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AHMAD AL SALMAN

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# Singular integral operators and maximal functions with Hardy space kernels

**Ahmad AL-SALMAN**[1,2,*]
[1]Department of Mathematics, Sultan Qaboos University, Sultanate of Oman
[2]Department of Mathematics, Yarmouk University, Irbid-Jordan

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**Abstract:** In this paper, we study singular integrals along compound curves with Hardy space kernels. We introduce a class of bidirectional generalized Hardy Littlewood maximal functions. We prove that the considered singular integrals and the maximal functions are bounded on $L^p, 1 < p < \infty$ provided that the compound curves are determined by generalized polynomials and convex increasing functions. The obtained results offer $L^p$ estimates that are not only new but also they generalize as well as improve previously known results.

**Key words:** Singular integrals, Hardy space, compound curves, Hardy Littlewood maximal function, convex functions

## 1. Introduction and statement of results

Let $\mathbb{R}^n$, $n \geq 2$, be the $n$-dimensional Euclidean space and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. For non zero $y \in \mathbb{R}^n$, we let $y' = |y|^{-1} y$. Suppose that $\Omega \in L^1(\mathbb{S}^{n-1})$ is a homogeneous functions of degree zero on $\mathbb{R}^n$ and satisfies the cancellation condition

$$\int_{\mathbb{S}^{n-1}} \Omega(y')d\sigma(y') = 0. \tag{1.1}$$

In 1979, Fefferman [12] introduced the following class of singular integral operators

$$\mathbf{T}_{\Omega,h} f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y) \frac{h(|y|)\Omega(y')}{|y|^n} dy, \tag{1.2}$$

where $h : \mathbb{R}_+ \to \mathbb{R}$ is a suitable measurable function. It is clear that if $h(t) = 1$, then the operator $\mathbf{T}_{\Omega,h}$ reduces to the classical Calderón–Zygmund singular integral operator, which will be denoted by $\mathbf{T}_\Omega$. In [6], Calderón and Zygmund showed that $\mathbf{T}_\Omega$ is bounded on $L^p$ for all $p \in (1,\infty)$ provided that $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$. Moreover, they showed that the condition $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$ is nearly optimal in the sense that the $L^p$ boundedness of $T_\Omega$ may not hold if $\Omega \in L(\log^+ L)^{1-\varepsilon}(\mathbb{S}^{n-1}) \backslash L\log^+ L(\mathbb{S}^{n-1})$ for some $\varepsilon > 0$. It was proved independently by Connett [7] and Ricci-Weiss [17] that the operator $T_\Omega$ is bounded on $L^p$ for all $p \in (1,\infty)$ if $\Omega \in H^1(\mathbb{S}^{n-1})$, the Hardy space in the sense of Coifman and Weiss [8]. Fefferman [12] proved that $\mathbf{T}_{\Omega,h}$ is bounded on $L^p$ for

*Correspondence: alsalman@squ.edu.om
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all $1 < p < \infty$ provided that $\Omega \in Lip_{\alpha}(\mathbb{S}^{n-1})$ for some $\alpha > 0$ and that $h \in L^{\infty}(\mathbb{R}_+)$. Here, $\mathbb{R}_+ = (0, \infty)$. In 1986, Namazi [15] showed that Fefferman's result still holds under the weaker condition $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q > 1$. Subsequently, the condition $h \in L^{\infty}(\mathbb{R}_+)$ was very much relaxed by Duoandikoetxea and Rubio de Francia [9]. In fact, they showed that the operator $\mathbf{T}_{\Omega,h}$ is bounded on $L^p$ for all $1 < p < \infty$ provided that $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q > 1$ and $h$ satisfies the condition

$$\|h\|_{\Delta_2} = \sup_{j \in \mathbb{Z}} \left( \int_{2^j}^{2^{j+1}} |h(t)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} < \infty. \tag{1.3}$$

In 1997, Fan and Pan [11] improved Duoandikoetxea and Rubio de Francia's result by showing that the operator $\mathbf{T}_{\Omega,h}$ is bounded on $L^p$ for all $1 < p < \infty$ provided that $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q > 1$ and $h$ lies in the class $\Delta_{\gamma}(\mathbb{R}_+)$ for some $\gamma > 1$ where $\Delta_{\gamma}(\mathbb{R}_+)$ is the class of all measurable functions $h : \mathbb{R}_+ \to \mathbb{R}$ satisfying (1.3) with 2 replaced by $\gamma$. It should be noted here that

$$L^{\infty}(\mathbb{R}_+) \subset \bigcap_{\gamma > 1} \Delta_{\gamma}(\mathbb{R}_+)$$

and that

$$\Delta_{\gamma_2}(\mathbb{R}_+) \subset \Delta_{\gamma_1}(\mathbb{R}_+) \text{ whenever } \gamma_1 \leq \gamma_2.$$

In [4], Al-Salman and Pan showed that the condition $\Omega \in L^q(\mathbb{S}^{n-1})$ can be replaced by the weaker condition $\Omega \in L \log L(\mathbb{S}^{n-1})$. Here, we remark that

$$Lip_{\alpha}(\mathbb{S}^{n-1}) \subsetneqq L^q(\mathbb{S}^{n-1}) \subsetneqq L(\log^+ L)(\mathbb{S}^{n-1}) \subsetneqq H^1(\mathbb{S}^{n-1}) \subsetneqq L^1(\mathbb{S}^{n-1})$$

for all $\alpha > 0$ and $q > 1$.

