Turkish Journal of Mathematics

Volume 45 | Number 5 Article 31

1-1-2021

# Diameter estimate for a class of compact generalized quasi-Einstein manifolds

DENG YIHUA

Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2021) 45: 2331 – 2340
© TÜBİTAK
doi:10.3906/mat-2106-93

# Diameter estimate for a class of compact generalized quasi-Einstein manifolds

**Yihua DENG***
College of Mathematics and Statistics, Hengyang Normal University, Hengyang, Hunan, P. R. China

**Received:** 23.06.2021 • **Accepted/Published Online:** 06.09.2021 • **Final Version:** 16.09.2021

**Abstract:** In this paper, we discuss the lower diameter estimate for a class of compact generalized quasi-Einstein manifolds which are closely related to the conformal geometry. Using the Bochner formula and the Hopf maximum principle, we get a gradient estimate for the potential function of the manifold. Based on the gradient estimate, we get the lower diameter estimate for this class of generalized quasi-Einstein manifolds.

**Key words:** Generalized quasi-Einstein manifolds, lower diameter estimate, the Bochner formula, maximum principle

## 1. Introduction

To extend the notion of quasi-Einstein, Catino [3] introduced the concept of generalized quasi-Einstein manifold. Let $(M,g)$ be an $n$-dimensional Riemannian manifold with $n \geq 3$. If there exist three smooth functions $f$, $\beta$ and $\lambda$ on $(M,g)$ such that the Ricci tensor satisfy

$$\mathrm{Ric} + \nabla^2 f - \beta df \otimes df = \lambda g, \tag{1.1}$$

then $(M,g)$ is called a generalized quasi-Einstein manifold, where $\nabla^2$ and $\otimes$ denote the Hessian and the tensorial product, respectively. The function $f$ in (1.1) is usually called potential function. If $m$ is a positive integer and $\beta = m^{-1}$, then $(M,g)$ is called generalized $m$-quasi-Einstein manifold (see [2]). Natural examples of generalized quasi-Einstein manifolds are given by Einstein manifolds, gradient Ricci solitons, gradient Ricci almost solitons and quasi-Einstein manifolds.

The classification of generalized $m$-quasi-Einstein manifolds is extensively studied, see for example [1, 2, 7, 9, 10, 11, 14]. Nowadays, the study of diameter estimate is an attractive topic in Riemannian geometry. Wei and Wylie [19] studied the upper diameter estimate and extended the Bonnet-Myers theorem to the Riemannian manifold with Bakry-Emery curvature bounded from below. Limoncu [12, 13], Soylu [16] and Tadano [17] improved the upper diameter estimate in [19]. Futaki and Sano [5] obtained a lower diameter bound for compact shrinking Ricci soliton. Futaki and Li [4] improved the diameter estimate in [5]. Wang [18] got a lower diameter bound for compact $\tau$-quasi-Einstein manifold. Hu, Mao and Wang [8] got a lower diameter estimate for compact generalized quasi-Einstein manifold satisfying $\lambda = \lambda(f)$, $\lambda'(t) \geq 0$ and $[t^{\frac{2}{\alpha_0(n-2)}}\lambda(t)]' \geq 0$.

Let $(M,g)$ be an $n$-dimensional Riemannian manifold with $n \geq 3$. If there exist smooth functions $f$

*Correspondence: dengchen4032@126.com
2010 *AMS Mathematics Subject Classification:* 53C25, 53C21

and $\lambda$ on $(M, g)$ such that the Ricci tensor satisfy

$$\mathrm{Ric} + \nabla^2 f - \frac{1}{2-n} df \otimes df = \lambda g, \tag{1.2}$$

then $M$ is called generalized $(2-n)$-quasi-Einstein manifold in this paper. Generalized $(2-n)$-quasi-Einstein manifolds are closely related to the conformal metric $\widetilde{g} = e^{-\frac{2}{n-2}f} g$ which is important in conformal geometry. Jauregui and Wylie [11] got the classification of generalized quasi-Einstein manifolds admitting a conformal diffeomorphism using the metric $\widetilde{g} = e^{-\frac{2}{n-2}f} g$. Remark 3.3 in [11] shows that the conformal metric $\widetilde{g} = e^{-\frac{2}{n-2}f} g$ is an Einstein metric if and only if $(M, g)$ is a generalized $(2-n)$-quasi-Einstein manifold. Catino [3] gave a local characterization of generalized quasi-Einstein manifolds with harmonic weyl tensor using $\widetilde{g} = e^{-\frac{2}{n-2}f} g$. Ribeiro and Tenenblat [15] provided a complete classification of generalized $(2-n)$-quasi-Einstein manifolds satisfying (1.2) with $\lambda = 0$.

