr{A}$. Since $\mathscr{F}([x,y]) \in \mathscr{Z}(\mathscr{A})$, for any $x, y \in \mathscr{A}$, it follows that $[\mathscr{F}(z), z] = 0$, for any $z \in \Lambda$. Thus Theorem 1.5 in [5] for $a = e$ applies. If the first case holds, that is, if $\mathscr{F}(x) = \lambda x$ for each $x \in \mathscr{A}$, where $\lambda \in \mathscr{C}$ is fixed, then $\mathscr{F} \neq 0$ and the primeness of $\mathscr{A}$ yields $[x,y] \in \mathscr{Z}(\mathscr{A})$, a contradiction. Hence, the second case must hold for $\mathscr{F}$, that is $\mathscr{A}$ satisfies $s_4(x_1, ..., x_4)$, the standard identity of degree four and there exist $b \in \mathscr{Q}_r$ and $\lambda \in \mathscr{C}$ such that $\mathscr{F}(x) = bx + xb + \lambda x$, for any $x \in \mathscr{A}$. If $b \in \mathscr{C}$, then this case reduces to the first one. Hence, we may assume that $b \notin \mathscr{C}$. From (2.20) it follows that for any $\alpha, \beta, \gamma \in \mathscr{A}$,

$$\left[\alpha, (b+\lambda)\Big((\beta\gamma)^m - \gamma^m \beta^m\Big) + \Big((\beta\gamma)^m - \gamma^m \beta^m\Big) b\right] = 0. \tag{2.21}$$

Consequently, $\mathscr{A}$ satisfies a generalized polynomial identity. Thus, by [3] $\mathscr{Q}_r$ also satisfies the same identity. In case $\mathscr{C}$ is infinite, we obtain that $\mathscr{Q}_r \bigotimes_{\mathscr{C}} \overline{\mathscr{C}}$ satisfies (2.21), where $\overline{\mathscr{C}}$ is the algebraic closure of $\mathscr{C}$. Since

both $\mathscr{Q}_r$ and $\mathscr{Q}_r \bigotimes_{\mathscr{C}} \overline{\mathscr{C}}$ are centrally closed (Theorems 2.5 and 3.5 in [6]), we may replace $\mathscr{Q}_r$ by either $\mathscr{Q}_r$ or $\mathscr{Q}_r \bigotimes_{\mathscr{C}} \overline{\mathscr{C}}$ depending whether $\mathscr{C}$ is finite or infinite. Thus, we may assume that $\mathscr{Q}_r$ is centrally closed over $\mathscr{C}$ which is either finite or algebraically closed. By Martindale's theorem [11], $\mathscr{Q}_r$ is a primitive ring having a nonzero socle with $\mathscr{C}$ as the associated division ring. In light of Jacobson's theorem ([8], p. 75) $\mathscr{A}$ is isomorphic to a dense ring of linear transformations on some vector space $\mathscr{V}$ over $\mathscr{C}$.
Since $b \notin \mathscr{C}$, there exists $v \in \mathscr{V}$ such that v, bv are linearly $\mathscr{C}$-independent. By the Jacobson density theorem there exist $\alpha_0, \beta_0, \gamma_0 \in \mathscr{Q}_r$ such that

$$\alpha_0 v = 0, \quad \alpha_0(bv) = v, \quad \beta_0 v = bv, \quad \beta_0(bv) = 0, \quad \gamma_0 v = 0, \quad \gamma_0(bv) = v.$$

However, then

$$\left[\gamma_0, (b+\lambda)\left((\alpha_0\beta_0)^n - \beta_0^n\alpha_0^n\right) + \left((\alpha_0\beta_0)^n - \beta_0^n\alpha_0^n\right)b\right]v = v \neq 0.$$