In this paper, we consider singular integrals along subvarities determined by compound curves. Let $\varphi : [0, \infty) \to \mathbb{R}$ be a $\mathcal{C}^1([0, \infty))$ function that satisfies $\varphi(0) = 0$. For a suitable function $\Gamma : [0, \infty) \to \mathbb{R}$, we consider the singular integral operator

$$\mathbf{T}_{\Omega,\Gamma,\varphi,h} f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - \Gamma(\varphi(|y|))y') \frac{h(|y|)\Omega(y')}{|y|^n} dy. \tag{1.4}$$

It is clear that if $\varphi(t) = \Gamma(t) := I(t) = t$, then the operator $\mathbf{T}_{\Omega,\Gamma,\varphi,h}$ reduces to the classical operator $\mathbf{T}_{\Omega,h}$ in (1.2). In the following few remarks, we shed some light on the history behind the consideration of the class of operators $\mathbf{T}_{\Omega,\Gamma,\varphi,h}$ in (1.4):

(i) When $h \in L^{\infty}(\mathbb{R}_+)$, $\varphi(t) = t$, and $\Gamma$ is a real valued polynomial, Al-Hasan and Fan [1] proved that the corresponding special operator

$$\mathbf{T}_{\Omega,\Gamma,h} f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - \Gamma(|y|)y') \frac{h(|y|)\Omega(y')}{|y|^n} dy. \tag{1.5}$$

is bounded on $L^p$ for all $p \in (1, \infty)$ if $\Omega \in H^1(\mathbb{S}^{n-1})$. Subsequently, when $h(t) = 1$ and $\Gamma(t)$ is convex increasing, Al-Salman (1.5) showed that the corresponding operator $\mathbf{T}_{\Omega,\Gamma} = \mathbf{T}_{\Omega,\Gamma,1}$ is bounded on $L^p$ for all $p \in (1, \infty)$ provided that $\Omega \in H^1(\mathbb{S}^{n-1})$ [5].

(ii) Let $\mathbf{T}_{\Omega,\Gamma}$ be the operator given by (1.4) with $\varphi(t) = t$ and $h(t) = 1$, i.e.

$$\mathbf{T}_{\Omega,\Gamma} f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - \Gamma(|y|)y') \frac{\Omega(y')}{|y|^n} dy.$$

In [3], Al-Salman and Al-Qassem generalized the $L^p$ boundedness result in [5] by proving that the operator $\mathbf{T}_{\Omega,\Gamma}$ is bounded on $L^p(\mathbb{R}^n)$ for every $1 < p < \infty$ provide that $\Omega \in H^1(\mathbb{S}^{n-1})$ and $\Gamma$ is either convex increasing with $\Gamma(0) = 0$ or a generalized polynomial. A mapping $\Gamma : \mathbb{R}_+ \to \mathbb{R}$ ia a generalized polynomial if it has the form

$$\Gamma(t) = \mu_1 t^{d_1} + \cdots + \mu_l t^{d_l} \tag{1.6}$$

for some $l \in \mathbb{N}$, distinct positive real numbers $d_1, \ldots, d_l$, and real numbers $\mu_1, \ldots, \mu_l$. In the case of generalized polynomials, Al-Salman and Al-Qassem showed that the bound for the operator norm $\|\mathbf{T}_{\Omega,\Gamma}\|_{p,p}$ is independent of the coefficients $\mu_1, \ldots, \mu_l$. The problem whether the $L^p$ estimates still hold in the case of kernels that are rough in the radial direction was left open.

(iii) In the recent paper [14], Liu and Zhang considered the operator $\mathbf{T}_{\Omega,\Gamma,\varphi,h}$ for compound polynomial mappings. They proved the following $L^2(\mathbb{R}^n)$ result:

**Theorem 1.1.([14]).** *Let $\mathbf{T}_{\Omega,\Gamma,\varphi,h}$ be the operator given by (1.5). Let $\varphi$ be a nonnegative (or non-positive) $\mathcal{C}^1(\mathbb{R}_+)$ monotonic function that satisfies $\left|\frac{\varphi(t)}{t\varphi'(t)}\right| \leq C_\varphi$ where $C_\varphi$ is a constant that depends only on $\varphi$. If $\Gamma$ is a real valued polynomial, $\Omega \in H^1(\mathbb{S}^{n-1})$, and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma > 1$, then*

$$\|\mathbf{T}_{\Omega,\Gamma,\varphi,h} f\|_{L^2} \leq C \|h\|_{\Delta_\gamma} \|\Omega\|_{H^1} \|f\|_{L^2}$$

*where $C > 0$ is independent of $h, \gamma, \Omega, f$ and the coefficients of the polynomial $\Gamma$ but depends on $\varphi$ and $\deg(\Gamma)$.*

The question whether the operator $\mathbf{T}_{\Omega,\Gamma,\varphi,h}$ is bounded for some $p \neq 2$ was left open in [14].

In light of the above remarks, it is our aim in this paper to consider the general operator $\mathbf{T}_{\Omega,\Gamma,\varphi,h}$ and to seek answers to the above stated problems. We shall assume that the function $h$ to be in the class of functions $\Lambda_\gamma^\eta$ introduced by Sato [18] (see also Seegeer [19] and [21]). In fact, for $\eta, \gamma > 0$, we let $\Lambda_\gamma^\eta$ be the class of all measurable functions $h : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\|h\|_{\Lambda_\gamma^\eta} = \|h\|_{\Delta_\gamma} + \|h\|_{\Lambda^\eta} < \infty,$$

where

$$\|h\|_{\Lambda^\eta} = \sup_{t \in (0,1)} t^{-\eta} \omega(h,t),$$

and

$$\omega(h,t) = \sup_{|s| < \frac{tR}{2}} \int_R^{2R} |h(r-s) - h(r)| \frac{dr}{r}, t \in (0,1].$$

The supremum is taken over all $s$ and $R$ such that $|s| < tR/2$. Our main result is the following:

