Turkish Journal of Mathematics

Volume 45 | Number 5 Article 30

1-1-2021

Some new uniqueness and Ulam stability results for a class of Multi-Terms fractional differential equations in the framework of Generalized Caputo Fractional Derivative using the Φ-fractional Bielecki-type norm

CHOUKRI DERBAZI

ZIDANE BAITICHE

MICHAL FECKAN

Follow this and additional works at: https://journals.tubitak.gov.tr/math

Part of the Mathematics Commons

Recommended Citation
DERBAZI, CHOUKRI; BAITICHE, ZIDANE; and FECKAN, MICHAL (2021) "Some new uniqueness and Ulam stability results for a class of Multi-Terms fractional differential equations in the framework of Generalized Caputo Fractional Derivative using the Φ-fractional Bielecki-type norm," *Turkish Journal of Mathematics*: Vol. 45: No. 5, Article 30. https://doi.org/10.3906/mat-2011-92
Available at: https://journals.tubitak.gov.tr/math/vol45/iss5/30

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact academic.publications@tubitak.gov.tr.

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 2307 – 2322
© TÜBİTAK
doi:10.3906/mat-2011-92

Some new uniqueness and Ulam stability results for a class of multiterms fractional differential equations in the framework of generalized Caputo fractional derivative using the Φ-fractional Bielecki-type norm

Choukri DERBAZI[1,*], **Zidane BAITICHE**[1], **Michal FEČKAN**[3,4]

[1]Laboratoire Equations Différentielles, Department of Mathematics, Faculty of Exact Sciences, University Frères Mentouri Constantine 25000, Algeria

[3]Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská Dolina, 842 48 Bratislava, Slovakia

[4]Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia

Received: 28.11.2020 • **Accepted/Published Online:** 01.09.2021 • **Final Version:** 16.09.2021

Abstract: In this research article, a novel Φ-fractional Bielecki-type norm introduced by Sousa and Oliveira [23] is used to obtain results on uniqueness and Ulam stability of solutions for a new class of multiterms fractional differential equations in the framework of generalized Caputo fractional derivative. The uniqueness results are obtained by employing Banach' and Perov's fixed point theorems. While the Φ-fractional Gronwall type inequality and the concept of the matrices converging to zero are implemented to examine different types of stabilities in the sense of Ulam–Hyers (UH) of the given problems. Finally, two illustrative examples are provided to demonstrate the validity of our theoretical findings.

Key words: Φ–Caputo fractional derivative, fixed point, uniqueness, Φ-fractional Bielecki-type norm, $\mathbb{E}_\mu$–Ulam stability

1. Introduction

Over these years, there has been a significant interest in the study of fractional differential equations (FDEs). This is due to their applications in many disciplines of science and engineering. For more details and applications about fractional calculus (FC), we refer the reader to [3, 8, 10, 17]. With the great progress of FC different definitions of fractional operators have appeared in the literature. In 2017, Almeida [2] formulated a new category of fractional operators called generalized fractional derivatives (GFDs) that contain kernels depending on a function Φ (or Φ–Caputo fractional derivative). Just one year later Sousa and Oliveira [21] extended Almeida's work to the Φ–Hilfer fractional derivative which yields a large class of fractional derivatives for some special cases of the function Φ. On the other hand, different techniques have been adopted to tackle the existence and uniqueness of solutions, as well as different types of stabilities in the sense of Ulam–Hyers (UH) for nonlinear ordinary differential equations (ODEs) and nonlinear FDEs involving various categories of fractional derivatives, for more details, see [1, 4–6, 11, 12, 15, 19, 23, 26–29], and the references cited therein. As far as we know, there are no contributions associated with the solutions of multiterms fractional differential

*Correspondence: choukriedp@yahoo.com

2010 *AMS Mathematics Subject Classification:* 34A08; 26A33

This work is licensed under a Creative Commons Attribution 4.0 International License.

equations in the frame of Φ–Caputo derivative, especially in generalized Banach spaces. Therefore, this paper comes to fill this gap. More specifically, we are interested in proving the uniqueness and the $\mathbb{E}_{\mu}$-UH stability of solutions for the following Φ–Caputo fractional multiterms differential equation (Φ–Caputo FMTDE) of the form

$$\begin{cases} {}^{c}\mathbb{D}_{u^+}^{\mu;\Phi}\mathfrak{m}(\ell) + \varrho\, {}^{c}\mathbb{D}_{u^+}^{\kappa;\Phi}\mathfrak{m}(\ell) = \mathbb{H}(\ell, \mathfrak{m}(\ell)), \ \ell \in \Sigma := [u, v], \\ \mathfrak{m}(u) = \theta, \end{cases} \tag{1.1}$$

where ${}^{c}\mathbb{D}_{u^+}^{\mu;\Phi}$ and ${}^{c}\mathbb{D}_{u^+}^{\kappa;\Phi}$ denote the Φ-Caputo fractional derivatives, with the orders μ and κ respectively such that $0 < \kappa < \mu \leq 1, \varrho > 0$, $\mathbb{H} \in C(\Sigma \times \mathbb{R}, \mathbb{R})$ and $\theta \in \mathbb{R}$.

Next, we switch onto the coupled system version of the above problem. Namely, we study the uniqueness and the UH stability of solutions for the Φ–Caputo fractional multiterms differential system (Φ–Caputo FMTDS) of the type

$$\begin{cases} {}^{c}\mathbb{D}_{u^+}^{\mu_1;\Phi}\mathfrak{m}_1(\ell) + \varrho_1\, {}^{c}\mathbb{D}_{u^+}^{\kappa_1;\Phi}\mathfrak{m}_1(\ell) = \mathbb{K}_1(\ell, \mathfrak{m}_1(\ell), \mathfrak{m}_2(\ell)), \\ {}^{c}\mathbb{D}_{u^+}^{\mu_2;\Phi}\mathfrak{m}_2(\ell) + \varrho_2\, {}^{c}\mathbb{D}_{u^+}^{\kappa_2;\Phi}\mathfrak{m}_2(\ell) = \mathbb{K}_2(\ell, \mathfrak{m}_1(\ell), \mathfrak{m}_2(\ell)), \end{cases} \quad \ell \in \Sigma, \tag{1.2}$$

augmented with the initial conditions given by:

$$\begin{cases} \mathfrak{m}_1(u) = \theta_1, \\ \mathfrak{m}_2(u) = \theta_2, \end{cases} \tag{1.3}$$

where $0 < \kappa_i < \mu_i \leq 1, \varrho_i > 0$, $\mathbb{K}_i \in C(\Sigma \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $\theta_i \in \mathbb{R}$, $i = 1, 2$.

It is worth noting that the results obtained in this paper are generalizations and partial continuation of some results obtained in [11, 12].

The rest of this paper is structured in the following way. Section 2 is dedicated to the primary definitions and notations. In Section 3 by the implementation of Banach's fixed point theorem associated with a new Φ-fractional Bielecki-type norm introduced by Sousa and Oliveira [23] and the Φ-fractional Gronwall's inequality we study the uniqueness and different kinds of the $\mathbb{E}_{\mu}$-UH stability of the proposed problem (1.1). In Section 4 a new uniqueness result for the Φ–Caputo FMTDS is guaranteed by Perov's fixed point theorem, while Urs's approach is utilized to obtain the UH-stability of solutions for the proposed problem system. Our main results are well illustrated by two particular examples presented in Section 5.

2. Preliminaries

In the current section, we state some basic concepts of fractional calculus, related to our work.

First of all, we recall the definition of the Mittag–Leffler functions (MLFs).

Definition 2.1 **([7])** *For* $\mathrm{p}, \mathrm{q} > 0$ *and* $\varpi \in \mathbb{R}$, *the Mittag–Leffler functions (MLFs) of one and two parameters are given by*

$$\mathbb{E}_{\mathrm{p}}(\varpi) = \sum_{k=0}^{\infty} \frac{\varpi^k}{\Gamma(\mathrm{p}k + 1)}, \quad \mathbb{E}_{\mathrm{p},\mathrm{q}}(\varpi) = \sum_{k=0}^{\infty} \frac{\varpi^k}{\Gamma(\mathrm{p}k + \mathrm{q})}. \tag{2.1}$$

Clearly, $\mathbb{E}_{\mathrm{p},1}(\varpi) = \mathbb{E}_{\mathrm{p}}(\varpi)$.

Lemma 2.2 ([7, 26]) *Let* $\mathrm{p} \in (0,1), \mathrm{q} > \mathrm{p}$ *be arbitrary and* $\varpi \in \mathbb{R}$. *The functions* $\mathbb{E}_{\mathrm{p}}, \mathbb{E}_{\mathrm{p,p}}$ *and* $\mathbb{E}_{\mathrm{p,q}}$ *are nonnegative and have the following properties:*

1. $\mathbb{E}_{\mathrm{p}}(\varpi) \leq 1, \mathbb{E}_{\mathrm{p,q}}(\varpi) \leq \frac{1}{\Gamma(\mathrm{q})}, \quad$ *for any* $\varpi < 0$,

2. $\mathbb{E}_{\mathrm{p,q}}(\varpi) = \varpi \mathbb{E}_{\mathrm{p,p+q}}(\varpi) + \frac{1}{\Gamma(\mathrm{q})}, \quad \mathrm{p,q} > 0, \varpi \in \mathbb{R}.$

Let $\Sigma = [u, v]$ $(0 \leq u < v < \infty)$ be a finite interval and $\Phi\colon \Sigma \longrightarrow \mathbb{R}$ be an increasing differentiable function such that $\Phi'(\ell) \neq 0$, for all $\ell \in \Sigma$.