From this contradiction, it therefore follows that $[x,y] \in \mathscr{Z}(\mathscr{A})$ for all $x, y \in \mathscr{A}$. Thus $\mathscr{A}$ is commutative by [7, Theorem 1.1.8], a contradiction. This complete the proof.
Proof of Theorem 1.4. By using the same techniques as in Proposition 2.1, we deduce that, for any $x \in \mathscr{A}$, either $\mathscr{F}(x) - \mathscr{G}(x) \in \mathscr{Z}(\mathscr{A})$ or $\mathscr{F}(x) + \mathscr{G}(x) \in \mathscr{Z}(\mathscr{A})$. Moreover, there is a positive integer $n > 1$ such that, for any $x, y \in \mathscr{A}$, either

$$\mathscr{F}((xy)^n) + \mathscr{G}(x^n y^n) \in \mathscr{Z}(\mathscr{A}) \tag{2.22}$$

or

$$\mathscr{F}((xy)^n) - \mathscr{G}(y^n x^n) \in \mathscr{Z}(\mathscr{A}). \tag{2.23}$$

Suppose there exists $x \in \mathscr{A}$ such that $\mathscr{F}(x) - \mathscr{G}(x) \in \mathscr{Z}(\mathscr{A})$ and $\mathscr{F}(x) + \mathscr{G}(x) \notin \mathscr{Z}(\mathscr{A})$. Let $y \in \mathscr{A}$ such that $\mathscr{F}(x+y) - \mathscr{G}(x+y) \in \mathscr{Z}(\mathscr{A})$. Then, also $\mathscr{F}(y) - \mathscr{G}(y) \in \mathscr{Z}(\mathscr{A})$ holds. On the other hand, if $\mathscr{F}(x+y) + \mathscr{G}(x+y) \in \mathscr{Z}(\mathscr{A})$, and since $\mathscr{F}(x) + \mathscr{G}(x) \notin \mathscr{Z}(\mathscr{A})$, then also $\mathscr{F}(y) + \mathscr{G}(y) \notin \mathscr{Z}(\mathscr{A})$. Therefore $\mathscr{F}(y) - \mathscr{G}(y) \in \mathscr{Z}(\mathscr{A})$, for any $y \in \mathscr{A}$.
By using a similar argument, one may prove that if there exists $x \in \mathscr{A}$, such that $\mathscr{F}(x) + \mathscr{G}(x) \in \mathscr{Z}(\mathscr{A})$ and $\mathscr{F}(x) - \mathscr{G}(x) \notin \mathscr{Z}(\mathscr{A})$, then $\mathscr{F}(y) + \mathscr{G}(y) \in \mathscr{Z}(\mathscr{A})$, for any $y \in \mathscr{A}$.
In other words, either

$$\mathscr{F}(y) - \mathscr{G}(y) \in \mathscr{Z}(\mathscr{A}), \text{for all y} \in \mathscr{A} \tag{2.24}$$

or

$$\mathscr{F}(y) + \mathscr{G}(y) \in \mathscr{Z}(\mathscr{A}), \text{for all y} \in \mathscr{A}. \tag{2.25}$$

For infinitely many real t, by relations (2.22)–(2.23), and for any $x, y \in \mathscr{A}$ we have that either

$$\mathscr{F}\left(((e+tx)y)^n\right) + \mathscr{G}\left((e+tx)^n y^n\right) \in \mathscr{Z}(\mathscr{A})$$

or

$$\mathscr{F}\left(((e+tx)y)^n\right) - \mathscr{G}\left(y^n(e+tx)^n\right) \in \mathscr{Z}(\mathscr{A}).$$

Hence, taking coefficient of t in the expansion of above equations and using Fact 1, it follows that for any $x, y \in \mathscr{A}$, either

$$\mathscr{F}(xy^n + \sum_{k=1}^{n-1} y^k x y^{n-k}) + n\mathscr{G}(xy^n) \in \mathscr{Z}(\mathscr{A}) \tag{2.26}$$

or

$$\mathscr{F}(xy^n + \sum_{k=1}^{n-1} y^k x y^{n-k}) - n\mathscr{G}(y^n x) \in \mathscr{Z}(\mathscr{A}). \tag{2.27}$$