Motivated by [8, 10, 11, 15, 18], we study the lower diameter estimate for generalized $(2-n)$-quasi-Einstein manifolds satisfying (1.2). As far as we know, the study of the diameter estimate for nontrivial generalized quasi-Einstein manifolds is very few up to now. Since $\lambda$ is a function in generalized quasi-Einstein manifolds, the diameter estimate of generalized quasi-Einstein manifolds is much more difficult than that of quasi-Einstein manifolds. To overcome this difficult, we need to use some new skills.

## 2. Some basic lemmas for generalized $(2-n)$-quasi-Einstein manifolds

Generalized $(2-n)$-quasi-Einstein manifolds are closely related to the conformal metric $\widetilde{g} = e^{-\frac{2}{n-2}f} g$ which is important in conformal geometry. In this section, we give some basic lemmas for generalized $(2-n)$-quasi-Einstein manifolds.

**Lemma 2.1** *If $(M, g)$ is a generalized $(2-n)$-quasi-Einstein manifold satisfying (1.2), then there exists a constant $\beta$ such that the following equality holds*

$$\triangle f - |\nabla f|^2 + (n-2)\lambda - \beta e^{\frac{2}{2-n}f} = 0. \tag{2.1}$$

**Proof** Similar to Lemma 2 in [2], we have

$$\nabla R = \frac{2(1-n)}{2-n}\mathrm{Ric}(\nabla f) + \frac{2}{2-n}(R - (n-1)\lambda)\nabla f + 2(n-1)\nabla\lambda \tag{2.2}$$

and

$$R + \triangle f - \frac{1}{2-n}|\nabla f|^2 = n\lambda. \tag{2.3}$$

It follows from (1.2) that

$$\mathrm{Ric}(\nabla f) = \lambda \nabla f - \frac{1}{2}\nabla |\nabla f|^2 + \frac{1}{2-n}|\nabla f|^2 \nabla f. \tag{2.4}$$

Putting (2.4) into (2.2), we obtain

$$\nabla R = \frac{2(1-n)}{2-n}[\lambda \nabla f - \frac{1}{2}\nabla|\nabla f|^2 + \frac{1}{2-n}|\nabla f|^2\nabla f] + \frac{2}{2-n}(R - (n-1)\lambda)\nabla f + 2(n-1)\nabla\lambda$$

$$= \frac{4(1-n)}{2-n}\lambda\nabla f - \frac{1-n}{2-n}\nabla|\nabla f|^2 + \frac{2(1-n)}{(2-n)^2}|\nabla f|^2\nabla f + \frac{2}{2-n}R\nabla f + 2(n-1)\nabla\lambda. \tag{2.5}$$

By (2.5), we have

$$\nabla R - \frac{2}{2-n}R\nabla f = 2(n-1)(\nabla\lambda - \frac{2}{2-n}\lambda\nabla f) - \frac{1-n}{2-n}(\nabla|\nabla f|^2 - \frac{2}{2-n}|\nabla f|^2\nabla f). \tag{2.6}$$

According to (2.6), we arrive at

$$\nabla(Re^{\frac{2}{n-2}f}) = 2(n-1)\nabla(\lambda e^{\frac{2}{n-2}f}) - \frac{1-n}{2-n}\nabla(|\nabla f|^2 e^{\frac{2}{n-2}f}).$$

Therefore, we conclude that there exists a constant $\beta$ such that

$$Re^{\frac{2}{n-2}f} - 2(n-1)(\lambda e^{\frac{2}{n-2}f}) + \frac{1-n}{2-n}(|\nabla f|^2 e^{\frac{2}{n-2}f}) = -\beta.$$

Thus

$$R - 2(n-1)\lambda + \frac{1-n}{2-n}|\nabla f|^2 + \beta e^{\frac{2}{2-n}f} = 0. \tag{2.7}$$

On the other hand, by (2.3) we have

$$R = -\triangle f + \frac{1}{2-n}|\nabla f|^2 + n\lambda. \tag{2.8}$$

Putting (2.8) into (2.7), we conclude that (2.1) is true. □

**Lemma 2.2** *Suppose that $(M, g)$ is a compact generalized $(2-n)$-quasi-Einstein manifold satisfying (1.2), $\lambda_{\max} = \max_{x\in M}\lambda(x)$. If $\beta > 0$, then $\lambda_{\max} > 0$.*

**Proof** Suppose that $x_0 \in M$ is the maximum point of $f$. Then $\triangle f(x_0) \le 0$ and $\nabla f(x_0) = 0$. Since $\beta > 0$, by (2.1) we have

$$(n-2)\lambda_{\max} \ge (n-2)\lambda(x_0) \ge \beta e^{\frac{2}{2-n}f(x_0)} > 0.$$

The proof of Lemma 2.2 is complete. □

## 3. Gradient estimate

To consider the lower diameter estimate for compact generalized $(2-n)$-quasi-Einstein manifold $M$ satisfying (1.2), we need to get a gradient estimate for $h = e^{\alpha f}$.