Definition 2.3 ([2, 10]) *The RL fractional integral of order* $\mu > 0$ *for an integrable function* $\mathfrak{m}\colon \Sigma \longrightarrow \mathbb{R}$ *with respect to* Φ *is described by*

$$\mathbb{I}_{u^+}^{\mu;\Phi}\mathfrak{m}(\ell) = \int_u^\ell \frac{\Phi'(\eta)(\Phi(\ell)-\Phi(\eta))^{\mu-1}}{\Gamma(\mu)}\mathfrak{m}(\eta)\mathrm{d}\eta,$$

where $\Gamma(\mu) = \int_0^{+\infty} \ell^{\mu-1}e^{-\ell}\mathrm{d}\ell, \quad \mu > 0$ is called the gamma function.

Definition 2.4 ([2]) *Let* $\Phi, \mathfrak{m} \in C^j(\Sigma, \mathbb{R})$. *The Caputo fractional derivative of* $\mathfrak{m}$ *of order* $j-1 < \mu < j$ *with respect to* Φ *is defined by*

$${}^{c}\mathbb{D}_{u^+}^{\mu;\Phi}\mathfrak{m}(\ell) = \mathbb{I}_{u^+}^{j-\mu;\Phi}\mathfrak{m}_{\Phi}^{[j]}(\ell),$$

where $j = [\mu]+1$ *for* $\mu \notin \mathbb{N}$, $j = \mu$ *for* $\mu \in \mathbb{N}$, *and* $\mathfrak{m}_{\Phi}^{[j]}(\ell) = \left(\frac{\frac{d}{d\ell}}{\Phi'(\ell)}\right)^j \mathfrak{m}(\ell)$.

Some basic properties of the Φ-fractional operators are listed in the following Lemma.

Lemma 2.5 ([2]) *Let* $\mu, \kappa, \beta > 0$, *and* $\mathfrak{m} \in \mathfrak{C} := C(\Sigma, \mathbb{R})$. *Then for each* $\ell \in \Sigma$,

1. ${}^{c}\mathbb{D}_{u^+}^{\mu;\Phi}\mathbb{I}_{u^+}^{\mu;\Phi}\mathfrak{m}(\ell) = \mathfrak{m}(\ell)$,

2. $\mathbb{I}_{u^+}^{\mu;\Phi}{}^{c}\mathbb{D}_{u^+}^{\mu;\Phi}\mathfrak{m}(\ell) = \mathfrak{m}(\ell) - \mathfrak{m}(u), \quad 0 < \mu \leq 1$,

3. $\mathbb{I}_{u^+}^{\mu;\Phi}(\Phi(\ell)-\Phi(u))^{\kappa-1} = \frac{\Gamma(\kappa)}{\Gamma(\kappa+\mu)}(\Phi(\ell)-\Phi(u))^{\kappa+\mu-1}$,

4. ${}^{c}\mathbb{D}_{u^+}^{\mu;\Phi}(\Phi(\ell)-\Phi(u))^{\kappa-1} = \frac{\Gamma(\kappa)}{\Gamma(\kappa-\mu)}(\Phi(\ell)-\Phi(u))^{\kappa-\mu-1}$,

5. $\mathbb{I}_{u^+}^{\mu;\Phi}\left(\mathbb{E}_{\mu}\big(\beta(\Phi(\ell)-\Phi(u))^{\mu}\big) = \frac{1}{\beta}\left(\mathbb{E}_{\mu}\big(\beta(\Phi(\ell)-\Phi(u))^{\mu}-1\right)$

Definition 2.6 ([9]) *A function* $\mathfrak{z} : [u, \infty) \to \mathbb{R}$ *is said to be of* $\Phi(\ell)$*-exponential order if there exist nonnegative constants* c_1, c_2, v *such that*

$$|\mathfrak{z}(\ell)| \leq c_1 e^{c_2(\Phi(\ell)-\Phi(u))}, \quad \ell \geq v.$$

Definition 2.7 ([9]) *Let $\mathfrak{m}, \Phi : [u, \infty) \to \mathbb{R}$ be real valued functions such that $\Phi(\ell)$ is continuous and $\Phi'(\ell) > 0$ on $[u, \infty)$. The generalized Laplace transform of $\mathfrak{m}$ is denoted by*

$$\mathbb{L}_{\Phi}\big\{\mathfrak{m}(\ell)\big\} = \int_u^{\infty} e^{-\lambda(\Phi(\ell)-\Phi(u))}\mathfrak{m}(\ell)\Phi'(\ell)\,\mathrm{d}\ell, \quad \textit{for all } \lambda > 0, \tag{2.2}$$

provided that integral on the right-hand side exists.

Definition 2.8 ([9]) *Let $\mathfrak{z}_1$ and $\mathfrak{z}_2$ be two functions which are piecewise continuous at each interval $[u, v]$ and of exponential order. We define the generalized convolution of $\mathfrak{z}_1$ and $\mathfrak{z}_2$ by*

$$(\mathfrak{z}_1 *_{\Phi} \mathfrak{z}_2)(\ell) = \int_u^{\ell} \Phi'(\eta)\mathfrak{z}_1(\eta)\mathfrak{z}_2\big(\Phi^{-1}\big(\Phi(\ell) + \Phi(u) - \Phi(\eta)\big)\big)\, d\eta.$$

Lemma 2.9 ([9]) *Let $\mathfrak{z}_1$ and $\mathfrak{z}_2$ be two functions which are piecewise continuous at each interval $[u, v]$ and of exponential order. Then*

$$\mathbb{L}_{\Phi}\big\{\mathfrak{z}_1 *_{\Phi} \mathfrak{z}_2\big\} = \mathbb{L}_{\Phi}\big\{\mathfrak{z}_1\big\}\mathbb{L}_{\Phi}\big\{\mathfrak{z}_2\big\}.$$

In the following lemma, we present the generalized Laplace transforms of some elementary functions as well as the generalized Laplace transforms of the generalized fractional integrals and derivatives.

Lemma 2.10 ([9]) *The following properties are satisfied:*

1. $\mathbb{L}_{\Phi}\big\{1\big\} = \frac{1}{\lambda}, \quad \lambda > 0,$
2. $\mathbb{L}_{\Phi}\big\{(\Phi(\ell) - \Phi(u))^{\mathrm{r}-1}\big\} = \frac{\Gamma(\mathrm{r})}{\lambda^{\mathrm{r}}}, \quad \mathrm{r}, \lambda > 0,$
3. $\mathbb{L}_{\Phi}\big\{\mathbb{E}_{\mathrm{p}}\big(\pm\varrho(\Phi(\ell) - \Phi(u))^{\mathrm{p}}\big)\big\} = \frac{\lambda^{\mathrm{p}-1}}{\lambda^{\mathrm{p}} \mp \varrho}, \quad \mathrm{p} > 0$ *and* $\big|\frac{\varrho}{\lambda^{\mathrm{p}}}\big| < 1,$
4. $\mathbb{L}_{\Phi}\big\{(\Phi(\ell) - \Phi(u))^{\mathrm{q}-1}\mathbb{E}_{\mathrm{p},\mathrm{q}}\big(\pm\varrho(\Phi(\ell) - \Phi(u))^{\mathrm{p}}\big)\big\} = \frac{\lambda^{\mathrm{p}-\mathrm{q}}}{\lambda^{\mathrm{p}} \mp \varrho}, \quad \mathrm{p} > 0$ *and* $\big|\frac{\varrho}{\lambda^{\mathrm{p}}}\big| < 1,$
5. $\mathbb{L}_{\Phi}\big\{\mathbb{I}_{u^+}^{\mu;\Phi}\mathfrak{m}(\ell)\big\} = \frac{\mathbb{L}_{\Phi}\{\mathfrak{m}(\ell)\}}{\lambda^{\mu}}, \quad \mu, \lambda > 0,$
6. $\mathbb{L}_{\Phi}\big\{{}^{c}\mathbb{D}_{u^+}^{\mu;\Phi}\mathfrak{m}(\ell)\big\} = \lambda^{\mu}\mathbb{L}_{\Phi}\big\{\mathfrak{m}(\ell)\big\} - \lambda^{\mu-1}\mathfrak{m}(u), \quad 0 < \mu \le 1$ *and* $\lambda > 0,$

Lemma 2.11 ([4]) *Let $\mu, \beta > 0$. Then for all $\ell \in \Sigma$ we have*

$$\mathbb{I}_{u^+}^{\mu;\Phi} e^{\beta(\Phi(\ell)-\Phi(u))} \le \frac{e^{\beta(\Phi(\ell)-\Phi(u))}}{\beta^{\mu}}.$$

The following lemma is a generalization of Gronwall's inequality.