Now, taking $\mathscr{F}((y(e' + tx))^n)$ in place of $\mathscr{G}(((e' + tx)y)^n)$, we see that for any $x, y \in \mathscr{A}$, either

$$\mathscr{F}(y^n x + \sum_{k=1}^{n-1} y^k x y^{n-k}) + n\mathscr{G}(xy^n) \in \mathscr{Z}(\mathscr{A}) \tag{2.28}$$

or

$$\mathscr{F}(y^n x + \sum_{k=1}^{n-1} y^k x y^{n-k}) - n\mathscr{G}(y^n x) \in \mathscr{Z}(\mathscr{A}). \tag{2.29}$$

Then, at least one of pairs of equations {(2.26), (2.28)}, {(2.26), (2.29)}, {(2.27), (2.28)} and {(2.27), (2.29)} must hold. By comparing the equations in these expressions, it follows that, for any $x, y \in \mathscr{A}$, one of the following holds:

$$\mathscr{F}([x, y^n]) \in \mathscr{Z}(\mathscr{A}) \tag{2.30}$$

$$\mathscr{F}([x, y^n]) + \mathscr{G}(nxy^n + ny^n x) \in \mathscr{Z}(\mathscr{A}) \tag{2.31}$$

$$\mathscr{F}([x, y^n]) - \mathscr{G}(nxy^n + ny^n x) \in \mathscr{Z}(\mathscr{A}). \tag{2.32}$$

By proceeding as in Proposition 2.1, and as a consequence of relations (2.30, 2.31 and 2.32) one has that, for any $x, y \in \mathscr{A}$, one of the following holds:

$$\mathscr{F}([x, y]) \in \mathscr{Z}(\mathscr{A}) \tag{2.33}$$

$$\mathscr{F}([x, y]) + \mathscr{G}(nxy + nyx) \in \mathscr{Z}(\mathscr{A}) \tag{2.34}$$

$$\mathscr{F}([x, y]) - \mathscr{G}(nxy + nyx) \in \mathscr{Z}(\mathscr{A}). \tag{2.35}$$

By the fact that (2.24) or (2.25) holds true, we deduce that for any $x, y \in \mathscr{A}$, one of the following holds:

$$\mathscr{F}([x,y]) \in \mathscr{Z}(\mathscr{A}) \tag{2.36}$$

$$\mathscr{F}([x,y] + nxy + nyx) \in \mathscr{Z}(\mathscr{A}) \tag{2.37}$$

$$\mathscr{G}([x,y] - nxy - nyx) \in \mathscr{Z}(\mathscr{A}). \tag{2.38}$$

Our aim is to prove that $\mathscr{F}([x,y]) \in \mathscr{Z}(\mathscr{A})$, for any $x, y \in \mathscr{A}$. To do this, we assume there exist $x_0, y_0 \in \mathscr{A}$ such that $\mathscr{F}([x_0, y_0]) \notin \mathscr{Z}(\mathscr{A})$. In this case, since $\mathscr{F}([x_0, y_0 + e']) = \mathscr{F}([x_0, y_0]) \notin \mathscr{Z}(\mathscr{A})$ and by the above relations, we have that either

$$\mathscr{F}([x_0, y_0 + e'] + nx_0(y_0 + e') + n(y_0 + e')x_0) \in \mathscr{Z}(\mathscr{A})$$

or

$$\mathscr{F}([x_0, y_0 + e'] - nx_0(y_0 + e') - n(y_0 + e')x_0) \in \mathscr{Z}(\mathscr{A})$$

that is, either

$$\mathscr{F}([x_0, y_0] + nx_0y_0 + ny_0x_0 + 2nx_0) \in \mathscr{Z}(\mathscr{A}) \tag{2.39}$$

or

$$\mathscr{F}([x_0, y_0] - nx_0y_0 - ny_0x_0 - 2nx_0) \in \mathscr{Z}(\mathscr{A}). \tag{2.40}$$

