[Turkish Journal of Mathematics](https://journals.tubitak.gov.tr/math)

[Volume 45](https://journals.tubitak.gov.tr/math/vol45) | [Number 5](https://journals.tubitak.gov.tr/math/vol45/iss5) Article 22

1-1-2021

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 2199 – 2210
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doi:10.3906/mat-2101-72

Bounded invertibility and separability of a parabolic type singular operator in the space $L_2(R^2)$

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Received: 17.01.2021 • **Accepted/Published Online:** 04.08.2021 • **Final Version:** 16.09.2021

Abstract: In this paper, we consider the operator of parabolic type

$$Lu = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + q(x)u,$$

in the space $L_2(R^2)$ with a greatly growing coefficient at infinity. The operator is originally defined on $C_0^\infty(R^2)$, where $C_0^\infty(R^2)$ is the set of infinitely differentiable and compactly supported functions.

Assume that the coefficient $q(x)$ is a continuous function in $R = (-\infty, \infty)$, and it can be a strongly increasing function at infinity.

The operator L admits closure in space $L_2(R^2)$, and the closure is also denoted by L.

In the paper, we proved the bounded invertibility of the operator L in the space $L_2(R^2)$ and the existence of the estimate

$$\left\| \frac{\partial u}{\partial t} \right\|_{L_2(R^2)} + \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L_2(R^2)} + \|q(x)u\|_{L_2(R^2)} \le C(\|Lu\|_{L_2(R^2)} + \|u\|_{L_2(R^2)}),$$

under certain restrictions on $q(x)$ in addition to the conditions indicated above.

Example. $q(x) = e^{100|x|}$, $-\infty < x < \infty$.

Key words: Coercive estimate, separability, singular operator, parabolic type operator, invertibility

1. Introduction

Consider the operator

$$(L + \lambda I)u = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + q(x)u + \lambda u \tag{1.1}$$

of parabolic type with an unbounded coefficient, originally defined on the set $C_0^\infty(R^2)$ of smooth and compactly supported arbitrary functions where $(t, x) \in R^2$, $\lambda \ge 0$.

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2010 *AMS Mathematics Subject Classification:* 35B65, 35K67, 47B38

Further, we assume that the coefficient $q(x)$ satisfies the conditions:

i) $q(x) \geq \delta > 0$ is a continuous function in R, $R = (-\infty, \infty)$;

ii) $\mu = \sup\limits_{|x-y|\leq 1} \frac{q(x)}{q(y)} < \infty$.

It is not difficult to verify that the operator $L + \lambda I$ admits closure in $L_2(R^2)$, and the closure is also denoted by $L + \lambda I$.

It should be noted that the question of the existence of a bounded operator $(L + \lambda I)^{-1}$ for a closed operator $(L + \lambda I)$ in the space $L_2(R^2)$ is equivalent to the following problem: find a unique solution $u(x, t)$ to an equation $(L + \lambda I)u = f \in L_2(R^2)$ belonging to the space $L_2(R^2)$, i.e. $u \in L_2(R^2)$ (Definition 2.10). In this case, it is easy to see that the closed operator $L + \lambda I$ generates a problem without initial conditions ([17], chapter III, Section 4).

Recently, there has been an increased interest in differential operators with unbounded coefficients [1–16]. Such operators appear in problems of quantum mechanics and stochastic heat equations [3, 6].

In contrast to the research of other authors, this paper studies the question of the separability of a parabolic type differential operator in the Hilbert space $L_2(R^2)$ (see Definition 1.1) with a greatly growing coefficient at infinity. Let us note that the behavior of the operator coefficient at infinity plays a significant role here.

J. Leray has also noted in his book [9] that an important and interesting problem is the study of differential operators in the whole space R^n $(n \geq 1)$. The behavior of the operator coefficients at infinity plays a significant role here.

In the paper, we are interested in the following items:

a) the existence of a resolvent;

b) the presence of the estimate

$$\left\| \frac{\partial u}{\partial t} \right\|_2 + \left\| \frac{\partial^2 u}{\partial x^2} \right\|_2 + \|q(x)u\|_2 \leq C(\|Lu\|_2 + \|u\|_2) \tag{1.2}$$

for the parabolic operator with a growing coefficient, where $u \in D(L)$ is the definition domain of the operator L, $\|\cdot\|_2$ is the norm in $L_2(R^2)$, $C > 0$ is a constant number.

Definition 1.1 *We will say that the parabolic operator L is separable if estimate (1.2) holds for all $u \in D(L)$.*

Here is a formulation of the main results.