**Lemma 3.1** *Let $h = e^{\alpha f}$. If $\beta$ is the constant in Lemma 2.1, then the following equality holds*

$$\triangle|\nabla h|^2 = 2|\nabla^2 h|^2 + (2+\frac{1}{\alpha})\frac{\nabla|\nabla h|^2\nabla h}{h} - (2+\frac{2}{\alpha})\frac{|\nabla h|^4}{h^2} - 2(n-2)\alpha\lambda|\nabla h|^2 + 2\lambda|\nabla h|^2$$

$$+2[1+\frac{2}{(2-n)\alpha}]\alpha\beta h^{\frac{2}{(2-n)\alpha}}|\nabla h|^2 + \frac{2+2(2-n)\alpha}{\alpha^2(2-n)}\frac{|\nabla h|^4}{h^2} - 2\alpha(n-2)h\nabla\lambda\nabla h \tag{3.1}$$

*on generalized $(2-n)$-quasi-Einstein manifolds.*

**Proof** Direct calculation shows that

$$\Delta h = \alpha^2 e^{\alpha f}|\nabla f|^2 + \alpha e^{\alpha f}\Delta f = h(\alpha^2|\nabla f|^2 + \alpha\Delta f). \tag{3.2}$$

Since $\nabla h = \alpha h \nabla f$, by (2.1) and (3.2) we conclude that

$$\Delta h = (1+\frac{1}{\alpha})\frac{|\nabla h|^2}{h} - (n-2)\alpha\lambda h + \alpha\beta h^{(1+\frac{2}{(2-n)\alpha})}. \tag{3.3}$$

Therefore

$$\begin{aligned} 2\nabla\Delta h\cdot\nabla h &= 2\nabla[(1+\frac{1}{\alpha})\frac{|\nabla h|^2}{h} - (n-2)\alpha\lambda h + \alpha\beta h^{(1+\frac{2}{(2-n)\alpha})}]\nabla h \\ &= (2+\frac{2}{\alpha})\frac{\nabla|\nabla h|^2\nabla h}{h} - (2+\frac{2}{\alpha})\frac{|\nabla h|^4}{h^2} - 2(n-2)\alpha\lambda|\nabla h|^2 \\ &\quad + 2(1+\frac{2}{(2-n)\alpha})\alpha\beta h^{\frac{2}{(2-n)\alpha}}|\nabla h|^2 - 2\alpha(n-2)h\nabla\lambda\nabla h. \end{aligned} \tag{3.4}$$

Since $\nabla h = \alpha h \nabla f$, by (1.2) we have

$$2\mathrm{Ric}(\nabla h,\nabla h) = 2\lambda|\nabla h|^2 + \frac{2+2(2-n)\alpha}{\alpha^2(2-n)}\frac{|\nabla h|^4}{h^2} - \frac{\nabla|\nabla h|^2\nabla h}{\alpha h}. \tag{3.5}$$

On the other hand, according to the Bochner formula we have

$$\triangle|\nabla h|^2 = 2|\nabla^2 h|^2 + 2\nabla\Delta h\cdot\nabla h + 2\mathrm{Ric}(\nabla h,\nabla h). \tag{3.6}$$

Putting (3.4) and (3.5) into (3.6), we conclude that (3.1) is true. □

In the following, we always suppose that $M$ is a compact generalized $(2-n)$-quasi-Einstein manifold, $\lambda_{\max} = \max_{x\in M}\lambda(x)$, $h = e^{\alpha f}$, $D = \{x \in M; \nabla h(x) \neq 0\}$, $\beta$ is the constant in Lemma 2.1 and

$$F(h) = |\nabla h|^2 + (n-2)\alpha\lambda_{\max}h^2 - \frac{(2-n)\alpha^2}{1+(2-n)\alpha}\beta h^{(2+\frac{2}{(2-n)\alpha})}. \tag{3.7}$$

**Lemma 3.2** *If $\alpha > 0$ and $\beta \geq 0$, then there exists a smooth vector field $X$ on $M$ such that*

$$\begin{aligned} \Delta F &\geq \nabla F\cdot X + \frac{[\alpha(n-2)-1]^2}{\alpha^2(2-n)(n-1)}\frac{|\nabla h|^4}{h^2} + 2\lambda|\nabla h|^2 - 2\alpha(n-2)\lambda|\nabla h|^2 \\ &\quad -2\alpha(n-2)h\nabla\lambda\nabla h + [\frac{2}{n-1}(1+\frac{1}{\alpha})^2 - (2+\frac{2}{\alpha}) + \frac{2+2(2-n)\alpha}{\alpha^2(2-n)}]\frac{|\nabla h|^4}{h^2} \end{aligned} \tag{3.8}$$