Lemma 2.12 ([22]) *Let Σ be the domain of the nonnegative integrable functions $\mathfrak{a}_1, \mathfrak{a}_2$. Also, $\mathfrak{a}_3$ is a continuous, nonnegative and nondecreasing function defined on Σ and $\Phi \in C^1(\Sigma, \mathbb{R}_+)$ is an increasing function with the restriction that $\Phi'(\ell) \ne 0, \forall \ell \in \Sigma$. If*

$$\mathfrak{a}_1(\ell) \le \mathfrak{a}_2(\ell) + \mathfrak{a}_3(\ell)\int_u^{\ell} \Phi'(\eta)(\Phi(\ell) - \Phi(\eta))^{\mu-1}\mathfrak{a}_1(\eta)\mathrm{d}\eta,\ \ell \in \Sigma.$$

Then

$$\mathfrak{a}_1(\ell) \leq \mathfrak{a}_2(\ell) + \int_u^\ell \sum_{n=0}^{\infty} \frac{(\mathfrak{a}_3(\ell)\Gamma(\mu))^n}{\Gamma(n\mu)} \Phi'(\eta)(\Phi(\ell) - \Phi(\eta))^{n\mu-1}\mathfrak{a}_2(\eta)\mathrm{d}\eta, \ \ell \in \Sigma.$$

Corollary 2.13 ([22]) *Under the conditions of Lemma 2.12, let* $\mathfrak{a}_2$ *be a nondecreasing function on* Σ*. Then we get that*

$$\mathfrak{a}_1(\ell) \leq \mathfrak{a}_2(\ell)\mathbb{E}_\mu\left(\Gamma(\mu)\mathfrak{a}_3(\ell)\big(\Phi(\ell) - \Phi(u)\big)^\mu\right), \ \ell \in \Sigma. \tag{2.3}$$

Now, we are ready to present our main results.

3. Uniqueness and $\mathbb{E}_\mu$-UH stability results for the $\Phi-$Caputo FMTDE (1.1)

Based on the work in [19, 28], we introduce different types of stabilities in the sense of $\mathbb{E}_\mu$–Ulam–Hyers ($\mathbb{E}_\mu$-UH) for problem (1.1).

Let $\varepsilon > 0, \mathbb{L}_\mathbb{H} > 0$ and $\zeta : \Sigma \to \mathbb{R}^+$, be a continuous function. We consider the following inequalities:

$$|{}^c\mathbb{D}^{\mu;\Phi}_{u^+}\mathfrak{n}(\ell) + \varrho\, {}^c\mathbb{D}^{\kappa;\Phi}_{u^+}\mathfrak{n}(\ell) - \mathbb{H}(\ell, \mathfrak{n}(\ell))| \leq \varepsilon, \quad \ell \in \Sigma; \tag{3.1}$$

$$|{}^c\mathbb{D}^{\mu;\Phi}_{u^+}\mathfrak{n}(\ell) + \varrho\, {}^c\mathbb{D}^{\kappa;\Phi}_{u^+}\mathfrak{n}(\ell) - \mathbb{H}(\ell, \mathfrak{n}(\ell))| \leq \zeta(\ell), \quad \ell \in \Sigma; \tag{3.2}$$

$$|{}^c\mathbb{D}^{\mu;\Phi}_{u^+}\mathfrak{n}(\ell) + \varrho\, {}^c\mathbb{D}^{\kappa;\Phi}_{u^+}\mathfrak{n}(\ell) - \mathbb{H}(\ell, \mathfrak{n}(\ell))| \leq \varepsilon\zeta(\ell), \quad \ell \in \Sigma. \tag{3.3}$$

Definition 3.1 ([28]) *Equation (1.1) is* $\mathbb{E}_\mu$*-UH stable if there exists a real number* $c > 0$ *such that, for each* $\varepsilon > 0$ *and for each solution* $\mathfrak{n} \in \mathfrak{C} := C(\Sigma, \mathbb{R})$ *of inequalities (3.1), there exists a solution* $\mathfrak{m} \in \mathfrak{C}$ *of (1.1) with*

$$\big|\mathfrak{n}(\ell) - \mathfrak{m}(\ell)\big| \leq c\varepsilon\mathbb{E}_\mu\big(\mathbb{L}_\mathbb{H}(\Phi(\ell) - \Phi(u))^\mu\big), \quad \ell \in \Sigma.$$

Definition 3.2 ([28]) *Equation (1.1) is generalized* $\mathbb{E}_\mu$*-UH stable if there exists* $\varphi : C(\mathbb{R}_+, \mathbb{R}_+)$ *with* $\varphi(0) = 0$ *such that, for each* $\varepsilon > 0$ *and for each solution* $\mathfrak{n} \in \mathfrak{C}$ *of inequalities (3.1), there exists a solution* $\mathfrak{m} \in \mathfrak{C}$ *of (1.1) with*

$$\big|\mathfrak{n}(\ell) - \mathfrak{m}(\ell)| \leq \varphi(\varepsilon)\mathbb{E}_\mu\big(\mathbb{L}_\mathbb{H}(\Phi(\ell) - \Phi(u))^\mu\big), \quad \ell \in \Sigma.$$

Definition 3.3 ([28]) *Equation (1.1) is* $\mathbb{E}_\mu$*–Ulam–Hyers–Rassias (*$\mathbb{E}_\mu$*-UHR) stable with respect to* ζ *if there exists a real number* $c_\zeta > 0$ *such that, for each* $\varepsilon > 0$ *and for each solution* $\mathfrak{n} \in \mathfrak{C}$ *of inequalities (3.3), there exists a solution* $\mathfrak{m} \in \mathfrak{C}$ *of (1.1) with*

$$\big|\mathfrak{n}(\ell) - \mathfrak{m}(\ell)\big| \leq c_\zeta\varepsilon\zeta(\ell)\mathbb{E}_\mu\big(\mathbb{L}_\mathbb{H}(\Phi(\ell) - \Phi(u))^\mu\big), \quad \ell \in \Sigma.$$

Definition 3.4 ([28]) *Equation (1.1) is generalized* $\mathbb{E}_\mu$*-UHR stable with respect to* ζ *if there exists a real number* $c_\zeta > 0$ *such that, for each solution* $\mathfrak{n} \in \mathfrak{C}$ *of inequalities (3.2), there exists a solution* $\mathfrak{m} \in \mathfrak{C}$ *of (1.1) with*

$$\big|\mathfrak{n}(\ell) - \mathfrak{m}(\ell)\big| \leq c_\zeta\zeta(\ell)\mathbb{E}_\mu\big(\mathbb{L}_\mathbb{H}(\Phi(\ell) - \Phi(u))^\mu\big), \quad \ell \in \Sigma.$$

Remark 3.5 ([28]) *It is clear that* *(i)* *Definition 3.1* $\Rightarrow$ *Definition 3.2,*

(ii) *Definition 3.3* $\Rightarrow$ *Definition 3.4,*

(iii) *Definition 3.3 for* $\zeta(\cdot) = 1 \Rightarrow$ *Definition 3.1.*

Remark 3.6 ([28]) *A function* $\mathfrak{n} \in \mathfrak{C}$ *is a solution of inequality (3.3) if and only if there exists a function* $\mathfrak{z} \in \mathfrak{C}$ *(which depends on solution* $\mathfrak{n}$*) such that*

(i) $|\mathfrak{z}(\ell)| \leq \varepsilon\zeta(\ell), \quad \ell \in \Sigma,$

(ii) ${}^{c}\mathbb{D}_{u^+}^{\mu;\Phi}\mathfrak{n}(\ell) + \varrho\, {}^{c}\mathbb{D}_{u^+}^{\kappa;\Phi}\mathfrak{n}(\ell) = \mathbb{H}(\ell, \mathfrak{n}(\ell)) + \mathfrak{z}(\ell), \quad \ell \in \Sigma.$

We shall prove our results concerning the Φ–Caputo FMTDE (1.1) under the following assumptions:

(H1) The function $\mathbb{H} : \Sigma \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

(H2) There exists $\mathbb{L}_{\mathbb{H}} > 0$ such that

$$|\mathbb{H}(\ell, \mathfrak{b}_2) - \mathbb{H}(\ell, \mathfrak{b}_1)| \leq \mathbb{L}_{\mathbb{H}}|\mathfrak{b}_2 - \mathfrak{b}_1|, \quad \ell \in \Sigma, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathbb{R}.$$

(H3) There exists an increasing function $\zeta \in C(\Sigma, \mathbb{R}_+)$ and there exists $\gamma_\zeta > 0$ such that for any $\ell \in \Sigma$

$$\mathbb{I}_{u^+}^{\mu;\Phi}\zeta(\ell) \leq \gamma_\zeta\zeta(\ell).$$

Before going to our main results, we state the following special linear cases of the Φ–Caputo FMTDE (1.1).

Lemma 3.7 *For a given* $\mathfrak{z} \in C(\Sigma, \mathbb{R}), 0 < \kappa < \mu \leq 1$ *and* $\varrho > 0$*, the linear* Φ*–Caputo FMTDE*

$$\begin{cases} {}^{c}\mathbb{D}_{u^+}^{\mu;\Phi}\mathfrak{m}(\ell) + \varrho\, {}^{c}\mathbb{D}_{u^+}^{\kappa;\Phi}\mathfrak{m}(\ell) = \mathfrak{z}(\ell), \ \ell \in \Sigma := [u, v], \\ \mathfrak{m}(u) = \theta, \end{cases} \tag{3.4}$$

has a unique solution given explicitly as

$$\mathfrak{m}(\ell) = \theta + \int_u^\ell \Phi'(\eta)(\Phi(\ell) - \Phi(\eta))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}\big(-\varrho(\Phi(\ell) - \Phi(\eta))^{\mu-\kappa}\big)\mathfrak{z}(\eta)\mathrm{d}\eta. \tag{3.5}$$

Proof Applying the generalized Laplace transform to both sides of Equation (3.4) and then using Lemma 2.10, one gets

$$\lambda^\mu\mathbb{L}_\Phi\big\{\mathfrak{m}(\ell)\big\} - \lambda^{\mu-1}\mathfrak{m}(u) + \varrho\lambda^\kappa\mathbb{L}_\Phi\big\{\mathfrak{m}(\ell)\big\} - \varrho\lambda^{\kappa-1}\mathfrak{m}(u) = \mathbb{L}_\Phi\big\{\mathfrak{z}(\ell)\big\}.$$