Moreover, since $\mathscr{F}([tx_0, e+ty_0]) \notin \mathscr{Z}(\mathscr{A})$, for any real element t, then, by relations (2.39) and (2.40), it follows that either

$$\mathscr{F}\Big([tx_0, e + ty_0] + n(tx_0)(e + ty_0) + n(e + ty_0)(tx_0) + 2n(tx_0)\Big) \in \mathscr{Z}(\mathscr{A}) \tag{2.41}$$

or

$$\mathscr{F}\Big([tx_0, e + ty_0] - n(tx_0)(e + ty_0) - n(e' + ty_0)(tx_0) - 2n(tx_0)\Big) \in \mathscr{Z}(\mathscr{A}). \tag{2.42}$$

By the computation of the coefficient of t and by Fact 1, we get in any case $\mathscr{F}(x_0) \in \mathscr{Z}(\mathscr{A})$.
On the other hand, let $x_1 \in \mathscr{A}$ such that $\mathscr{F}([x_1, y_0]) \in \mathscr{Z}(\mathscr{A})$. Since $\mathscr{F}([x_1 + x_0, y_0]) \notin \mathscr{Z}(\mathscr{A})$, then $\mathscr{F}(x_1 + x_0) \in \mathscr{Z}(\mathscr{A})$ by the above argument, that is $\mathscr{F}(x_1) \in \mathscr{Z}(\mathscr{A})$.
Hence, it is shown that either $\mathscr{F}([\mathscr{A}, \mathscr{A}]) \subseteq \mathscr{Z}(\mathscr{A})$ or $\mathscr{F}(\mathscr{A}) \subseteq \mathscr{Z}(\mathscr{A})$. In any case, we may assume that $\mathscr{F}([x,y]) \in \mathscr{Z}(\mathscr{A})$, for any $x, y \in \mathscr{A}$. Moreover, since $\mathscr{F}([x^n, y^n]) \in \mathscr{Z}(\mathscr{A})$, by relations (2.22), (2.23), (2.24) and (2.25), it follows, for any $x, y \in \mathscr{A}$, that either

$$\mathscr{F}((xy)^n) + \mathscr{F}(y^n x^n) \in \mathscr{Z}(\mathscr{A}), \tag{2.43}$$

or

$$\mathscr{F}((xy)^n) - \mathscr{F}(y^n x^n) \in \mathscr{Z}(\mathscr{A}). \tag{2.44}$$

Starting from (2.43) and (2.44), here we follow the same argument used in the proof of Theorem 1.3, so we omit some details for brevity. Suppose, by contradiction, $\mathscr{A}$ is not commutative, that is $[x,y] \notin \mathscr{Z}(\mathscr{A})$, for some $x,y \in \mathscr{A}$. Therefore, there is a commutator which does not commute with $[x,y]$, so that $\Lambda = [\mathscr{A},\mathscr{A}]$ is a noncommutative Lie ideal of $\mathscr{A}$. Since $\mathscr{F}([x,y]) \in \mathscr{Z}(\mathscr{A})$, for any $x,y \in \mathscr{A}$, it follows that $[\mathscr{F}(z),z] = 0$, for any $z \in \Lambda$. Thus by Theorem 1.5 in [5] (for $a = e$) either $\mathscr{F}(x) = \mu x$, where $\mu \in \mathscr{Z}(\mathscr{A})$ or $\mathscr{A}$ satisfies $s_4(x_1,...,x_4)$, the standard identity of degree four and there exist $b \in \mathscr{Q}_r$ and $\lambda \in \mathscr{C}$ such that $\mathscr{F}(x) = bx + xb + \lambda x$ for all $x \in \mathscr{A}$.
If the first case occurs, since $\mathscr{F} \neq 0$, so the primeness of $\mathscr{A}$ yields $[x,y] \in \mathscr{Z}(\mathscr{A})$, a contradiction. Hence, the second case must hold and we may assume that $b \notin \mathscr{C}$. By relations (2.43) and (2.44), it follows for any $\alpha,\beta,\gamma \in \mathscr{A}$ that either