*holds on $D$.*

**Proof** Direct calculation shows that

$$\nabla F = \nabla|\nabla h|^2 + 2(n-2)\alpha\lambda_{\max}h\nabla h - 2\alpha\beta h^{(1+\frac{2}{(2-n)\alpha})}\nabla h. \tag{3.9}$$

Therefore, we have

$$\triangle F = \triangle|\nabla h|^2 + 2(n-2)\alpha\lambda_{\max} h|\nabla h|^2 + 2(n-2)\alpha\lambda_{\max} h\triangle h$$

$$-2\alpha\beta h^{(1+\frac{2}{(2-n)\alpha})}\triangle h - 2[1+\frac{2}{(2-n)\alpha}]\alpha\beta h^{\frac{2}{(2-n)\alpha}}|\nabla h|^2. \quad (3.10)$$

For the purpose of convenience, we let

$$G(h) = (n-2)\alpha\lambda_{\max} h - \alpha\beta h^{(1+\frac{2}{(2-n)\alpha})}, \quad L(h) = (n-2)\alpha\lambda h - \alpha\beta h^{(1+\frac{2}{(2-n)\alpha})}. \quad (3.11)$$

Putting (3.1) into (3.10), we obtain

$$\triangle F = 2|\nabla^2 h|^2 + (2+\frac{1}{\alpha})\frac{\nabla|\nabla h|^2\nabla h}{h} - (2+\frac{2}{\alpha})\frac{|\nabla h|^4}{h^2} - 2(n-2)\alpha\lambda|\nabla h|^2 + 2\lambda|\nabla h|^2$$

$$+\frac{2+2(2-n)\alpha}{\alpha^2(2-n)}\frac{|\nabla h|^4}{h^2} - 2\alpha(n-2)h\nabla\lambda\nabla h + 2(n-2)\alpha\lambda_{\max} h|\nabla h|^2 + 2G(h)\triangle h. \quad (3.12)$$

Consider a point $O \in D$. Rotating the orthonormal frame at $O$ so that $|\nabla h|(O) = h_1(O) \neq 0$. According to (2.10) in [18], we have

$$2|\nabla^2 h|^2 + (2+\frac{1}{\alpha})\frac{\nabla|\nabla h|^2\nabla h}{h} \geq \frac{2n}{n-1}h_{11}^2 - \frac{4}{n-1}h_{11}\triangle h + \frac{2}{n-1}(\triangle h)^2 + \frac{4\alpha+2}{\alpha}\frac{h_{11}|\nabla h|^2}{h}. \quad (3.13)$$

According to (3.9), we get

$$h_{11} = \frac{\nabla F\nabla h}{2|\nabla h|^2} - (n-2)\alpha\lambda_{\max} h + \alpha\beta h^{(1+\frac{2}{(2-n)\alpha})} = \frac{\nabla F\nabla h}{2|\nabla h|^2} - G(h). \quad (3.14)$$

Putting (3.14) into (3.13), we conclude that there exists a smooth vector field $X$ on $M$ such that

$$2|\nabla^2 h|^2 + (2+\frac{1}{\alpha})\frac{\nabla|\nabla h|^2\nabla h}{h} \geq \nabla F\cdot X + \frac{2n}{n-1}[G(h)]^2 + \frac{4}{n-1}G(h)\triangle h + \frac{2}{n-1}(\triangle h)^2 - \frac{4\alpha+2}{\alpha}\frac{|\nabla h|^2}{h}G(h). \quad (3.15)$$

On the other hand, by (3.3) we have

$$2G(h)\triangle h + \frac{2n}{n-1}[G(h)]^2 + \frac{4}{n-1}G(h)\triangle h + \frac{2}{n-1}(\triangle h)^2$$

$$= \frac{2n+2}{n-1}G(h)[(1+\frac{1}{\alpha})\frac{|\nabla h|^2}{h} - L(h)] + \frac{2n}{n-1}[G(h)]^2 + \frac{2}{n-1}[(1+\frac{1}{\alpha})\frac{|\nabla h|^2}{h} - L(h)]^2$$

$$= \frac{2n+2}{n-1}G(h)(1+\frac{1}{\alpha})\frac{|\nabla h|^2}{h} - \frac{2n+2}{n-1}G(h)L(h) + \frac{2n}{n-1}[G(h)]^2 + \frac{2}{n-1}(1+\frac{1}{\alpha})^2\frac{|\nabla h|^4}{h^2}$$

$$-\frac{4}{n-1}(1+\frac{1}{\alpha})\frac{|\nabla h|^2}{h}L(h) + \frac{2}{n-1}[L(h)]^2. \quad (3.16)$$