So,

$$\begin{aligned} \mathbb{L}_\Phi\big\{\mathfrak{m}(\ell)\big\} &= \varrho\frac{\lambda^{-1}}{\lambda^{\mu-\kappa} + \varrho}\theta + \frac{\lambda^{\mu-\kappa-1}}{\lambda^{\mu-\kappa} + \varrho}\theta + \frac{\lambda^{-\kappa}}{\lambda^{\mu-\kappa} + \varrho}\mathbb{L}_\Phi\big\{\mathfrak{z}(\ell)\big\} \\ &= \varrho\mathbb{L}_\Phi\big\{(\Phi(\ell) - \Phi(u))^{\mu-\kappa}\mathbb{E}_{\mu-\kappa,\mu-\kappa+1}\big(-\varrho(\Phi(\ell) - \Phi(u))^{\mu-\kappa}\big)\big\}\theta \\ &+ \mathbb{L}_\Phi\big\{\mathbb{E}_{\mu-\kappa}\big(-\varrho(\Phi(\ell) - \Phi(u))^{\mu-\kappa}\big)\big\}\theta \\ &+ \mathbb{L}_\Phi\big\{(\Phi(\ell) - \Phi(u))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}\big(-\varrho(\Phi(\ell) - \Phi(u))^{\mu-\kappa}\big)\big\}\mathbb{L}_\Phi\big\{\mathfrak{z}(\ell)\big\}. \end{aligned}$$

Taking the inverse generalized Laplace transform to both sides of the last expression, we get

$$\begin{aligned}\mathfrak{m}(\ell) &= \big(\mathbb{E}_{\mu-\kappa}\big(-\varrho(\Phi(\ell)-\Phi(u))^{\mu-\kappa}\big) + \varrho(\Phi(\ell)-\Phi(u))^{\mu-\kappa}\mathbb{E}_{\mu-\kappa,\mu-\kappa+1}\big(-\varrho(\Phi(\ell)-\Phi(u))^{\mu-\kappa}\big)\big)\,\theta \\ &\quad + \mathfrak{z}(\ell) *_{\Phi} (\Phi(\ell)-\Phi(u))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}\big(-\varrho(\Phi(\ell)-\Phi(u))^{\mu-\kappa}\big) \\ &= \theta + \int_u^{\ell} \Phi'(\eta)(\Phi(\ell)-\Phi(\eta))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}\big(-\varrho(\Phi(\ell)-\Phi(\eta))^{\mu-\kappa}\big)\mathfrak{z}(\eta)\mathrm{d}\eta.\end{aligned}$$

This ends the proof of Lemma 3.7. □

As a result of Lemma 3.7, the Φ–Caputo FMTDE (1.1) can be converted to an integral equation which takes the following form

$$\mathfrak{m}(\ell) = \theta + \int_u^{\ell} \Phi'(\eta)(\Phi(\ell)-\Phi(\eta))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}\big(-\varrho(\Phi(\ell)-\Phi(\eta))^{\mu-\kappa}\big)\mathbb{H}(\eta,\mathfrak{m}(\eta))\mathrm{d}\eta. \tag{3.6}$$

Now, we transform the integral representation (3.6) of the Φ–Caputo FMTDE (1.1) into a fixed point problem as follows:

$$\mathfrak{m} = \mathbb{Q}\mathfrak{m}, \quad \mathfrak{m} \in \mathfrak{C} := C(\Sigma,\mathbb{R}),$$

where $\mathbb{Q} : \mathfrak{C} \longrightarrow \mathfrak{C}$ is defined by

$$\mathbb{Q}\mathfrak{m}(\ell) = \theta + \int_u^{\ell} \Phi'(\eta)(\Phi(\ell)-\Phi(\eta))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}\big(-\varrho(\Phi(\ell)-\Phi(\eta))^{\mu-\kappa}\big)\mathbb{H}(\eta,\mathfrak{m}(\eta))\mathrm{d}\eta. \tag{3.7}$$

Clearly, the operator $\mathbb{Q}$ is well-defined. Moreover, the existence of a fixed point for the operator $\mathbb{Q}$ will ensure the existence of the solution of the Φ–Caputo FMTDE (1.1).

Theorem 3.8 *If assumptions* (H1) *and* (H2) *are satisfied, then there exists a unique solution of the Φ–Caputo FMTDE* (1.1) *on Σ. Furthermore, if the hypotheses* (H3) *holds, then the Φ–Caputo FMTDE* (1.1) *is $\mathbb{E}_\mu$-UHR stable.*

Proof The proof of this theorem is divided into two parts. In the first part, we shall prove that the Φ–Caputo FMTDE (1.1) has a unique solution by using the Banach contraction principle, and the second one is devoted to the $\mathbb{E}_\mu$-UHR stability of solutions for the mentioned problem.

Part 1: In this part we shall prove that the Φ–Caputo FMTDE (1.1) has a unique solution. For this end, let us consider on the space $\mathfrak{C} := C(\Sigma,\mathbb{R})$ the Φ-fractional Bielecki-type norm $\|\cdot\|_{\mathfrak{C},\mathfrak{B},\mu}$ given by previous studies [20, 23] and defined by

$$\|\mathfrak{m}\|_{\mathfrak{C},\mathfrak{B},\mu} := \sup_{\ell\in\Sigma} \frac{|\mathfrak{m}(\ell)|}{\mathbb{E}_\mu\big(\beta(\Phi(\ell)-\Phi(u))^{\mu}\big)}, \quad \beta > 0. \tag{3.8}$$

Consequently, $\big(\mathfrak{C}, \|\cdot\|_{\mathfrak{C},\mathfrak{B},\mu}\big)$ is a Banach space. Our aim is to show that $\mathbb{Q}$ is a contraction operator with respect to the Φ-fractional Bielecki-type norm. To do this, let $\mathfrak{m},\mathfrak{n} \in \mathfrak{C}$ and $\ell \in \Sigma$, then, by (H2) and Lemmas 2.2 and 2.5 one has

$$\begin{aligned}|\mathbb{Q}\mathfrak{m}(\ell) - \mathbb{Q}\mathfrak{n}(\ell)| &\leq \mathbb{L}_{\mathbb{H}}\|\mathfrak{m}-\mathfrak{n}\|_{\mathfrak{C},\mathfrak{B},\mu}\int_u^{\ell} \frac{\Phi'(\eta)(\Phi(\ell)-\Phi(\eta))^{\mu-1}}{\Gamma(\mu)}\mathbb{E}_\mu\big(\beta(\Phi(\eta)-\Phi(u))^{\mu}\big)\mathrm{d}\eta \\ &\leq \frac{\mathbb{L}_{\mathbb{H}}}{\beta}\big(\mathbb{E}_\mu\big(\beta(\Phi(\ell)-\Phi(u))^{\mu} - 1\big)\,\|\mathfrak{m}-\mathfrak{n}\|_{\mathfrak{C},\mathfrak{B},\mu},\end{aligned}$$

that is,

$$\|\mathbb{Q}\mathfrak{m}-\mathbb{Q}\mathfrak{n}\|_{\mathfrak{C},\mathfrak{B},\mu}\leq\frac{\mathbb{L}_{\mathbb{H}}}{\beta}\|\mathfrak{m}-\mathfrak{n}\|_{\mathfrak{C},\mathfrak{B},\mu}.$$

Undoubtedly, the mapping $\mathbb{Q}$ is a contraction for β sufficiently large as we wish. Then, by using the well-known Banach fixed point theorem, we get a unique fixed point $\mathfrak{m}$ of $\mathbb{Q}$. Consequently, the Φ–Caputo FMTDE (1.1) has a unique solution.

Part 2: Now, we discuss the $\mathbb{E}_{\mu}$-UHR stability of solutions for the Φ–Caputo FMTDE (1.1). The arguments are based on the Φ-fractional Gronwall's inequality Eq. (2.3).

Let $\varepsilon>0$ and let $\mathfrak{n}\in\mathfrak{C}$ be a function which satisfies the inequality (3.3) and let $\mathfrak{m}\in\mathfrak{C}$ the unique solution of the following problem

$$\begin{cases}{}^{c}\mathbb{D}_{u^{+}}^{\mu;\Phi}\mathfrak{m}(\ell)+\varrho\,{}^{c}\mathbb{D}_{u^{+}}^{\kappa;\Phi}\mathfrak{m}(\ell)=\mathbb{H}(\ell,\mathfrak{m}(\ell)),\ \ell\in\Sigma,\\ \mathfrak{m}(u)=\theta.\end{cases}$$

By Lemma 3.7, we have

$$\mathfrak{m}(\ell)=\theta+\int_{u}^{\ell}\Phi'(\eta)(\Phi(\ell)-\Phi(\eta))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}\big(-\varrho(\Phi(\ell)-\Phi(\eta))^{\mu-\kappa}\big)\mathbb{H}(\eta,\mathfrak{m}(\eta))\mathrm{d}\eta.$$

Since we have assumed that $\mathfrak{n}$ is a solution of (3.3), hence by Remark 3.6 we can get

$$\begin{cases}{}^{c}\mathbb{D}_{u^{+}}^{\mu;\Phi}\mathfrak{n}(\ell)+\varrho\,{}^{c}\mathbb{D}_{u^{+}}^{\kappa;\Phi}\mathfrak{n}(\ell)=\mathbb{H}(\ell,\mathfrak{n}(\ell))+\mathfrak{z}(\ell),\ \ell\in\Sigma,\\ \mathfrak{n}(u)=\theta.\end{cases}$$