$$\left[\alpha,(b+\lambda)\Big((\beta\gamma)^n+\gamma^n\beta^n\Big)+\Big((\beta\gamma)^n+\gamma^n\beta^n\Big)b\right]=0, \tag{2.45}$$

or

$$\left[\alpha,(b+\lambda)\Big((\beta\gamma)^n-\gamma^n\beta^n\Big)+\Big((\beta\gamma)^n-\gamma^n\beta^n\Big)b\right]=0. \tag{2.46}$$

In any case, $\mathscr{A}$ satisfies a generalized polynomial identity. As in the proof of Theorem 1.3, we may assume that $\mathscr{Q}_r$ is centrally closed over $\mathscr{C}$ which is either finite or algebraically closed. Moreover, $\mathscr{Q}_r$ is a primitive ring having a nonzero socle with $\mathscr{C}$ as the associated division ring. In light of Jacobson's theorem ([8], p. 75) $\mathscr{A}$ is isomorphic to a dense ring of linear transformations on some vector space $\mathscr{V}$ over $\mathscr{C}$.
Since $b \notin \mathscr{C}$, there exists $v \in \mathscr{V}$ such that v, bv are linearly $\mathscr{C}$-independent. By the Jacobson density theorem there exist $\alpha,\beta,\gamma \in \mathscr{Q}_r$ such that

$$\alpha v = 0, \quad \alpha(bv) = v, \quad \beta v = bv, \quad \beta(bv) = 0, \quad \gamma v = 0, \quad \gamma(bv) = v.$$

However, both

$$\left[\gamma,(b+\lambda)\Big((\alpha\beta)^n+\beta^n\alpha^n\Big)+\Big((\alpha\beta)^n+\beta^n\alpha^n\Big)b\right]v = v \neq 0$$

and

$$\left[\gamma,(b+\lambda)\Big((\alpha\beta)^n-\beta^n\alpha^n\Big)+\Big((\alpha\beta)^n-\beta^n\alpha^n\Big)b\right]v = v \neq 0.$$

These last two relations contradict the fact that one of (2.45) and (2.46) must hold. From this contradiction it follows that $[x,y] \in \mathscr{Z}(\mathscr{A})$ for all $x,y \in \mathscr{A}$, this leads to a contradiction again. Thus $\mathscr{A}$ satisfies $s_4(x_1,...,x_4)$, the standard identity of degree four.
We conclude our paper with following open questions:

Open Questions. Let $\mathscr{A}$ be a semisimple Banach algebra (unital or not) with centre $\mathscr{Z}(\mathscr{A})$, Θ_1,Θ_2 be open subsets of $\mathscr{A}$. Suppose $\mathscr{F}:\mathscr{A}\to\mathscr{A}$ and $\mathscr{G}:\mathscr{A}\to\mathscr{A}$ two continuous linear b-generalized skew derivations of $\mathscr{A}$. What can be said about the structure of $\mathscr{A}$ and the form of $\mathscr{F}$ and $\mathscr{G}$ in the following cases:

1. For each $x \in \Theta_1, y \in \Theta_2$, there exists an integer $m = m(x,y) > 1$ such that $\mathscr{F}((xy)^m) - \mathscr{G}(x^m y^m) \in \mathscr{Z}(\mathscr{A})$ or $\mathscr{F}((xy)^m) - \mathscr{G}(y^m x^m) \in \mathscr{Z}(\mathscr{A})$.

2. For each $x \in \Theta_1, y \in \Theta_2$, there exists an integer $m = m(x,y) > 1$ such that $\mathscr{F}((xy)^m) + \mathscr{G}(x^m y^m) \in \mathscr{Z}(\mathscr{A})$ or $\mathscr{F}((xy)^m) - \mathscr{G}(y^m x^m) \in \mathscr{Z}(\mathscr{A})$.

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