Theorem 1.1 *Let the condition i) be fulfilled.Then there exists a continuous inverse operator $(L + \lambda I)^{-1}$ for $\lambda \geq 0$ defined in the space $L_2(R^2)$.*

Theorem 1.2 *Let the conditions i) - ii) be fulfilled. Then the operator L is separable.*

Example. Consider the operator

$$Lu = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + e^{100|x|} \cdot u, \quad u \in D(L),$$

$-\infty < t < \infty$, $-\infty < x < \infty$.

It is easy to verify that all conditions of Theorems 1.1–1.2 are satisfied.

Therefore, there exists a bounded inverse operator and the estimate

$$\left\|\frac{\partial u}{\partial t}\right\|_2 + \left\|\frac{\partial^2 u}{\partial x^2}\right\|_2 + \left\|e^{100|x|}\cdot u\right\|_2 \leq C(\|Lu\|_2 + \|u\|_2),$$

holds, where $C>0$ is a constant number.

2. Existence of a resolvent

2.1. Auxiliary estimates and lemmas

Consider the operator

$$(l_\tau + \lambda I)u(x) = -u''(x) + (i\tau + q(x) + \lambda)u$$

originally defined on the set $C_0^\infty(R)$, where $\tau \in (-\infty, \infty)$.

It is not difficult to verify that the operator $l_\tau + \lambda I$ admits closure in $L_2(R^2)$ and the closure is also denoted by $l_\tau + \lambda I$.

Lemma 2.1 *Let the condition i) be fulfilled. Then the following estimates*

$$\|(l_\tau + \lambda I)u\|_2 \geq (\delta + \lambda)\|u\|_2,$$

$$\|(l_\tau + \lambda I)u\|_2 \geq |\tau|\,\|u\|_2, \tau \neq 0$$

hold for $\lambda \geq 0$ *and for all* $u \in D(l_\tau)$.

Proof Let $u \in C_0^\infty(R)$. Then the equality

$$< (l_\tau + \lambda I)u, u >= \int_R (-u''(x) + (i\tau + q(x) + \lambda)u)\overline{u}dx = \int_R |u'(x)|^2 dx + \int_R (i\tau + q(x) + \lambda)|u|^2 dx$$

holds, where $<\cdot,\cdot>$ is scalar product in $L_2(R)$.

Hence, we have

$$| < (l_\tau + \lambda I)u, u > | \geq \int_R |u'|^2 + (q(x) + \lambda)|u|^2 dx, \tag{2.1}$$

$$| < (l_\tau + \lambda I)u, u > | \geq |\int_R i\tau|u|^2 dx|, \quad for \ \ \tau \neq 0. \tag{2.2}$$

From these inequalities, we obtain that

$$\|(l_\tau + \lambda I)u\|_2 \geq (\delta + \lambda)\|u\|_2, \tag{2.3}$$

$$\|(l_\tau + \lambda I)u\|_2 \geq |\tau| \cdot \|u\|_2. \tag{2.4}$$

From inequalities (2.3) and (2.4), we obtain the proof of Lemma 2.1. □

Take the collection of non-negative functions $\{\varphi_j\}$ from $C_0^\infty(R)$, such that

$$\sum_j \varphi_j^2(x) \equiv 1, \;\; \sup p\varphi_j \subseteq \Delta_j, \;\; \bigcup_j \Delta_j = R,$$

where $\Delta_j = (j-1, j+1)$, $j \in Z$ [13, 14].

Let us extend $q(x)$ from Δ_j to the whole R. So that its continuation $q_j(x)$ is a bounded and periodic function of the same period.

We denote by $(l_{\tau,j} + \lambda I)$ the closure of the operator

$$(l_{\tau,j} + \lambda I)u = -u''(x) + (i\tau + q_j(x) + \lambda)u$$

defined in $C_0^\infty(R)$.

Lemma 2.2 *Let the condition i) be fulfilled. Then the following estimates*

$$\|(l_{\tau,j} + \lambda I)u\|_2 \geq (\delta + \lambda)\|u\|_2, \tag{2.5}$$

$$\|(l_{\tau,j} + \lambda I)u\|_2 \geq |\tau| \cdot \|u\|_2, \tau \neq 0 \tag{2.6}$$

hold for $\lambda \geq 0$ *and any* $u \in D(l_\tau)$, *and for all* $\tau \in R \setminus \{0\}$.

Proof Repeating the computations and arguments used in the proof of Lemma 2.1, we obtain the proof of Lemma 2.2. □

Lemma 2.3 *The operator* $l_{\tau,j} + \lambda I$ *is continuously invertible in the space* $L_2(R)$ *for* $\lambda \geq 0$.