Another application of Lemma 3.7, it yields

$$\mathfrak{n}(\ell)=\theta+\int_{u}^{\ell}\Phi'(\eta)(\Phi(\ell)-\Phi(\eta))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}\big(-\varrho(\Phi(\ell)-\Phi(\eta))^{\mu-\kappa}\big)\mathbb{H}(\eta,\mathfrak{n}(\eta))\mathrm{d}\eta.$$

On the other side, for each $\ell\in\Sigma$ we obtain

$$\begin{aligned}|\mathfrak{n}(\ell)-\mathfrak{m}(\ell)|\leq&\int_{u}^{\ell}\Phi'(\eta)(\Phi(\ell)-\Phi(\eta))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}\big(-\varrho(\Phi(\ell)-\Phi(\eta))^{\mu-\kappa}\big)|\mathfrak{z}(\eta)|\mathrm{d}\eta\\&+\int_{u}^{\ell}\Phi'(\eta)(\Phi(\ell)-\Phi(\eta))^{\mu-1}\mathbb{E}_{\mu-\kappa,\mu}\big(-\varrho(\Phi(\ell)-\Phi(\eta))^{\mu-\kappa}\big)\\&\times|\mathbb{H}(\eta,\mathfrak{n}(\eta))-\mathbb{H}(\eta,\mathfrak{m}(\eta))|\mathrm{d}\eta.\end{aligned}$$

Using part (i) of Remark 3.6, (H2) and (H3) we can arrive at

$$\begin{aligned}|\mathfrak{n}(\ell)-\mathfrak{m}(\ell)|\leq&\varepsilon\int_{u}^{\ell}\frac{\Phi'(\eta)(\Phi(\ell)-\Phi(\eta))^{\mu-1}}{\Gamma(\mu)}\zeta(\eta)\mathrm{d}\eta\\&+\mathbb{L}_{\mathbb{H}}\int_{u}^{\ell}\frac{\Phi'(\eta)(\Phi(\ell)-\Phi(\eta))^{\mu-1}}{\Gamma(\mu)}|\mathfrak{n}(\eta)-\mathfrak{m}(\eta)|\mathrm{d}\eta\\\leq&\varepsilon\gamma_{\zeta}\zeta(\ell)+\frac{\mathbb{L}_{\mathbb{H}}}{\Gamma(\mu)}\int_{u}^{\ell}\Phi'(\eta)(\Phi(\ell)-\Phi(\eta))^{\mu-1}|\mathfrak{n}(\eta)-\mathfrak{m}(\eta)|\mathrm{d}\eta.\end{aligned}$$

Applying Corollary 2.13 (the Φ-fractional Gronwall's inequality Eq. (2.3)), to above inequality with $\mathfrak{a}_1(\ell) = |\mathfrak{n}(\ell) - \mathfrak{m}(\ell)|, \mathfrak{a}_2(\ell) = \varepsilon\gamma_\zeta\zeta(\ell)$ and $\mathfrak{a}_3(\ell) = \frac{\mathbb{L}_{\mathbb{H}}}{\Gamma(\mu)}$. Since $\mathfrak{a}_2(\ell)$ is nondecreasing function on Σ, we conclude that

$$|\mathfrak{n}(\ell) - \mathfrak{m}(\ell)| \leq \varepsilon\gamma_\zeta\zeta(\ell)\mathbb{E}_\mu\left(\mathbb{L}_{\mathbb{H}}\big(\Phi(\ell) - \Phi(u)\big)^\mu\right), \quad \ell \in \Sigma. \tag{3.9}$$

Thus, the Φ–Caputo FMTDE (1.1) is $\mathbb{E}_\mu$-UHR stable. □

Remark 3.9 *It is important to notice that by combining the results of the above theorem along with Remark 3.5 the $\mathbb{E}_\mu$-UH stability, generalized $\mathbb{E}_\mu$-UH stability, and generalized $\mathbb{E}_\mu$-UHR stability of the Φ–Caputo FMTDE (1.1) can be obtained as corollaries.*

Remark 3.10 *Notice that in our analysis we do not assume that $\frac{\mathbb{L}_{\mathbb{H}}(\Phi(v)-\Phi(u))^\mu}{\Gamma(\mu+1)} < 1$ in Theorem 3.8, while it is required in Theorem 22 in the article of Liu et al. [11].*

4. Uniqueness and stability results for the coupled systems (1.2)–(1.3)

We start this section by recalling the basic results of matrix analysis.

Let $x, y \in \mathbb{R}^m$ with $x = (x_1, x_2, \dots, x_m), y = (y_1, y_2, \dots, y_m)$.

By $x \leq y$ we mean $x_i \leq y_i, i = 1, \dots, m$. Also,

$$|x| = (|x_1|, |x_2|, \dots, |x_m|),$$
$$\max(x, y) = (\max(x, y), \max(\bar{x}, \bar{y}), \dots, \max(x_m, y_m)),$$

and

$$\mathbb{R}^m_+ = \left\{x \in \mathbb{R}^m \colon x_i \in \mathbb{R}_+, i = 1, \dots, m\right\}.$$

If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c, i = 1, \dots, m$.

Definition 4.1 ([13]) *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d\colon X \times X \to \mathbb{R}^m$ with the following properties:*

(i) $\mathrm{d}(x, y) \geq 0$ *for all* $x, y \in X$*, and if* $\mathrm{d}(x, y) = 0$*, then* $x = y$*;*

(ii) $\mathrm{d}(x, y) = \mathrm{d}(y, x)$ *for all* $x, y \in X$*;*

(iii) $\mathrm{d}(x, z) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z)$ *for all* $x, y, z \in X$*.*

We call the pair (X, d) a generalized metric space with

$$\mathrm{d}(x, y) := \begin{pmatrix} \mathrm{d}_1(x, y) \\ \mathrm{d}_2(x, y) \\ \vdots \\ \mathrm{d}_m(x, y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if $\mathrm{d}_i, i = 1, \dots, m$, are metrics on X.

Definition 4.2 **([25])** *A square matrix* $\mathbb{B}$ *of real numbers is said to be convergent to zero if and only if its spectral radius* $\rho(\mathbb{B})$ *is strictly less than* 1. *In other words, this means that all the eigenvalues of* $\mathbb{B}$ *are in the open unit disc, i.e.,* $|\alpha| < 1$ *for every* $\alpha \in \mathbb{C}$ *with* $\det(\mathbb{B} - \alpha\mathbb{I}) = 0$, *where* $\mathbb{I}$ *denotes the unit matrix of* $\mathbb{B}_{m\times m}(\mathbb{R})$.

Theorem 4.3 **([25])** *For any nonnegative square matrix* $\mathbb{B}$, *the following properties are equivalent*

(*i*) $\mathbb{B}$ *is convergent to zero;*

(*ii*) $\rho(\mathbb{B}) < 1$;

(*iii*) *the matrix* $\mathbb{I} - \mathbb{B}$ *is nonsingular and*

$$(\mathbb{I} - \mathbb{B})^{-1} = \mathbb{I} + \mathbb{B} + \cdots + \mathbb{B}^n + \cdots ;$$

(*iv*) $\mathbb{I} - \mathbb{B}$ *is nonsingular and* $(\mathbb{I} - \mathbb{B})^{-1}$ *is a nonnegative matrix.*

Definition 4.4 **([16, 18])** *Let* $(\mathbb{E}, \mathrm{d})$ *be a generalized metric space. An operator* $\mathbb{T}\colon \mathbb{E} \to \mathbb{E}$ *is said to be contractive if there exists a matrix* $\mathbb{B}$ *convergent to zero such that*

$$\mathrm{d}(\mathbb{T}(x), \mathbb{T}(y)) \leq \mathbb{B}\mathrm{d}(x, y), \quad \textit{for all } x, y \in \mathbb{E}.$$

We employ the following fixed point theorem as a basic tool for proving our main existence result for a coupled system of the proposed problem.

Theorem 4.5 **([14, 16])** *Let* $(\mathbb{E}, \mathrm{d})$ *be a complete generalized metric space and* $\mathbb{T}\colon \mathbb{E} \to \mathbb{E}$ *be a contractive operator with Lipschitz matrix* $\mathbb{B}$. *Then* $\mathbb{T}$ *has a unique fixed point* x_0, *and for each* $x \in \mathbb{E}$, *we have*

$$\mathrm{d}(\mathbb{T}^k(x), x_0) \leq \mathbb{B}^k(\mathbb{I} - \mathbb{B})^{-1}\mathrm{d}(x, \mathbb{T}(x)) \quad \textit{for all } k \in \mathbb{N}.$$

Theorem 4.6 *Let the following assumptions hold:*

(H_1') $\mathbb{K}_1, \mathbb{K}_2 : \Sigma \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *are continuous functions.*

(H_2') *There exist a positive cantants* c_i, d_i, $i = 1, 2$, *such that*

$$|\mathbb{K}_i(\ell, \mathfrak{m}_1, \mathfrak{m}_2) - \mathbb{K}_i(\ell, \mathfrak{n}_1, \mathfrak{n}_1)| \leq c_i|\mathfrak{m}_1 - \mathfrak{n}_1| + d_i|\mathfrak{m}_2 - \mathfrak{n}_1|, \ \ell \in \Sigma \textit{ and } \mathfrak{m}_1, \mathfrak{n}_1, \mathfrak{m}_2, \mathfrak{n}_2 \in \mathbb{R},$$

then the Φ*–Caputo FMTDS* (1.2)–(1.3) *has a unique solution.*

Proof Consider the Banach space $\mathfrak{C} := C(\Sigma, \mathbb{R})$ equipped with the Φ-fractional Bielecki-type norm defined in (3.8) with specific value of $\mu = 1$. As results, the product space $\mathbb{Y} := \mathfrak{C} \times \mathfrak{C}$ is a generalized Banach space, endowed with the Bielecki vector-valued norm

$$\|(\mathfrak{m}_1, \mathfrak{m}_2)\|_{\mathbb{Y},\mathfrak{B},1} = \begin{pmatrix} \|\mathfrak{m}_1\|_{\mathfrak{C},\mathfrak{B},1} \\ \|\mathfrak{m}_2\|_{\mathfrak{C},\mathfrak{B},1} \end{pmatrix}.$$