Since $G(h) \geq L(h)$, then

$$\frac{2n}{n-1}[G(h)]^2 - \frac{2n+2}{n-1}G(h)L(h) + \frac{2}{n-1}[L(h)]^2$$

$$= \frac{2n}{n-1}[G(h)]^2 - \frac{2n}{n-1}G(h)L(h) + \frac{2}{n-1}[L(h)]^2 - \frac{2}{n-1}G(h)L(h)$$

$$= \frac{2n}{n-1}G(h)[G(h) - L(h)] - \frac{2}{n-1}L(h)[G(h) - L(h)]$$

$$= \frac{2}{n-1}[G(h) - L(h)][nG(h) - L(h)] \geq 0. \tag{3.17}$$

By (3.16) and (3.17), we obtain

$$2G(h)\triangle h + \frac{2n}{n-1}[G(h)]^2 + \frac{4}{n-1}G(h)\triangle h + \frac{2}{n-1}(\triangle h)^2$$

$$\geq \frac{2n+2}{n-1}G(h)(1+\frac{1}{\alpha})\frac{|\nabla h|^2}{h} + \frac{2}{n-1}(1+\frac{1}{\alpha})^2\frac{|\nabla h|^4}{h^2} - \frac{4}{n-1}(1+\frac{1}{\alpha})\frac{|\nabla h|^2}{h}L(h). \tag{3.18}$$

According to (3.12), (3.15) and (3.18), we get

$$\triangle F \geq \nabla F \cdot X - \frac{4\alpha+2}{\alpha}\frac{|\nabla h|^2}{h}G(h) + \frac{2n+2}{n-1}G(h)(1+\frac{1}{\alpha})\frac{|\nabla h|^2}{h} + \frac{2}{n-1}(1+\frac{1}{\alpha})^2\frac{|\nabla h|^4}{h^2}$$

$$-\frac{4}{n-1}(1+\frac{1}{\alpha})\frac{|\nabla h|^2}{h}L(h) - (2+\frac{2}{\alpha})\frac{|\nabla h|^4}{h^2} - 2(n-2)\alpha\lambda|\nabla h|^2 + \frac{2+2(2-n)\alpha}{\alpha^2(2-n)}\frac{|\nabla h|^4}{h^2}$$

$$+2\lambda|\nabla h|^2 - 2\alpha(n-2)h\nabla\lambda\nabla h + 2(n-2)\alpha\lambda_{\max}h|\nabla h|^2. \tag{3.19}$$

Since $G(h) \geq L(h)$, $\alpha > 0$ and $\beta \geq 0$, then

$$[-\frac{4\alpha+2}{\alpha}G(h) + \frac{2n+2}{n-1}G(h)(1+\frac{1}{\alpha}) - \frac{4}{n-1}(1+\frac{1}{\alpha})L(h)]\frac{|\nabla h|^2}{h} + 2(n-2)\alpha\lambda_{\max}h|\nabla h|^2$$

$$= -2G(h)\frac{|\nabla h|^2}{h} + \frac{4}{n-1}(1+\frac{1}{\alpha})[G(h) - L(h)]\frac{|\nabla h|^2}{h} + 2(n-2)\alpha\lambda_{\max}h|\nabla h|^2$$

$$= 2\alpha\beta h^{\frac{2}{(2-n)\alpha}}|\nabla h|^2 + \frac{4}{n-1}(1+\frac{1}{\alpha})[G(h) - L(h)]\frac{|\nabla h|^2}{h} \geq 0. \tag{3.20}$$

By (3.20) and (3.19), we conclude that (3.8) is true. □

**Lemma 3.3** *Suppose that $M$ is a compact generalized $(2-n)$-quasi-Einstein manifold, $\alpha > 0$ and $\beta \geq 0$. If $\nabla\lambda\nabla f \leq 0$ holds on $M$, then there exists a constant $\delta > 0$ such that if $|\alpha - \frac{1}{n-2}| < \delta$ then*

$$\Delta F \geq \nabla F \cdot X - \alpha(n-2)h\nabla\lambda\nabla h \tag{3.21}$$

*holds on $D$.*

**Proof** Since $M$ is a compact generalized $(2-n)$-quasi-Einstein manifold, then it is obvious that

$$\lim_{n\to\frac{1}{n-2}}[\frac{2}{n-1}(1+\frac{1}{\alpha})^2-(2+\frac{2}{\alpha})+\frac{2+2(2-n)\alpha}{\alpha^2(2-n)}]\frac{|\nabla h|^4}{h^2}=0$$

and

$$\lim_{n\to\frac{1}{n-2}}[2\lambda|\nabla h|^2-2\alpha(n-2)\lambda|\nabla h|^2]=0.$$