**Theorem 1.2.** *Let* $\mathbf{T}_{\Omega,\Gamma,\varphi,h}$ *be the operator given by (1.4). Let* $\Omega \in H^1(\mathbb{S}^{n-1})$ *be a homogeneous functions of degree zero on* $\mathbb{R}^n$ *and satisfies the cancellation condition (1.1). Suppose that*
*(i)* $h \in \Lambda_1^\eta$ *for some* $\eta > 0$;
*(ii)* $\Gamma : [0,\infty) \to \mathbb{R}$ *is a non-constant generalized polynomial of the form (1.6)*;
*(iii)* $\varphi$ *is a* $\mathcal{C}^2([0,\infty))$ *convex increasing function with* $\varphi(0) = 0$;
*Then*

$$\|\mathbf{T}_{\Omega,\Gamma,\varphi,h} f\|_{L^p} \leq C \|h\|_{\Lambda_1^\eta} \|\Omega\|_{H^1} \|f\|_{L^p}$$

for all $1 < p < \infty$ *where* $C > 0$ *is independent of* $h, \eta, \Omega, f$ *and the coefficients of the generalized polynomial* $\Gamma$ *but depends on the function* $\varphi$ *and the numbers* $d_1, \dots, d_l$.

It is clear that Theorem 1.2 is a substantial improvement of the corresponding result in [3]. Furthermore, it substantially generalizes the result in Theorem 1.2 as far as the range of the parameter $p$ is concerned.

The proof of Theorem 1.2 involves a key idea, which is characterized by introducing a new maximal function that is more general than the directional Hardy–Littlewood maximal function. We shall refer to this maximal function by the generalized bidirectional Hardy–Littlewood maximal function. For suitable mappings $\Gamma, \Lambda, \varphi : [0,\infty) \to \mathbb{R}$, a suitable measurable function $h : \mathbb{R}_+ \to \mathbb{R}$, and two vectors $z_1, z_1 \in \mathbb{R}^n$, consider the maximal function

$$H_{\Gamma,\Lambda,\varphi,h}^{(z_1,z_2)}(g)(x) = \sup_{j\in\mathbb{Z}} \int_{2^{j-1}}^{2^j} g\left(x - \Gamma(\varphi(t))z_1 - \Lambda(\varphi(t))z_2\right) \frac{h(t)}{t} dt. \tag{1.7}$$

It is clear that if $\Gamma(t) = \Lambda(t) = \varphi(t) := I(t) = t$ and $h(t) = 1$, then the operator $H_{\Gamma,\Lambda,\varphi,h}^{(z_1,z_2)}$ reduces to the classical directional Hardy Littlewood maximal function in the direction of the vector $z = z_1 + z_2$. The classical directional Hardy–Littlewood maximal function in the direction of a vector $z$ will be denoted by $H^{(z)} = H_{I,I,I,1}^{(\frac{z}{2},\frac{z}{2})}$. It is well known that the maximal function $H^{(z)}$ is bounded on $L^p$ for all $1 < p < \infty$ with $L^p$ bounds independent of the vector $z$. If the function $h$ is in $L^\infty(\mathbb{R}_+)$ and $\Gamma(t) = t$, then the special operator $H_{\varphi,h}^{(z)} = H_{I,\Lambda,\varphi,h}^{(z,0)}$ is dominated by the maximal function.

$$H_\varphi^{(z)}(g)(x) = \sup_{j\in\mathbb{Z}} \int_{2^{j-1}}^{2^j} g\left(x - \varphi(t)z\right) \frac{1}{t} dt. \tag{1.8}$$

The $L^p$ boundedness of the operator $H_\varphi^{(z)}$ has been discussed by several authors if the function $\varphi$ is of special form. In particular, if $\varphi$ is a polynomial mapping, then the $L^p$ boundedness of $H_\varphi^{(z)}$ follows by a well known result on page 477 of [20]. On the other hand, if $\varphi$ is convex increasing, then the $L^p$ boundedness of $H_\varphi^{(z)}$ was discussed in [2], [9], among others. However, for general functions $\Gamma, \varphi$, and $h$, the boundedness of the general operators $H_{\Gamma,\varphi,h}^{(z)} = H_{\Gamma,\Lambda,\varphi,h}^{(z,0)}$ is not known. Our main result concerning the maximal function $H_{\Gamma,\Lambda,\varphi,h}^{(z_1,z_2)}$ is the following:

**Theorem 1.3**. Let $\Gamma$ *and* $\Lambda$ *be generalized polynomials of the form in (ii) in Theorem 1.2. Let* $\varphi$ *and* $h$ *be as in the statement of Theorem 1.2. Let* $z_1, z_2 \in \mathbb{R}^n$ *and let* $H_{\Gamma,\Lambda,\varphi,h}^{(z_1,z_2)}$ *be given as in (1.7). Suppose that*

$h \in \Lambda_1^\eta (\eta > 0)$. *Then*

$$\left\| H^{(z_1,z_2)}_{\Gamma,\Lambda,\varphi,h}(g) \right\|_p \le C_p \left\| h \right\|_{\Lambda_1^\eta} \left\| g \right\|_p ,$$

$1 < p < \infty$ *with constant* $C_p$ *independent of* $h, \eta, g,\ z_1, z_2$, *and the coefficients of the generalized polynomials* $\Gamma$ *and* $\Lambda$, *but depends on the function* $\varphi$, *and the numbers representing the powers of the monomials involved in the representations of the generalized polynomials* $\Gamma$ *and* $\Lambda$.