Next, in light of Lemma 3.7 we define an operator $\mathfrak{F}=\big(\mathfrak{F}_1,\mathfrak{F}_2\big)\colon \mathbb{Y}\to\mathbb{Y}$ as:

$$\mathfrak{F}(\mathfrak{m}_1,\mathfrak{m}_2)=\big(\mathfrak{F}_1(\mathfrak{m}_1,\mathfrak{m}_2),\mathfrak{F}_2(\mathfrak{m}_1,\mathfrak{m}_2)\big), \tag{4.1}$$

where

$$\begin{aligned}\mathfrak{F}_i(\mathfrak{m}_1,\mathfrak{m}_2)(\ell)=&\theta_i+\int_u^\ell \Phi'(\eta)(\Phi(\ell)-\Phi(\eta))^{\mu_i-1}\mathbb{E}_{\mu_i-\kappa_i,\mu_i}\big(-\varrho_i(\Phi(\ell)-\Phi(\eta))^{\mu_i-\kappa_i}\big)\\&\times\mathbb{K}_i(\eta,\mathfrak{m}_1(\eta),\mathfrak{m}_2(\eta))\mathrm{d}\eta,\quad i=1,2.\end{aligned} \tag{4.2}$$

It should be noted that $\mathfrak{F}$ is well-defined since both $\mathbb{K}_1$ and $\mathbb{K}_2$ are continuous. Now, we apply Perov's fixed point theorem to prove that $\mathfrak{F}$ has a unique fixed point. In this moment, we must show that $\mathfrak{F}$ is a contraction mapping on $\mathbb{Y}$ with the Bielecki vector-valued norm. Note that by definition of operator $\mathfrak{F}$, for any $(\mathfrak{m}_1,\mathfrak{m}_2),(\bar{\mathfrak{m}}_1,\bar{\mathfrak{m}}_2)\in\mathbb{Y}$ and $\ell\in\Sigma$, using (H_2'), and Lemmas 2.2, 2.11 we can get

$$\begin{aligned}&\big|\mathfrak{F}_i(\mathfrak{m}_1,\mathfrak{m}_2)(\ell)-\mathfrak{F}_i(\bar{\mathfrak{m}}_1,\bar{\mathfrak{m}}_2)(\ell)\big|\\ \le&\big(c_i\|\mathfrak{m}_1-\bar{\mathfrak{m}}_1\|_{\mathfrak{C},\mathfrak{B},1}+d_i\|\mathfrak{m}_2-\bar{\mathfrak{m}}_2\|_{\mathfrak{C},\mathfrak{B},1}\big)\int_u^\ell\frac{\Phi'(\eta)(\Phi(\ell)-\Phi(\eta))^{\mu_i-1}}{\Gamma(\mu_i)}e^{\beta(\Phi(\eta)-\Phi(u))}\mathrm{d}\eta\\ \le&\frac{e^{\beta(\Phi(\ell)-\Phi(u))}}{\beta^{\mu_i}}\big(c_i\|\mathfrak{m}_1-\bar{\mathfrak{m}}_1\|_{\mathfrak{C},\mathfrak{B},1}+d_i\|\mathfrak{m}_2-\bar{\mathfrak{m}}_2\|_{\mathfrak{C},\mathfrak{B},1}\big).\end{aligned}$$

Hence

$$\big\|\mathfrak{F}_i(\mathfrak{m}_1,\mathfrak{m}_2)-\mathfrak{F}_i(\bar{\mathfrak{m}}_1,\bar{\mathfrak{m}}_2)\big\|_{\mathfrak{C},\mathfrak{B},1}\le\frac{1}{\beta^{\mu_i}}\big(c_i\|\mathfrak{m}_1-\bar{\mathfrak{m}}_1\|_{\mathfrak{C},\mathfrak{B},1}+d_i\|\mathfrak{m}_2-\bar{\mathfrak{m}}_2\|_{\mathfrak{C},\mathfrak{B},1}\big).$$

This leads to

$$\big\|\mathfrak{F}(\mathfrak{m}_1,\mathfrak{m}_2)-\mathfrak{F}(\bar{\mathfrak{m}}_1,\bar{\mathfrak{m}}_2)\big\|_{\mathfrak{C},\mathfrak{B},1}\le\mathbb{M}_\beta\|(\mathfrak{m}_1,\mathfrak{m}_2)-(\bar{\mathfrak{m}}_1,\bar{\mathfrak{m}}_2)\|_{\mathfrak{C},\mathfrak{B},1},$$

where

$$\mathbb{M}_\beta=\begin{pmatrix}\frac{c_1}{\beta^{\mu_1}} & \frac{d_1}{\beta^{\mu_1}}\\ \frac{c_2}{\beta^{\mu_2}} & \frac{d_2}{\beta^{\mu_2}}\end{pmatrix}. \tag{4.3}$$

It is easy to see that the matrix $\mathbb{M}_\beta$ converges to zero for β large enough. Thus, our conclusion follows from Perov's fixed point theorem.

Now we close this section by studying the UH stability of solutions of the proposed Φ–Caputo FMTDS (1.2)–(1.3).

For some $\varepsilon_1,\varepsilon_2>0$, we consider the following inequalities:

$$\begin{cases}|{}^c\mathbb{D}_{u^+}^{\mu_1;\Phi}\tilde{\mathfrak{m}}_1(\ell)+\varrho_1\,{}^c\mathbb{D}_{u^+}^{\kappa_1;\Phi}\tilde{\mathfrak{m}}_1(\ell)-\mathbb{K}_1(\ell,\tilde{\mathfrak{m}}_1(\ell),\tilde{\mathfrak{m}}_2(\ell))|\le\varepsilon_1,\\ |{}^c\mathbb{D}_{u^+}^{\mu_2;\Phi}\tilde{\mathfrak{m}}_2(\ell)+\varrho_2\,{}^c\mathbb{D}_{u^+}^{\kappa_2;\Phi}\tilde{\mathfrak{m}}_2(\ell)-\mathbb{K}_2(\ell,\tilde{\mathfrak{m}}_1(\ell),\tilde{\mathfrak{m}}_2(\ell))|\le\varepsilon_2.\end{cases}\quad \ell\in\Sigma, \tag{4.4}$$

Motivated by the work of Urs [24] we give the following definition

Definition 4.7 ([24]) *The Φ–Caputo FMTDS (1.2)–(1.3) is UH stable if we can find a positive constants $\nu_i, i=\overline{1,4}$ such that for every $\varepsilon_1,\varepsilon_1>0$ and for each solution $(\tilde{\mathfrak{m}}_1,\tilde{\mathfrak{m}}_2)\in\mathbb{Y}$ of the inequalities (4.4), there exists a solution $(\mathfrak{m}_1,\mathfrak{m}_2)\in\mathbb{Y}$ of the Φ–Caputo FMTDS (1.2)–(1.3) with*

$$\begin{cases}|\tilde{\mathfrak{m}}_1(\ell)-\mathfrak{m}_1(\ell)|\le\nu_1\varepsilon_1+\nu_2\varepsilon_2,\\ |\tilde{\mathfrak{m}}_2(\ell)-\mathfrak{m}_2(\ell)|\le\nu_3\varepsilon_1+\nu_4\varepsilon_2.\end{cases}\quad\ell\in\Sigma,$$

Remark 4.8 *A function* $(\mathfrak{m}_1, \mathfrak{m}_2) \in \mathbb{Y}$ *is a solution of the inequalities* (4.4) *if and only if there exist a functions* $\mathfrak{z}_1, \mathfrak{z}_2 \in C(\Sigma, \mathbb{R})$ *(which depend upon* $\tilde{\mathfrak{m}}_1$ *and* $\tilde{\mathfrak{m}}_2$ *respectively), such that*

(i) $|\mathfrak{z}_1(\ell)| \leq \varepsilon_1, \quad |\mathfrak{z}_2(\ell)| \leq \varepsilon_2, \quad \ell \in \Sigma;$

(ii) and

$$\begin{cases} {}^c\mathbb{D}_{u^+}^{\mu_1;\Phi}\tilde{\mathfrak{m}}_1(\ell) + \varrho_1\, {}^c\mathbb{D}_{u^+}^{\kappa_1;\Phi}\tilde{\mathfrak{m}}_1(\ell) = \mathbb{K}_1(\ell, \tilde{\mathfrak{m}}_1(\ell), \tilde{\mathfrak{m}}_2(\ell)) + \mathfrak{z}_1(\ell), \\ {}^c\mathbb{D}_{u^+}^{\mu_2;\Phi}\tilde{\mathfrak{m}}_2(\ell) + \varrho_2\, {}^c\mathbb{D}_{u^+}^{\kappa_2;\Phi}\tilde{\mathfrak{m}}_2(\ell) = \mathbb{K}_2(\ell, \tilde{\mathfrak{m}}_1(\ell), \tilde{\mathfrak{m}}_2(\ell)) + \mathfrak{z}_2(\ell). \end{cases} \quad \ell \in \Sigma,$$

.