Since $\nabla\lambda\nabla f\leq 0$, then $\alpha(n-2)h\nabla\lambda\nabla h\leq 0$. Therefore, there exists a constant $\delta>0$ such that if $|\alpha-\frac{1}{n-2}|<\delta$ then

$$[\frac{2}{n-1}(1+\frac{1}{\alpha})^2-(2+\frac{2}{\alpha})+\frac{2+2(2-n)\alpha}{\alpha^2(2-n)}]\frac{|\nabla h|^4}{h^2}+2\lambda|\nabla h|^2-2\alpha(n-2)\lambda|\nabla h|^2-\alpha(n-2)h\nabla\lambda\nabla h\geq 0. \quad (3.22)$$

By (3.22) and (3.8), we conclude that (3.21) is true. □

**Lemma 3.4** *Suppose that $M$ is a compact generalized $(2-n)$-quasi-Einstein manifold, $\alpha>0$ and $\beta\geq 0$. If $\nabla\lambda\nabla f\leq 0$ and $\nabla\lambda\neq 0$ holds on $M$, then there exists a constant $\delta>0$ such that if $|\alpha-\frac{1}{n-2}|<\delta$ then*

$$|\nabla h|^2(x)\leq \mathcal{G}(h(x_0))-\mathcal{G}(h(x)) \quad (3.23)$$

*holds for all $x\in M$, where $x_0$ is the maximum point of $F(h(x))$ on $M$ and*

$$\mathcal{G}(h)=(n-2)\alpha\lambda_{\max}h^2-\frac{(2-n)\alpha^2}{1+(2-n)\alpha}\beta h^{(2+\frac{2}{(2-n)\alpha})}. \quad (3.24)$$

**Proof** By Lemma 3.3, there exists a constant $\delta>0$ such that if $|\alpha-\frac{1}{n-2}|<\delta$ then (3.21) holds. If $x_0\in D$, then there exists a neighborhood $\mathcal{U}$ of $x_0$ so that $\mathcal{U}\subset D$. Moreover, $x_0$ is the maximum point of $F(h(x))$ on $\mathcal{U}$. By Lemma 3.3, we conclude that $\Delta F\geq\nabla F\cdot X$ holds on $\mathcal{U}$. Therefore, by the Hopf maximum principle in [6] we conclude that $F$ is constant on $\mathcal{U}$. Since $\Delta F(x_0)\leq 0$, $\nabla F(x_0)=0$, $\nabla\lambda\nabla f\leq 0$ and $\nabla\lambda\neq 0$, (3.21) tells us that $\nabla h(x_0)=0$, which is a contradiction with $x_0\in D$. Therefore, $x_0$ is not in $D$, which means that $\nabla h(x_0)=0$. Thus $|\nabla h|^2(x)+\mathcal{G}(h(x))\leq\mathcal{G}(h(x_0))$. □

## 4. Diameter estimate and main result

In this section, we consider the lower diameter estimate for compact generalized $(2-n)$-quasi-Einstein manifold $M$ using the gradient estimate obtained in Lemma 3.4. If $\beta\leq 0$ and $\lambda\geq 0$, by (2.1) and the Maximum principle in [6] we conclude that $M$ is an Einstein manifold. Under this consideration, we only discuss the diameter estimate for compact generalized $(2-n)$-quasi-Einstein manifold with $\beta>0$. The main result of this paper is

**Theorem 4.1** *Suppose that $M$ is a compact generalized $(2-n)$-quasi-Einstein manifold satisfying (1.2). Let $\omega_f=\max_{x\in M}f(x)-\min_{x\in M}f(x)$ and*

$$d_1=\frac{1}{\sqrt{\lambda_{\max}-\lambda_{\min}e^{\frac{2}{2-n}w_f}}}(\frac{\pi}{2}-\arcsin e^{\frac{w_f}{2-n}}),\quad d_2=\frac{1}{\sqrt{\lambda_{\max}}}(\frac{\pi}{2}-\arcsin e^{\frac{\omega_f}{2-n}}).$$

*If $\nabla\lambda\nabla f \leq 0$ holds on $M$, $n \geq 3$, $\beta > 0$, then the diameter of $M$ satisfies*

$$\mathrm{diam}M \geq \max\{d_1, d_2\}.$$

**Proof** Suppose that $\delta$ is the constant mentioned in Lemma 3.4. Let $|\alpha - \frac{1}{n-2}| < \delta$. Then (3.23) holds. Assume that $x_0$ is the maximum point of $F(h(x))$ on $M$. By Lemma 3.4, for all $x \in M$,

$$\mathcal{G}(h(x)) \leq |\nabla h|^2(x) + \mathcal{G}(h(x)) \leq \mathcal{G}(h(x_0)).$$