Proof The inequality (2.6) implies that there exists a bounded operator $(l_{\tau,j} + \lambda I)^{-1}$ on $R(l_{\tau,j} + \lambda I)$ and $\left\|(l_{\tau,j} + \lambda I)^{-1}\right\|_{2\to 2} \to 0$ for $|\tau| \to \infty$. From this and the inequality (2.5), it follows that to prove Lemma 2.3, it is sufficient to prove that the range of values $R(l_{\tau,j} + \lambda I)$ of the operator $l_{\tau,j} + \lambda I$ coincides with $L_2(R)$.

Assume on the contrary that $R(l_{\tau,j} + \lambda I)$ does not coincide with the space $L_2(R)$. Then there exists an element $\vartheta \in L_2(R)$, $\vartheta \neq 0$ such that the equality

$$< (l_{\tau,j} + \lambda I)u, \vartheta >= 0$$

holds for any $u \in D(l_{\tau,j})$.

Hence, we get

$$(l_{\tau,j} + \lambda I)^*\vartheta = -\vartheta'' + (-i\tau + q_j(x) + \lambda)\vartheta = 0 \tag{2.7}$$

in the sense of distributions. It is easy to verify that, due to the boundedness $q_j(x)$, we have $q_j(x)\vartheta \in L_2(R)$. This and the equality (2.7) imply that $\vartheta'' \in L_2(R)$ for a finite τ. Taking this into account and repeating the computations and arguments used in Lemma 2.1, it is easy to obtain the estimate

$$\|(l_{\tau,j} + \lambda I)^*\vartheta\|_2 \geq C\|\vartheta\|_2,$$

where $C > 0$ is a constant number.

Since $(l_{\tau,j} + \lambda I)^*\vartheta = 0$, then the last inequality implies $\vartheta = 0$. Lemma 2.3 is proved. □

Lemma 2.4 *The following estimates*

a) $\left\|(l_{\tau,j}+\lambda I)^{-1}\right\|_{2\to 2}\le \frac{1}{(\delta+\lambda)}$;

b) $\left\|\frac{d}{dx}(l_{\tau,j}+\lambda I)^{-1}\right\|_{2\to 2}\le \frac{1}{(\delta+\lambda)^{\frac{1}{2}}}$

hold for the operator $(l_{\tau,j}+\lambda I)^{-1}$.

Proof The proof of item a) of Lemma 2.4 follows from estimate (2.5). Item b) of Lemma 2.4 is proved by transforming the functional $<(l_{\tau,j}+\lambda I)u,u>$ and using the computations used in the proof of Lemma 2.1. □

Assume

$$K_\lambda f=\sum_{\{j\}}\varphi_j(l_{\tau,j}+\lambda I)^{-1}(\varphi_j f), \tag{2.8}$$

where $f\in L_2(R)$.

Let us make sure by direct computation that

$$(l_\tau+\lambda I)K_\lambda f=f-B_\lambda f, \tag{2.9}$$

where $B_\lambda f=\sum\limits_{\{j\}}\varphi_j''(l_{\tau,j}+\lambda I)^{-1}(\varphi_j f)+2\sum\limits_{\{j\}}\varphi_j'\frac{d}{dx}(l_{\tau,j}+\lambda I)^{-1}(\varphi_j f)$.

Lemma 2.5 *Let the condition i) be fulfilled. Then there exists a number* $\lambda_0>0$ *for the operator* B_λ *such that*

$$\|B_\lambda\|_{2\to 2}<1,\quad for\ \ all\ \ \lambda\ge\lambda_0.$$

Proof Let $f\in C_0^\infty(R)$. Now, taking into account that only the functions φ_{j-1}, φ_j, φ_{j+1} are nonzero on the interval $\Delta_j (j\in Z)$, we obtain

$$\|B_\lambda f\|_2^2=\int_{-\infty}^{\infty}|\sum_{\{j\}}\varphi_j''(l_{\tau,j}+\lambda I)^{-1}\varphi_j f+2\sum_{\{j\}}\varphi_j'\frac{d}{dx}(l_{\tau,j}+\lambda I)^{-1}\varphi_j f|^2dx\le$$

$$\le\sum_{j=-\infty}^{\infty}\int_{\Delta_j}|\sum_{k=j-1}^{j+1}\varphi_k''(l_{\tau,k}+\lambda I)^{-1}\varphi_k f+2\varphi_k'\frac{d}{dx}(l_{\tau,j}+\lambda I)^{-1}\varphi_k f|^2dx.$$

From this and using the obvious inequality $(a+b+c)^2\le 3(a^2+b^2+c^2)$, as well as the results of Lemma 2.4, we find

$$\|B_\lambda f\|_2^2\le C_0[\frac{1}{(\delta_0+\lambda)^2}+\frac{1}{(\delta_0+\lambda)}]\cdot\|f\|_2^2,$$

where $C_0=72\sup\limits_{j\in Z}\max\{\|\varphi_j'\|_\infty^2,\|\varphi_j''\|_\infty^2\}$.