Lemma 4.9 *Let* $(\tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2) \in \mathbb{Y}$ *be a solution of the inequalities* (4.4)*, then the following inequalities will be satisfied:*

$$\left|\tilde{\mathfrak{m}}_i(\ell) - \mathfrak{F}_i(\tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2)(\ell)\right| \leq \varepsilon_i \mathbb{A}_{\mu_i,\Phi}, \quad i = 1, 2.$$

where $\mathfrak{F}_i, i = 1, 2$ *are defined by* (4.2)*, and* $\mathbb{A}_{\mu_i,\Phi} = \frac{(\Phi(v)-\Phi(u))^{\mu_i}}{\Gamma(\mu_i+1)}$.

Proof By Remark 4.8 (ii), we have

$$\begin{cases} {}^c\mathbb{D}_{u^+}^{\mu_1;\Phi}\tilde{\mathfrak{m}}_1(\ell) + \varrho_1\, {}^c\mathbb{D}_{u^+}^{\kappa_1;\Phi}\tilde{\mathfrak{m}}_1(\ell) = \mathbb{K}_1(\ell, \tilde{\mathfrak{m}}_1(\ell), \tilde{\mathfrak{m}}_2(\ell)) + \mathfrak{z}_1(\ell), \\ {}^c\mathbb{D}_{u^+}^{\mu_2;\Phi}\tilde{\mathfrak{m}}_2(\ell) + \varrho_2\, {}^c\mathbb{D}_{u^+}^{\kappa_2;\Phi}\tilde{\mathfrak{m}}_2(\ell) = \mathbb{K}_2(\ell, \tilde{\mathfrak{m}}_1(\ell), \tilde{\mathfrak{m}}_2(\ell)) + \mathfrak{z}_2(\ell), \end{cases} \quad \ell \in \Sigma, \tag{4.5}$$

with the following initial conditions

$$\begin{cases} \tilde{\mathfrak{m}}_1(u) = \theta_1, \\ \tilde{\mathfrak{m}}_2(u) = \theta_2. \end{cases} \tag{4.6}$$

Thanks to Lemma 3.7, the integral representation of (4.5)–(4.6) is expressed as

$$\begin{aligned} \tilde{\mathfrak{m}}_i(\ell) =&\theta_i + \int_u^\ell \Phi'(\eta)(\Phi(\ell) - \Phi(\eta))^{\mu_i-1}\mathbb{E}_{\mu_i-\kappa_i,\mu_i}\big(-\varrho_i(\Phi(\ell) - \Phi(\eta))^{\mu_i-\kappa_i}\big) \\ &\times \big(\mathbb{K}_i(\eta, \tilde{\mathfrak{m}}_1(\eta), \tilde{\mathfrak{m}}_2(\eta)) + \mathfrak{z}_i(\eta)\big)\mathrm{d}\eta, \quad i = 1, 2. \end{aligned} \tag{4.7}$$

It follows from (4.7), together with Remark 4.8 (i), and Lemma 2.2 that

$$\left|\tilde{\mathfrak{m}}_i(\ell) - \mathfrak{F}_i(\tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2)(\ell)\right| \leq \int_u^\ell \frac{\Phi'(\eta)(\Phi(\ell) - \Phi(\eta))^{\mu_i-1}}{\Gamma(\mu_i)}|\mathfrak{z}_i(\eta)|\mathrm{d}\ell \leq \varepsilon_i \mathbb{A}_{\mu_i,\Phi}, \quad i = 1, 2.$$

□

Theorem 4.10 *Let the assumptions of Theorem 4.6 be fulfilled. Then the* Φ*–Caputo FMTDS* (1.2)–(1.3) *is UH stable with respect to the Bielecki's norm.*

Proof Let $(\mathfrak{m}_1, \mathfrak{m}_2)$ be the be a unique solution of the Φ–Caputo FMTDS (1.2)–(1.3) and $(\tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2)$ be any solution satisfying (4.4), then by (H_2') and Lemmas 2.11, 4.9 we can get

$$\begin{aligned} \left|\tilde{\mathfrak{m}}_i(\ell) - \mathfrak{m}_i(\ell)\right| &\leq \left|\tilde{\mathfrak{m}}_i(\ell) - \mathfrak{F}_i(\tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2)(\ell)\right| + \left|\mathfrak{F}_i(\tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2)(\ell) - \mathfrak{F}_i(\mathfrak{m}_1, \mathfrak{m}_2)(\ell)\right| \\ &\leq \varepsilon_i \mathbb{A}_{\mu_i,\Phi} + \frac{1}{\beta^{\mu_i}}\big(c_i\|\mathfrak{m}_1 - \tilde{\mathfrak{m}}_1\|_{\mathfrak{C},\mathfrak{B},1} + d_i\|\mathfrak{m}_2 - \tilde{\mathfrak{m}}_2\|_{\mathfrak{C},\mathfrak{B},1}\big), \ i = 1, 2. \end{aligned}$$

Thus

$$\|\tilde{\mathfrak{m}}_i - \mathfrak{m}_i\|_{\mathfrak{C},\mathfrak{B},1} \leq \varepsilon_i \mathbb{A}_{\mu_i,\Phi} + \frac{1}{\beta^{\mu_i}}\big(c_i\|\mathfrak{m}_1 - \bar{\mathfrak{m}}_1\|_{\mathfrak{C},\mathfrak{B},1} + d_i\|\mathfrak{m}_2 - \bar{\mathfrak{m}}_2\|_{\mathfrak{C},\mathfrak{B},1}\big), \ i = 1, 2.$$

The matrix representation of the aforementioned inequalities is as follows:

$$(\mathbb{I} - \mathbb{M}_\beta)\begin{pmatrix} \|\tilde{\mathfrak{m}}_1 - \mathfrak{m}_1\|_{\mathfrak{C},\mathfrak{B},1} \\ \|\tilde{\mathfrak{m}}_2 - \mathfrak{m}_2\|_{\mathfrak{C},\mathfrak{B},1} \end{pmatrix} \leq \begin{pmatrix} \mathbb{A}_{\mu_1,\Phi}\varepsilon_1 \\ \mathbb{A}_{\mu_2,\Phi}\varepsilon_2 \end{pmatrix}. \tag{4.8}$$

Where $\mathbb{M}_\beta$ is the matrix given by (4.3). Of course, the matrix $\mathbb{M}_\beta$ is converges to zero for sufficiently large value of β. So, by Theorem 4.3 we conclude that the matrix $(\mathbb{I} - \mathbb{M}_\beta)$ is nonsingular and $(\mathbb{I} - \mathbb{M}_\beta)^{-1}$ has nonnegative elements. Therefore, (4.8) is equivalent to

$$\begin{pmatrix} \|\tilde{\mathfrak{m}}_1 - \mathfrak{m}_1\|_{\mathfrak{C},\mathfrak{B},1} \\ \|\tilde{\mathfrak{m}}_2 - \mathfrak{m}_2\|_{\mathfrak{C},\mathfrak{B},1} \end{pmatrix} \leq (\mathbb{I} - \mathbb{M}_\beta)^{-1}\begin{pmatrix} \mathbb{A}_{\mu_1,\Phi}\varepsilon_1 \\ \mathbb{A}_{\mu_2,\Phi}\varepsilon_2 \end{pmatrix},$$

which yields that

$$\begin{cases} \|\tilde{\mathfrak{m}}_1 - \mathfrak{m}_1\|_{\mathfrak{C},\mathfrak{B},1} \leq \sigma_1 \mathbb{A}_{\mu_1,\Phi}\varepsilon_1 + \sigma_2 \mathbb{A}_{\mu_2,\Phi}\varepsilon_2, \\ \|\tilde{\mathfrak{m}}_2 - \mathfrak{m}_2\|_{\mathfrak{C},\mathfrak{B},1} \leq \sigma_3 \mathbb{A}_{\mu_1,\Phi}\varepsilon_1 + \sigma_4 \mathbb{A}_{\mu_2,\Phi}\varepsilon_2, \end{cases}$$

where $\sigma_i, i = \overline{1,4}$ are the elements of the matrix $(\mathbb{I} - \mathbb{M}_\beta)^{-1}$.

Consequently, the Φ–Caputo FMTDS (1.2)–(1.3) is UH stable with respect to Bielecki's norm $\|\cdot\|_{\mathfrak{C},\mathfrak{B},1}$. □

Remark 4.11 *Importing the same logic as in Theorem 4.10. One can easily show that the Φ–Caputo FMTDS (1.2)–(1.3) is generalized UH, UHR and generalized UHR stable with respect to Bielecki's norm $\|\cdot\|_{\mathfrak{C},\mathfrak{B},1}$.*

□

5. Applications

In this section, we present two examples where we apply Theorems 3.8 and 4.6 to some particular cases.