As a consequence of Theorem 1.3, we obtain the following result:

**Corollary 1.4**. *Let* $\Omega \in L^1(\mathbb{S}^{n-1})$ *be a homogeneous functions of degree zero on* $\mathbb{R}^n$. Let $\Gamma$ *and* $\Lambda$ *be generalized polynomials of the form in (ii) in Theorem 1.2. Let* $\varphi$ *and* $h$ *be as in the statement of Theorem 1.2. For two mappings* $\Phi_1, \Phi_2 : \mathbb{S}^{n-1} \to \mathbb{R}^n$, *let* $M^{(\Phi_1,\Phi_2)}_{\Omega,\Gamma,\Lambda,\varphi,h}$ *be given by*

$$M^{(\Phi_1,\Phi_2)}_{\Omega,\Gamma,\Lambda,\varphi,h}(f)(x) = \sup_{j\in\mathbb{Z}} \int\limits_{2^{j-1}\le|y|<2^j} f\left(x - \Gamma(\varphi(t))\Phi_1(y') - \Lambda(\varphi(t))\Phi_2(y')\right)\Omega(y')\frac{h(|y|)}{|y|^n}dy.$$

*Suppose that* $h \in \Lambda_1^\eta (\eta > 0)$. *Then*

$$\left\| M^{(\Phi_1,\Phi_2)}_{\Omega,\Gamma,\Lambda,\varphi,h}(f) \right\|_p \le C_p \left\| \Omega \right\|_{L^1} \left\| h \right\|_{\Lambda_1^\eta} \left\| f \right\|_p ,$$

$1 < p < \infty$ *with constant* $C_p$ *independent of* $h, \eta, g, \Phi_1, \Phi_2, z_1, z_2$, *and the coefficients of the generalized polynomials* $\Gamma$ *and* $\Lambda$, *but depends on the function* $\varphi$, *and the numbers representing the powers of the monomials involved in the representations of the generalized polynomials* $\Gamma$ *and* $\Lambda$.

It is clear that Corollary 1.4 generalizes as well as improves the corresponding result on page 477 of [20].

Throughout this paper, the letter $C$ will stand for a positive constant that may vary at each occurrence, but it is independent of the essential variables.

## 2. $L^p$ Bounds of generalized bidirectional Hardy–Littlewood maximal functions

The main aim of this section is to prove the key result of Theorem 1.3. We shall start by establishing the following lemma:

**Lemma 2.1**. Let $\Gamma$ *and* $\varphi$ *be as in the statement of Theorem 1.3. Let* $z \in \mathbb{R}^n$ *and let* $H^{(z)}_{\Gamma,\varphi,h}$ *be given by (1.7) with* $z_1 = z$ *and* $z_2 = 0$. *Suppose that* $h \in \Lambda_1^\eta (\eta > 0)$. *Then*

$$\left\| H^{(z)}_{\Gamma,\varphi,h}(g) \right\|_p \le C_p \left\| h \right\|_{\Lambda_1^\eta} \left\| g \right\|_p ,$$

$1 < p < \infty$ *with constant* $C_p$ *independent of* $h, \eta, g,\ z$, *and the coefficients of the generalized polynomial* $\Gamma$, *but depends on the function* $\varphi$ *and the numbers* $d_1, \ldots, d_l$.

Proof. Suppose that

$$\Gamma(t) = \mu_1 t^{d_1} + \cdots + \mu_l t^{d_l} \tag{2.1}$$

for some $l \in \mathbb{N}$, distinct positive real numbers $d_1, \ldots, d_l$ and real numbers $\mu_1, \ldots, \mu_l$. We shall argue by induction on the number of terms $l$. We start by assuming that $l = 1$. Let $\varphi(t) = (\varphi(t))^{d_1}$ and $\tilde{z} = \mu_1 z$. Since $\Gamma$ is not constant, then $d_1 \neq 0$ and $\mu_1 \neq 0$. For $j \in \mathbb{Z}$, define the measure $\mu_j$ by

$$\int g d\mu_j = \int_{2^{j-1}}^{2^j} g(\varphi(t)\tilde{z}) \frac{h(t)}{t} dt. \tag{2.2}$$

Then

$$\hat{\mu}_j(\xi) = \int_{2^{j-1}}^{2^j} e^{-i\varphi(t)\xi \cdot \tilde{z}} \frac{h(t)}{t} dt = \int_{\frac{1}{2}}^{1} e^{-i\varphi(2^j t)\xi \cdot \tilde{z}} \frac{h(2^j t)}{t} dt.$$

Choose a function $\psi \in \mathcal{C}^\infty(\mathbb{R})$ such that $supp(\psi) \subset (0, 10^{-9})$, $\psi \geq 1$, and $\int_{-\infty}^{\infty} \psi(s) ds = 1$. Set

$$k_j(r) = \int_0^{\frac{r}{2}} h(2^j(r-s))\psi_u(s) ds, r > 0, \tag{2.3}$$

where $\psi_u(s) = \frac{1}{u}\psi(\frac{s}{u})$. Define the measure $\nu_j$ by

$$\int g d\nu_j = \int_{\frac{1}{2}}^{1} \frac{k_j(t)}{t} g(\varphi(2^j t)\tilde{z}) dt.$$

Thus,

$$|\hat{\mu}_j(\xi)| \leq |\hat{\mu}_j(\xi) - \hat{\nu}_j(\xi)| + |\hat{\nu}_j(\xi)|.$$

Now, we use the properties of the function $h$ to estimate $|\hat{\mu}_j(\xi) - \hat{\nu}_j(\xi)|$. In fact,

$$\begin{aligned}
|\hat{\mu}_j(\xi) - \hat{\nu}_j(\xi)| &\leq \int_{\frac{1}{2}}^{1} \left| h(2^j t) - k_j(t) \right| \frac{dt}{t} \\
&= \int_{\frac{1}{2}}^{1} \left| \int_{r<t/2} (h(2^j(t-r) - h(2^j t))\psi_u(r) dr \right| \frac{dt}{t} \\
&\leq \int_{r<1/4} \int_{\frac{1}{2}}^{1} \left| h(2^j(t-r) - h(2^j t) \right| \frac{dt}{t} |\psi_u(r)| dr \\
&\leq \int_{r<1/4} \int_{2^{j-1}}^{2^j} \left| h(t - 2^j r) - h(t) \right| \frac{dt}{t} |\psi_u(r)| dr \\
&\leq C\omega(h, u) \leq u^\eta C \|h\|_{\Lambda^\eta}.
\end{aligned} \tag{2.4}$$