Therefore, $x_0$ is the maximum point of $\mathcal{G}(h(x))$. According to (3.24), we get

$$\mathcal{G}'(t) = -2[(n-2)\alpha\lambda_{\max}t - \alpha\beta t^{(1+\frac{2}{(2-n)\alpha})}].$$

Let $x_1$ and $x_2$ be the maximum and minimum points of $h(x)$ on $M$, respectively. Similar to the proof of Theorem 1.3 in [18], we conclude that $x_0 = x_1$ or $x_0 = x_2$. we only consider the case that $x_0 = x_1$. Choosing a minimizing geodesic $\gamma$ jointing $x_1$ and $x_2$. Let $h_1 = h(x_1)$, $h_2 = h(x_2)$. Similar to the proof of Theorem 1.3 in [18], we have

$$\begin{aligned}\mathrm{diam}M &\geq \int_{h_2}^{h_1} \frac{dh}{\sqrt{\mathcal{G}(h(x_1)) - \mathcal{G}(h(x))}} \\ &= \int_{h_2}^{h_1} \frac{dh}{\sqrt{(n-2)\alpha\lambda_{\max}(h_1^2 - h^2) - \frac{(2-n)\alpha^2\beta}{1+(2-n)\alpha}[h_1^{(2+\frac{2}{(2-n)\alpha})} - h^{(2+\frac{2}{(2-n)\alpha})}]}} \\ &= \int_{\frac{h_2}{h_1}}^{1} \frac{d\sigma}{\sqrt{(n-2)\alpha\lambda_{\max}(1-\sigma^2) + \frac{(n-2)\alpha^2\beta}{1+(2-n)\alpha}h_1^{\frac{2}{(2-n)\alpha}}[1 - \sigma^{(2+\frac{2}{(2-n)\alpha})}]}} \qquad (4.1)\end{aligned}$$

Since $\beta > 0$, by Lemma 2.2 we have $\lambda_{\max} > 0$. If $\alpha > \frac{1}{n-2}$, then $1 + (2-n)\alpha < 0$. Therefore, by (4.1) we conclude that

$$\mathrm{diam}M \geq \frac{1}{\sqrt{(n-2)\alpha\lambda_{\max}}} \int_{\frac{h_2}{h_1}}^{1} \frac{d\sigma}{\sqrt{1-\sigma^2}} = \frac{1}{\sqrt{(n-2)\alpha\lambda_{\max}}}(\frac{\pi}{2} - \arcsin e^{-\alpha w_f}).$$

Let $\alpha \to \frac{1}{n-2}$ from the right side of $\frac{1}{n-2}$. Then we have

$$\mathrm{diam}M \geq \frac{1}{\sqrt{\lambda_{\max}}}(\frac{\pi}{2} - \arcsin e^{\frac{\omega_f}{2-n}}). \qquad (4.2)$$

If $0 < \alpha < \frac{1}{n-2}$, we consider the function

$$S(\sigma) = -[1 + (2-n)\alpha](1-\sigma^2) + \sigma^{(2+\frac{2}{(2-n)\alpha})} - 1$$

on $[\frac{h_1}{h_0}, 1]$. Since $0 < \alpha < \frac{1}{n-2}$, then $(n-2)\alpha < 1$, $\sigma^{\frac{2}{(2-n)\alpha}} \geq 1$, $2 + \frac{2}{(2-n)\alpha} < 0$. Therefore, we have

$$S'(\sigma) = [2 + \frac{2}{(2-n)\alpha}][\sigma^{\frac{2}{(2-n)\alpha}} - (n-2)\alpha]\sigma < 0.$$

Thus, we conclude that $S(\sigma) > S(1) = 0$. Then

$$1 - \sigma^{(2+\frac{2}{(2-n)\alpha})} < -[1 + (2-n)\alpha](1 - \sigma^2). \tag{4.3}$$

Since $\beta > 0$ and $1 + (2-n)\alpha > 0$, by (4.3) we get

$$\frac{(n-2)\alpha^2\beta}{1+(2-n)\alpha} h_1^{\frac{2}{(2-n)\alpha}} [1 - \sigma^{(2+\frac{2}{(2-n)\alpha})}] < -(n-2)\alpha^2\beta h_1^{\frac{2}{(2-n)\alpha}} (1 - \sigma^2). \tag{4.4}$$

Since $\alpha > 0$, then $x_2$ is a minimum point of $f(x)$. Let $\lambda_{\min} = \min_{x\in M} \lambda(x)$. According to (2.2) we have

$$\beta e^{\frac{2}{2-n} f(x_2)} \geq (n-2)\lambda(x_2) \geq (n-2)\lambda_{\min}. \tag{4.5}$$