Example 5.1 *Let us consider problem (1.1) with specific data:*

$$\mu = 0.5,\ \kappa = 0.4,\ \varrho = 0.5,\ \theta = 1. \tag{5.1}$$

In order to illustrate Theorem 3.8, we take

$$\mathbb{H}(\ell, \mathfrak{m}(\ell)) = e^{\ell^2}\left(1 + \frac{|\mathfrak{m}(\ell)|}{1 + |\mathfrak{m}(\ell)|}\right), \tag{5.2}$$

in (1.1). Clearly, the function $\mathbb{H}$ is continuous. Moreover, For any $\mathfrak{m}, \mathfrak{n} \in \mathbb{R}$ and $\ell \in [u, v]$ we have

$$|\mathbb{H}(\ell, \mathfrak{m}) - \mathbb{H}(\ell, \mathfrak{n})| \leq e^{v^2}|\mathfrak{m} - \mathfrak{n}|.$$

Hence the condition (H2) holds with $\mathbb{L}_{\mathbb{H}} = e^2$. Moreover, by letting $\zeta(\ell) = \Phi(\ell) - \Phi(u)$, we have

$$\mathbb{I}_{u^+}^{\mu;\Phi}\zeta(\ell) = \frac{(\Phi(\ell) - \Phi(u))^{1.5}}{\Gamma(2.5)} \leq 2\sqrt{\frac{\Phi(v) - \Phi(u)}{\pi}} := \gamma_\zeta \zeta(\ell).$$

*So, condition (*H3) is satisfied with $\zeta(\ell) = \Phi(\ell) - \Phi(u)$ and $\gamma_\zeta = 2\sqrt{\frac{\Phi(v)-\Phi(u)}{\pi}}$. It follows from Theorem 3.8 that the problem (1.1) with the data (5.1) and (5.2) has a unique solution on $[u, v]$ and is $\mathbb{E}_\mu$-UHR stable.

Remark 5.2 *It is worth noting that in the previous example β can be determined according to the assumptions of Theorem 3.8. for example, we can choose $\beta > e^v$.*

Example 5.3 *To illustrate Theorem 4.6, consider the Φ–Caputo FMTDS (1.2)–(1.3) with specific values*

$$\begin{cases} \mu_1 = 0.6, \mu_2 = 0.9, \kappa_1 = 0.5, \kappa_2 = 0.7, \varrho_1 = 1.8, \varrho_2 = 2, \theta_1 = 0.5, \theta_2 = 1.5, \\ \mathbb{K}_1(\ell, \mathfrak{m}_1(\ell), \mathfrak{m}_2(\ell)) = \ell^2 \arctan(|\mathfrak{m}_1(\ell)|) + e^\ell \sin(\mathfrak{m}_2(\ell)) + 1, \\ \mathbb{K}_1(\ell, \mathfrak{m}_1(\ell), \mathfrak{m}_2(\ell)) = \frac{\sin \ell}{2}\left(\mathfrak{m}_1(\ell) + \sqrt{1 + \mathfrak{m}_1^2(\ell)}\right) + \frac{\ell}{1+|\mathfrak{m}_1(\ell)|+|\mathfrak{m}_2(\ell)|}. \end{cases} \tag{5.3}$$

It is clear that the functions $\mathbb{K}_1$ and $\mathbb{K}_2$ are continuous. Ferthmore, for all $\ell \in \Sigma$ and $\mathfrak{m}_1, \mathfrak{n}_1, \mathfrak{m}_2, \mathfrak{n}_2 \in \mathbb{R}$ we have

$$|\mathbb{K}_i(\ell, \mathfrak{m}_1, \mathfrak{m}_2) - \mathbb{K}_i(\ell, \mathfrak{n}_1, \mathfrak{n}_1)| \le c_i|\mathfrak{m}_1 - \mathfrak{n}_1| + d_i|\mathfrak{m}_2 - \mathfrak{n}_1|, \ i = 1, 2,$$

where

$$c_1 = v^2, \ c_2 = 1, \ d_1 = e^v, \ d_2 = v.$$

Hence, all conditions of Theorem 4.6 are satisfied and consequently the Φ–Caputo FMTDS (1.2)–(1.3) with the given values (5.3) has a unique solutions on $[u, v]$. Moreover, Theorem 4.10 garuntates that the Φ–Caputo FMTDS (1.2)–(1.3) is UH with respect to Bielecki's norm $\|\cdot\|_{\mathfrak{C},\mathfrak{B},1}$. Furthermore, the matrix $\mathbb{M}_\beta$ given by (4.3) has the following form

$$\mathbb{M}_\beta = \begin{pmatrix} \frac{v^2}{\beta^{0.6}} & \frac{e^v}{\beta^{0.6}} \\ \frac{1}{\beta^{0.9}} & \frac{v}{\beta^{0.9}} \end{pmatrix},$$

which is convergent for β sufficiently large as we wish.

Acknowledgment

M.F. was partially supported by the Slovak Research and Development Agency under the contract No. APVV-18-0308, and the Slovak Grant Agency VEGA No. 1/0358/20 and No. 2/0127/20.

References

[1] Abbas S, Benchohra M, Samet B, Zhou Y. Coupled implicit Caputo fractional q-difference systems. Advances in Difference Equations 2019; 2019: 527.

[2] Almeida R. A Caputo fractional derivative of a function with respect to another function. Communications in Nonlinear Science and Numerical Simulation 2017; 44: 460-481.

[3] Almeida R, Bastos NRO, Monteiro MTT. Modeling some real phenomena by fractional differential equations. Mathematical Methods in the Applied Science 2016; 39 (16): 4846-4855.

[4] Baitiche Z, Derbazi C, Matar, MM. Ulam-stability results for a new form of nonlinear fractional Langevin differential equations involving two fractional orders in the ψ–Caputo sense. Applicable Analysis 2021; 2021. doi: 10.1080/00036811.2021.1873300

[5] Derbazi C, Baitiche Z. Coupled systems of ψ-Caputo differential equations with initial conditions in Banach spaces. Mediterranean Journal of Mathematics 2020; 17 (5): 1-13.

[6] Derbazi C, Baitiche Z, Benchohra M, N'Guérékata G. Existence, uniqueness, and Mittag–Leffler–Ulam stability results for Cauchy problem involving ψ-Caputo derivative in Banach and Fréchet spaces. International Journal of Differential Equations 2020; 2020. doi: 10.1155/2020/6383916

[7] Gorenflo R, Kilbas AA, Mainardi F, Rogosin SV. Mittag–Leffler Functions, Related Topics and Applications. Springer, New York 2014.

[8] Hilfer R. Applications of Fractional Calculus in Physics. World Scientific, Singapore, 2000.

[9] Jarad F, Abdeljawad T. Generalized fractional derivatives and Laplace transform. Discrete and Continuous Dynamical Systems. Series S 2020; 13 (3): 709-722.

[10] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Sudies, Vol. 204. Amsterdam, Netherlands: Elsevier Science B.V., 2006.

[11] Liu K, Fečkan M, Wang J. Hyers–Ulam stability and existence of solutions to the generalized Liouville–Caputo fractional differential equations. Symmetry 2020; 2(6): 955. doi: 10.3390/sym12060955.

[12] Peng S, Wang J. Existence and Ulam-Hyers stability of ODEs involving two Caputo fractional derivatives. Electronic Journal of Qualitative Theory of Differential Equations. 2015; 2015: 52.

[13] Petre IR, Petruşel A. Krasnoselskii's theorem in generalized Banach spaces and applications. Electronic Journal of Qualitative Theory of Differential Equations 2012; 2012: 85.

[14] Perov AI. On the Cauchy problem for a system of ordinary differential equations. Pviblizhen. Met. Reshen. Differ. Uvavn. 1964; 2: 115-134. (in Russian).

[15] Precup R. The role of matrices that are convergent to zero in the study of semilinear operator systems. Mathematical and Computer Modelling 2009; 49: 703-708.

[16] Precup R, Viorel A. Existence results for systems of nonlinear evolution equations. International Journal of Pure and Applied Mathematics 2008; 47(2): 199-206.

[17] Podlubny I. Fractional Differential Equations. Mathematics in Science and Engineering. New York, NY, USA: Academic Press, 1999.

[18] Rus IA. Generalized contractions and applications. Cluj University Press, Cluj-Napoca, 2001.

[19] Rus IA. Ulam stability of ordinary differential equations. Studia. Universitatis Babeş-Bolyai. Mathematica 2009; 54 (4): 125-133.

[20] Vanterler da Costa Sousa J. Existence results and continuity dependence of solutions for fractional equations. Differential Equations & Applications 2020; 12 (4): 377-396.

[21] Vanterler da C. Sousa J, Capelas de Oliveira E. On the ψ-Hilfer fractional derivative. Communications in Nonlinear Science and Numerical Simulation 2018; 60: 72-91.

[22] Vanterler da C. Sousa, J, Capelas de Oliveira E. A Gronwall inequality and the Cauchy-type problem by means of Φ–Hilfer operator. Differential Equations & Applications 2019; 11(1): 87-106.

[23] Vanterler da C. Sousa J, Capelas de Oliveira E. Existence, uniqueness, estimation and continuous dependence of the solutions of a nonlinear integral and an integrodifferential equations of fractional order. 2018; arXiv:1806.01441.

[24] Urs C. Coupled fixed point theorems and applications to periodic boundary value problems. Miskolc Mathematical Notes 2013; 14 (1): 323-333.

[25] Varga RS. Matrix iterative analysis, second revised and expanded edition, Springer Series in Computational Mathematics, 27, Springer-Verlag, Berlin, 2000.

[26] Wang J, Fečkan M, Zhou Y. Presentation of solutions of impulsive fractional Langevin equations and existence results. The European Physical Journal Special Topics 2013; 222 (8): 1857-1874.

[27] Wang J, Lv L, Zhou Y. Ulam stability and data dependence for fractional differential equations with Caputo derivative. Electronic Journal of Qualitative Theory of Differential Equations 2011; 2011: 63.

[28] Wang J, Li X. E_α-Ulam type stability of fractional order ordinary differential equations. Journal of Applied Mathematics and Computing 2014; 45: 449-459.

[29] Wang J, Zhang Y. Ulam-Hyers-Mittag-Leffler stability of fractional-order delay differential equations. Optimization 2014; 63 (8): 1181-1190.