Since $\varphi$ is convex increasing and $\varphi(0)=0$, we have

$$\begin{aligned} \varphi(2r) &\geq 2\varphi(r) \qquad (2.5)\\ r\varphi'(r) &\geq \varphi(r) \qquad (2.6) \end{aligned}$$

for every $r>0$. Thus, for $1/2 \leq t < r/2^j \leq 1$, we can easily show that

$$\begin{aligned} \left|\frac{d}{dt}\left(\varphi(2^j t)\right)\right| &= \left|d_1\left(\varphi(2^j t)\right)^{d_1-1} 2^j \varphi'(2^j t)\right| \\ &= \left|d_1\left(\varphi(2^j t)\right)^{d_1-1} 2^j t\varphi'(2^j t)\right| \\ &\geq \frac{d_1}{t}\left(\varphi(2^j t)\right)^{d_1} \geq d_1\varphi(2^{j-1}). \qquad (2.7) \end{aligned}$$

Thus, since $\varphi$ is increasing, by the inequality (2.7) along with van der Corput Lemma [20], we have

$$\begin{aligned} \left|\int_{2^{j-1}}^{r} e^{-i\varphi(t)\xi\cdot\tilde{z}}\frac{dt}{t}\right| &\leq \frac{1}{d_1}\left|\varphi(2^{j-1})\xi\cdot\tilde{z}\right|^{-1}\left(\frac{1}{r}+\int_{2^{j-1}}^{r}\frac{1}{t^2}dt\right) \\ &\leq \frac{1}{d_1}\left|\varphi(2^{j-1})\xi\cdot\tilde{z}\right|^{-1}. \qquad (2.8) \end{aligned}$$

for all $2^{j-1} \leq r \leq 2^j$ uniformly in $r$. Therefore, we have

$$|\hat{\nu}_j(\xi)| \leq \frac{1}{d_1}\left|\varphi(2^{j-1})\xi\cdot\tilde{z}\right|^{-1}\left(|k_j(1)| + \int_{\frac{1}{2}}^{1}\left|k_j'(r)\right|dr\right) \leq \frac{C}{u}\left|\varphi(2^{j-1})\xi\cdot\tilde{z}\right|^{-1}. \qquad (2.9)$$

Now, if we take $u = \left|\varphi(2^{j-1})\xi\cdot\tilde{z}\right|^{-\frac{1}{\eta+1}}$, then we have

$$|\hat{\mu}_j(\xi)| \leq |\hat{\mu}_j(\xi) - \hat{\nu}_j(\xi)| + |\hat{\nu}_j(\xi)| \leq C\left|\varphi(2^{j-1})\xi\cdot\tilde{z}\right|^{-\frac{\eta}{\eta+1}}. \qquad (2.10)$$

Next, let

$$A_j = \int_{2^{j-1}}^{2^j}\frac{h(t)}{t}dt.$$

Then $|A_j| \leq \|h\|_{\Delta_1}$ and

$$|\hat{\mu}_j(\xi) - A_j| = \left|\int_{2^{j-1}}^{2^j}(e^{-i\varphi(t)\xi\cdot\tilde{z}} - 1)h(t)\frac{dt}{t}\right| \leq \|h\|_{\Delta_1}\left|\varphi(2^j)\xi\cdot\tilde{z}\right|. \qquad (2.11)$$

Now choose $\theta \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\theta}(\xi) = 1$ if $|\xi| < \frac{1}{4}$ and $\hat{\theta}(\xi) = 0$ if $|\xi| > 1$. Let $\hat{\pi}_j(\xi) = \hat{\theta}(\varphi(2^j)\xi)$ and define $\sigma_j$ by

$$\sigma_j = \mu_j - A_j\pi_j. \qquad (2.12)$$

Thus, by (2.10), (2.11), and the properties of the function $\theta$, we have

$$|\hat{\sigma}_j(\xi)| \leq C \|h\|_{\Lambda_1^\eta} \min\{\left|\varphi(2^{j-1})\xi \cdot \tilde{z}\right|^{-\frac{\eta}{\eta+1}}, \left|\varphi(2^j)\xi \cdot \tilde{z}\right|\}. \tag{2.13}$$

Moreover, by (2.12), we arrive at the following:

$$\begin{aligned} H^{(z)}_{\Gamma,\varphi,h}g(x) &\leq \sup_{j\in\mathbb{Z}} |\sigma_j * g(x)| + \sup_{j\in\mathbb{Z}} \left|A_j \pi_j * g(x)\right| \\ &\leq (\sum_j |\sigma_j * g(x)|^2)^{\frac{1}{2}} + \|h\|_{\Delta_1} Mg(x) \\ &= S_{z,h}(g)(x) + \|h\|_{\Delta_1} Mg(x), \end{aligned} \tag{2.14}$$

where $M$ is the Hardy–Littlewood maximal function. Hence, the $L^p$ boundedness of the operator follows by a bootstrapping argument as in [9].

Next, we assume that $H^{(z)}_{\Gamma,\varphi,h}$ is bounded on $L^p$ for all $1 < p < \infty$ provided that the number of terms $l$ of the generalized polynomial $\Gamma$ is less than $M \in \mathbb{N}$. Let $\Gamma$ be given by (2.1) with $l = M+1$. Assume that $d_1 \leq d_2 \leq ... \leq d_{M+1}$. Let $l_0 = \max\{1 \leq l \leq M : \mu_l \neq 0\}$ and let

$$\Gamma_{l_0}(t) = \mu_1 t^{d_1} + \cdots + \mu_{l_0} t^{d_{l_0}}. \tag{2.15}$$

For $j \in \mathbb{Z}$, define the measure $\mu_{\Gamma,j}$ and $\mu_{\Gamma_{l_0},j}$ by

$$\int g d\mu_{\Gamma,j} = \int_{2^{j-1}}^{2^j} g(\Gamma(\varphi(t))\tilde{z}) \frac{h(t)}{t} dt \tag{2.16}$$

and

$$\int g d\mu_{\Gamma_{l_0},j} = \int_{2^{j-1}}^{2^j} g(\Gamma_{l_0}(\varphi(t))\tilde{z}) \frac{h(t)}{t} dt. \tag{2.17}$$