By (4.5), we obtain

$$\beta h_1^{\frac{2}{(2-n)\alpha}} = \beta e^{\frac{2}{2-n} w_f} e^{\frac{2}{2-n} f(x_2)} \geq (n-2)\lambda_{\min} e^{\frac{2}{2-n} w_f}. \tag{4.6}$$

According to (4.1), (4.4) and (4.5), we have

$$\mathrm{diam} M \geq \frac{1}{\sqrt{(n-2)\alpha\lambda_{\max} - (n-2)^2\alpha^2\lambda_{\min} e^{\frac{2}{2-n} w_f}}} (\frac{\pi}{2} - \arcsin e^{-\alpha w_f}).$$

Let $\alpha \to \frac{1}{n-2}$ from the left side of $\frac{1}{n-2}$. Then

$$\mathrm{diam} M \geq \frac{1}{\sqrt{\lambda_{\max} - \lambda_{\min} e^{\frac{2}{2-n} w_f}}} (\frac{\pi}{2} - \arcsin e^{\frac{w_f}{2-n}}). \tag{4.7}$$

According to (4.2) and (4.7), we conclude that $\mathrm{diam} M \geq \max\{d_1, d_2\}$. □


## Acknowledgment

This work is supported by the discipline special research projects of Hengyang Normal University (XKZX21002, KYZX21003), Application-oriented Special Disciplines, Double First-Class University Project of Hunan Province (Xiangjiaotong[2018]469), the science and technology plan project of Hunan province (2016TP1020).


## References


[1] Barros A, Gomes J. A compact gradient generalized quasi-Einstein metric with constant scalar curvature. Journal of Mathematical Analysis and Applications 2013; 401: 702-705.

[2] Barros A, Ribeiro JE. Characterizations and integral formulae for generalized $m$-quasi-Einstein metrics. Bulletin Brazilian Mathematical Society (N.S) 2014; 45: 325-341.

[3] Catino G. Generalized quasi-Einstein manifolds with harmonic weyl tensor. Mathematische Zeitschrift 2012; 271: 751-756.

[4] Futaki A, Li H, Li X. On the first eigenvalue of the Witten-Laplacian and the diameter of compact shrinking solitons. Annals of Global Analysis and Geometry 2013; 44: 105-114.

[5] Futaki A, Sano Y. Lower diameter bounds for compact shrinking Ricci solitons. Asian Journal of Mathematics 2013; 17: 17-32.

[6] Gilbarg D, Trudinger N. Elliptic Partial Differential Equations of Second Order. Springer, Berlin, 1977

[7] Hu Z, Li D, Zhai S. On generalized $m$-quasi-Einstein manifolds with constant Ricci curvatures. Journal of Mathematical Analysis and Applications 2017; 446: 843-851.

[8] Hu L, Mao J, Wang L. Lower diameter estimate for a spacial quasi-almost-Einstein metric. Journal of East China Normal University (Natural Science) 2014; 27: 27-35.

[9] Huang G, Wei Y. The classification of $(m, \rho)$-quasi-Einstein manifolds. Annals of Global Analysis and Geometry 2013; 44: 269-282.

[10] Huang G, Zeng F. A note on gradient generalized quasi-Einstein manifolds. Journal of Geometry 2015; 106: 297-311.

[11] Jauregui J, Wylie W. Conformal diffeomorphisms of gradient Ricci solitons and generalized quasi-Einstein manifolds. Journal of Geometric Analysis 2015; 25: 668-708.

[12] Limoncu M. Modifications of the Ricci tensor and applications. Archiv der Mathematik 2010; 95: 191-199.

[13] Limoncu M. The Bakry-Émery Ricci tensor and its applications to some compactness theorems. Mathematische Zeitschrift 2012; 271: 715-722.

[14] Neto B. Generalized quasi-Einstein manifolds with harmonic anti-self dual Weyl tensor. Archiv der Mathematik (Basel) 2016; 106: 489-499.

[15] Ribeiro JE, Tenenblat K. Noncompact quasi-Einstein manifolds conformal to a Euclidean space. arXiv:1904.11283

[16] Soylu Y. A Myers-type compactness theorem by the use of Bakry-Émery Ricci tensor. Differential Geometry and its Applications 2017; 54: 245-250.

[17] Tadano H. Remark on a diameter bound for complete Riemannian manifolds with positive Bakry-Émery Ricci curvature. Differential Geometry and its Applications 2016; 44: 136-143.

[18] Wang L. Diameter estimate for compact quasi-Einstein metrics. Mathematische Zeitschrift 2012; 273: 801-809.

[19] Wei G, Wylie W. Comparison geometry for the Bakry-Émery Ricci tensor. Journal of Differential Geometry 2009; 83: 377-405.