Let $k_j$, $\psi$, and $\psi_u$ be as above. Let $\nu_{\Gamma,j}$ be given by

$$\int g \nu_{\Gamma,j} = \int_{\frac{1}{2}}^{1} \frac{k_j(t)}{t} g(\Gamma(\varphi(t))\tilde{z}) dt.$$

Then by similar argument as that led to (2.4), we obtain

$$|\hat{\mu}_{\Gamma,j}(\xi) - \hat{\nu}_{\Gamma,j}(\xi)| \leq u^\eta C \|h\|_{\Lambda^\eta}. \tag{2.18}$$

Now, for $2^{j-1} \leq r \leq 2^j$, by proposition on page 184 in [16]( van der Corput Lemma for generalized polynomials), we have

$$\left| \int_{\varphi(2^{j-1})}^{\varphi(r)} e^{-i\Gamma(s)\xi\cdot\tilde{z}} ds \right| = \varphi(r) \left| \int_{\frac{\varphi(2^{j-1})}{\varphi(r)}}^{1} e^{-i\Gamma(\varphi(r)s)\xi\cdot\tilde{z}} ds \right| \leq C\varphi(r) \left|(\varphi(2^{j-1})^{d_{M+1}} L_{d,z}(\xi)\right|^{-\varepsilon} \tag{2.19}$$

for some $0 < \varepsilon < \min\{\frac{1}{\mu_{M+1}}, \frac{1}{M+1}\}$, with bound $C$ independent of $j, r, \mu_2, ..., \mu_{M+1}$. Here,

$$L_{d,z}(\xi) = (\mu_{M+1})^{d_{M+1}} \xi \cdot \tilde{z}.$$

Thus, by using proper change of variables, we obtain

$$\begin{aligned}
\left| \int_{2^{j-1}}^{r} e^{-i\Gamma(\varphi(t))\xi\cdot\tilde{z}} \frac{dt}{t} \right| &= \left| \int_{\varphi(2^{j-1})}^{\varphi(r)} e^{-i\Gamma(s)\xi\cdot\tilde{z}} \frac{ds}{\varphi^{-1}(s)\varphi'(\varphi^{-1}(s))} \right| \\
&\leq \frac{C\varphi(r)}{2^j\varphi'(2^j)} \left|(\varphi(2^{j-1})^{d_{M+1}} L_{d,z}(\xi)\right|^{-\varepsilon} \\
&\leq \frac{C\varphi(r)}{\varphi(2^j)} \left|(\varphi(2^{j-1})^{d_{M+1}} L_{d,z}(\xi)\right|^{-\varepsilon} \qquad (2.20) \\
&\leq C \left|(\varphi(2^{j-1})^{d_{M+1}} L_{d,z}(\xi)\right|^{-\varepsilon} \qquad (2.21)
\end{aligned}$$

for all $2^{j-1} \leq r \leq 2^j$ uniformly in $r$. Therefore, by similar argument as in (2.9), we have

$$|\hat{\nu}_{\Gamma,j}(\xi)| \leq \frac{C}{u} \left|(\varphi(2^{j-1})^{d_{M+1}} L_{d,z}(\xi)\right|^{-\varepsilon}. \qquad (2.22)$$

By (2.22) and (2.18) with

$$u = \left|(\varphi(r))^{d_{M+1}} L_{d,z}(\xi)\right|^{-\frac{1}{\eta+1}},$$

we get

$$|\hat{\mu}_{\Gamma,j}(\xi)| \leq C \left|(\varphi(2^{j-1})^{d_{M+1}} L_{d,z}(\xi)\right|^{-\frac{\eta}{\eta+1}}. \qquad (2.23)$$

Next, it can be easily seen that

$$|\hat{\mu}_{\Gamma,j}(\xi) - \hat{\nu}_{\Gamma,j}(\xi)| \leq \|h\|_{\Delta_1} \left|\left(\varphi(2^j)\right)^{d_{M+1}} L_{d,z}(\xi)\right|. \qquad (2.24)$$

Again, we choose $\theta \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\theta}(\xi) = 1$ if $|\xi| < \frac{1}{4}$ and $\hat{\theta}(\xi) = 0$ if $|\xi| > 1$. Let $\hat{\pi}_j(\xi) = \hat{\theta}(\left(\varphi(2^j)\right)^{d_{M+1}} \xi)$ and define $\sigma_{\Gamma,j}$ by

$$\sigma_{\Gamma,j} = \mu_{\Gamma,j} - \pi_j * \mu_{\Gamma_{l_0},j}. \qquad (2.25)$$

Thus, by (2.23), (2.24), and the properties of function $\theta$, we have

$$|\sigma_{\Gamma,j}(\xi)| \leq C \|h\|_{\Lambda_1^\eta} \min\{\left|\left(\varphi(2^{j-1})\right)^{d_{M+1}} L_{d,z}(\xi)\right|^{-\frac{\eta}{\eta+1}}, \left|\left(\varphi(2^j)\right)^{d_{M+1}} L_{d,z}(\xi)\right|\}. \qquad (2.26)$$

Moreover, by (2.25), we obtain

$$\begin{aligned}
H^{(z)}_{\Gamma,\varphi,h} g(x) &\leq \sup_{j\in\mathbb{Z}} |\sigma_{\Gamma,j} * g(x)| + \sup_{j\in\mathbb{Z}} \left|\pi_j * \mu_{\Gamma_{l_0},j} * g(x)\right| \\
&\leq \left(\sum_j |\sigma_{\Gamma,j} * g(x)|^2\right)^{\frac{1}{2}} + \|h\|_{\Delta_1} \mu^*_{\Gamma_{l_0}} g(x) \\
&= G_{z,h}(g)(x) + \|h\|_{\Delta_1} \mu^*_{\Gamma_{l_0}} g(x), \qquad (2.27)
\end{aligned}$$

where $\mu^*_{\Gamma_{l_0}}$ is the maximal function

$$\mu^*_{\Gamma_{l_0}}(g)(x) = \sup_j \left| \left| \mu_{\Gamma_{l_0},j} \right| * g(x) \right|. \tag{2.28}$$

Therefore, by induction assumption, we have

$$\left\| \mu^*_{\Gamma_{l_0}}(g) \right\|_p \leq C_p \left\| h \right\|_{\Lambda_1^\eta} \left\| g \right\|_p \tag{2.29}$$

for all $1 < p < \infty$. Hence, the $L^p$ boundedness of the operator $H^{(z)}_{\Gamma,\varphi,h}$ follows by a bootstrapping argument as in [9]. This completes the proof.

Now, we prove Theorem 1.3:

**Proof (of Theorem 1.3)**. Let $\Gamma, \Lambda, \varphi, z_1, z_2$, and $h$ be as in the statement of Theorem 1.3. If $z_1 = 0$ or $z_2 = 0$, then the result follows by Lemma 2.1. Thus, we assume that $z_1 \neq 0$ and $z_2 \neq 0$. We shall argue by induction on the number of terms of $\Gamma$. Assume that $\Gamma$ is given by (2.1) with $l = 1$ and let $H^{(z_2)}_{\Lambda,\varphi,h}$ be the operator given by (1.7) with $z_1 = 0$. Then by Lemma 2.1, we have

$$\left\| H^{(z_2)}_{\Lambda,\varphi,h}(g) \right\|_p \leq C_p \left\| h \right\|_{\Lambda_1^\eta} \left\| g \right\|_p \tag{2.30}$$

for $1 < p < \infty$ with constant $C_p$ independent of $h, \eta, g$ and the coefficients of the generalized polynomial $\Lambda$. For each $j \in \mathbb{Z}$, let $\nu_j$ and $\vartheta_j$ be the measures defined by

$$\int f d\nu_j = \int_{2^{j-1}}^{2^j} f\left(\Gamma(\varphi(t)) z_1 + \Lambda(\varphi(t)) z_2\right) \frac{h(t)}{t} dt \tag{2.31}$$

and

$$\int f d\vartheta_j = \int_{2^{j-1}}^{2^j} f\left(\Lambda(\varphi(t)) z_2\right) \frac{h(t)}{t} dt. \tag{2.32}$$

Then

$$H^{(z_1,z_2)}_{\Gamma,\Lambda,\varphi,h} f(x) = \sup_{j\in\mathbb{Z}} \left| |\nu_j| * f(x) \right| \tag{2.33}$$

and

$$H^{(z_2)}_{\Lambda,\varphi,h} f(x) = \sup_{j\in\mathbb{Z}} \left| |\vartheta_j| * f(x) \right|. \tag{2.34}$$

By (2.30) and repeating the same steps (2.16)-(2.29) with the proper modifications, we obtain the desired estimates for $H^{(z_1,z_2)}_{\Gamma,\Lambda,\varphi,h}$.

Next, we assume that $H^{(z_1,z_2)}_{\Gamma,\Lambda,\varphi,h}$ has the $L^p$ estimates stated in Theorem 1.3 whenever $\Gamma$ has $l$ terms with $l \leq M$. Let $\Gamma$ be given by (2.1) with $l = M + 1$ and let

$$\Gamma_M(t) = \Gamma(t) - \mu_{M+1} t^{d_{M+1}}. \tag{2.35}$$

For each $j \in \mathbb{Z}$, let $\nu_{M+1,j}$ and $\vartheta_{M,j}$ be the measures defined by

$$\int f d\nu_{M+1,j} = \int_{2^{j-1}}^{2^j} f\left(\Gamma(\varphi(t))z_1 + \Lambda(\varphi(t))z_2\right)\frac{h(t)}{t}dt \tag{2.36}$$

and

$$\int f d\vartheta_{M,j} = \int_{2^{j-1}}^{2^j} f\left(\Gamma_M(\varphi(t))z_1 + \Lambda(\varphi(t))z_2\right)\frac{h(t)}{t}dt. \tag{2.37}$$

Then

$$H^{(z_1,z_2)}_{\Gamma,\Lambda,\varphi,h} f(x) = \sup_{j\in\mathbb{Z}} \left| |\nu_{M+1,j}| * f(x) \right|. \tag{2.38}$$

Let

$$(\vartheta_M)^* f(x) = \sup_{j\in\mathbb{Z}} \left| |\vartheta_{M,j}| * f(x) \right|. \tag{2.39}$$

By induction assumption, we have

$$\left\|(\vartheta_M)^*(f)\right\|_p \leq C_p \left\|h\right\|_{\Lambda_1^\eta} \left\|f\right\|_p \tag{2.40}$$

$1 < p < \infty$ with constant $C_p$ independent of $h, \eta, f$ and the coefficients of the generalized polynomial $\Gamma$ and $\Lambda$. Thus, the desired $L^p$ boundedness of $H^{(z_1,z_2)}_{\Gamma,\Lambda,\varphi,h}$ follows by similar argument as in the first step of the induction argument with minor modifications. This completes the proof.

### 2.1. Proof of main results

**Proof of Theorem 1.3**. Since $\Omega \in H^1(\mathbb{S}^{n-1})$, there exists complex numbers $\lambda_j$ and functions $b_j$ on $\mathbb{S}^{n-1}$ such that

$$\Omega = \sum_j \lambda_j b_j \tag{2.41}$$

*and*

$$\|f\|_{H^1(\mathbb{S}^{n-1})} \approx \sum_j |\lambda_j|,$$

where $b_j$ is either in $L^\infty(\mathbb{S}^{n-1})$ and $\|b_j\|_\infty \leq 1$ or $b_j(\cdot)$ satisfies the following properties:

$$supp(b_j) \subset \mathbb{S}^{n-1} \cap \mathbf{B}(\zeta,\rho), \text{ where } \mathbf{B}(\zeta,\rho) = \{y \in \mathbb{R}^n : |y-\zeta| < \rho\}; \tag{2.42}$$

$$\|b_j\|_\infty \leq \rho^{-n+1}; \tag{2.43}$$

$$\int_{\mathbb{S}^{n-1}} b_j(y')d\sigma(y') = 0 \tag{2.44}$$

for some $\zeta \in \mathbb{S}^{n-1}$ and $\rho \in (0,2]$. If $b_j$ satisfies (2.42)-(2.44), then it is called a regular atom. Otherwise, it is called an exceptional atom. (see [17]). By the decomposition (2.41), we only need to show that the theorem holds for regular atoms with $L^p$ norms independent of the particular atom . Let $b$ be a regular atom. By using a proper rotation, we may assume that $supp(b) \subset \mathbb{S}^{n-1} \cap \mathbf{B}(\mathbf{e}, \rho)$ such that $\mathbf{e} = (0, \cdots, 1)$. We shall also assume that $\rho$ is very small. The case for large $\rho$ follows by similar( but easier) argument. Let $\Gamma$ be given as in (2.1). For $1 \leq s \leq l$, let $\Gamma_s$ be given by (2.15) with $l_0$ is replaced by $s$ . Also, for $1 \leq s \leq l$, let $\Psi_s : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$\Psi_s(t, y) = \Gamma_s(t)y' - \left( \sum_{j=s+1}^{l} \mu_j t^{d_j} \right) \mathbf{e}.$$

Here, we use the convention $\sum_{j \in \varnothing} = 0$. We shall let $\Gamma_0(t) = 0$.

For $0 \leq s \leq l$ and $k \in \mathbb{Z}$, let $\sigma_{s,k}$ be the measure that is defined in the Fourier transform side by

$$\hat{\sigma}_{s,k}(\xi) = \int\limits_{2^k \leq |y| < 2^{k+1}} e^{i\Psi_s(\varphi(t), y') \cdot \xi} \frac{h(|y|)b(y')}{|y|^n} dy. \tag{2.45}$$

By the cancellation condition (2.44), we have

$$\hat{\sigma}_{0,k}(\xi) = 0.$$

Moreover,

$$\mathbf{T}_{\Omega, \Gamma, \varphi, h} f(x) = \sum_k \sigma_{s,k} * f(x). \tag{2.46}$$

Let

$$(\sigma_s)^* (f)(x) = \sup_{k \in \mathbb{Z}} ||\sigma_{s,k}| * f(x)| .$$

By Corollary 1.4, we obtain

$$\left\| (\sigma_s)^* (f) \right\|_p \leq C_p \left\| b \right\|_{L^1} \left\| h \right\|_{\Lambda_1^\eta} \left\| f \right\|_p \tag{2.47}$$

$1 < p < \infty$ with constant $C_p$ independent of $h, \eta, g, \Phi_1, \Phi_2$, and the coefficients of the generalized polynomials $\Gamma$ and $\Lambda$, but it depends on the function $\varphi$, and the numbers representing the powers of the monomials involved in the representations of the generalized polynomials $\Gamma$ and $\Lambda$.

Now, it is straightforward to see that

$$|\hat{\sigma}_{s,k}(\xi)| \leq \rho^{-n+1} \int\limits_{\mathbf{B}(\mathbf{e}, \rho)} |\mathbf{I}_k(y', z')| \, d\sigma(y') d\sigma(y'), \tag{2.48}$$

where

$$\mathbf{I}_{k,s}(y^{'}, \xi) = \int\limits_{2^{j-1}}^{2^j} e^{-i\Psi_s(\varphi(t), y') \cdot \xi} \frac{h(t)dt}{t}. \tag{2.49}$$

By similar argument as that led to (2.23), we have

$$\left| \mathbf{I}_{k,s}(y^{'}, \xi) \right| \leq C \left| (\varphi(2^{j-1})^{d_s} \mu_s \xi \cdot y' \right|^{-\frac{\eta}{\eta+1}} . \tag{2.50}$$

By (2.48) and (2.50), we obtain

$$|\hat{\sigma}_{s,k}(\xi)| \le C \left|(\varphi(2^{j-1})^{d_s}\mu_s\rho\xi\right|^{-\frac{\eta}{\eta+1}} \tag{2.51}$$

with constant $C$ independent of the essential variables.

On the other hand, it is not hard to see that

$$|\hat{\sigma}_{s,k}(\xi) - \hat{\sigma}_{s-1,k}(\xi)| \le C \left|(\varphi(2^{j})^{d_s}\mu_s\rho\xi\right|. \tag{2.52}$$

Hence, the result follows by (2.46), (2.47), (2.51), (2.52), and Lemma 5.2 in ([10]) □

Now we show that Corollary 1.4 is an immediate consequence of Theorem 1.3. In fact, by generalized Minkowsk's inequality and Theorem 1.3, we have

$$\begin{aligned}\left\|M^{(\Phi_1,\Phi_2)}_{\Omega,\Gamma,\Lambda,\varphi,h}(f)\right\|_p &\le \int_{\mathbb{S}^{n-1}} \left|\Omega(y')\left\|H^{(\Phi(y_1'),\Phi_2(y'))}_{\Gamma,\Lambda,\varphi,h}f(x)\right\|_p\right| d\sigma(y') \\ &\le C_p \left\|h\right\|_{\Lambda_1^\eta}\left\|\Omega\right\|_{L^1}\left\|f\right\|_p.\end{aligned}$$

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