The last inequality implies that there exists a number $\lambda_0>0$ such that $\|B_\lambda\|_{2\to 2}<1$ for $\lambda\ge\lambda_0$. Lemma 2.5 is proved. □

Lemma 2.6 *Let the condition i) be fulfilled. Then the operator* $l_\tau+\lambda I$ *is boundedly invertible for* $\lambda\ge\lambda_0>0$, *and the equality*

$$(l_\tau+\lambda I)^{-1}=K_\lambda(I-B_\lambda)^{-1}$$

holds for the inverse operator $(l_\tau+\lambda I)^{-1}$.

Proof Using representation (2.9) and Lemmas 2.1, 2.2, and 2.5, we obtain the proof of Lemma 2.6. □

Lemma 2.7 *Let the condition i) be fulfilled. Then the operator* $l_\tau + \lambda I$ *is boundedly invertible in the space* $L_2(R)$ *for all* $\lambda \geq 0$*.*

The proof of Lemma 2.7 follows from Lemmas 2.1, 2.6 and the following lemma.

Lemma 2.8 *[2]. Let the operator* $l_\tau + \lambda_0 I$ $(\lambda_0 > 0)$ *be boundedly invertible in* $L_2(R)$ *and the estimate* $\|(l_\tau + \lambda I)u\|_2 \geq C\|u\|_2$ *be fulfilled for* $\lambda \in [0, \lambda_0]$, $u \in D(l_\tau)$ *. Then the operator* $l_\tau : L_2(R) \to L_2(R)$ *is also boundedly invertible.*

2.2. Proof of Theorem 1.1

First, we prove the following lemma.

Lemma 2.9 *Let the condition i) be fulfilled and* $\lambda \geq 0$ *. Then the inequality*

$$\|(L + \lambda I)u\|_2 \geq (\delta + \lambda)\|u\|_2, \tag{2.10}$$

holds for any $u \in D(L + \lambda I)$*.*

Proof Since the operator L has the real-valued coefficient, it is sufficient to prove the estimate (2.10) for real-valued functions. Let $u \in C_0^\infty(R^2)$ be a real-valued function.

Consider the following scalar product

$$< (L + \lambda I)u, u >= \int\limits_{R^2} \frac{\partial u}{\partial t} u dx dt + \int\limits_{R^2} (|u'|^2 + (q(x) + \lambda)|u|^2) dx dt. \tag{2.11}$$

It is easy to verify that

$$\int\limits_{R^2} \frac{\partial u}{\partial t} u dx dt = 0.$$

Hence, using the equality (2.11), the Cauchy-Bunyakovsky inequality and condition *i)*, we find

$$\|(L + \lambda I)u\|_2 \geq (\delta + \lambda)\|u\|_2.$$

Since the operator is closable, then the last estimate also holds for all $u \in D(L + \lambda I)$. Lemma 2.9 is proved. □

Now, let us prove the existence of the operator $(L + \lambda I)^{-1}$.

Definition 2.10 *The function* $u \in L_2(R^2)$ *is called a solution of the equation* $(L + \lambda I)u = f$ *if there exists a sequence* $\{u_n\}_{n=1}^\infty \subset C_0^\infty(R^2)$ *such that*

$$\|u_n - u\|_2 \to 0, \quad \|(L + \lambda I)u_n - f\|_2 \to 0, \quad as \;\; n \to \infty.$$

Hence, it is clear that the inverse operator $(L+\lambda I)^{-1}$ coincides with the closure of the operator $(L+\lambda I)^{-1}$ in $L_2(R^2)$, defined on $C_0^\infty(R^2)$.

Consider the equation

$$(L+\lambda I)u = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + q(x)u + \lambda u = f \in C_0^\infty(R^2). \tag{2.12}$$

Applying the Fourier transform to the equation (2.12) with respect to t, we obtain the following equation

$$(l_\tau + \lambda I)u = -\widetilde{u}''(\tau, x) + (-i\tau + q(x) + \lambda)\widetilde{u} = \widetilde{f}(\tau, x), \tag{2.13}$$

where $\widetilde{u}(\tau, x)$, $\widetilde{f}(\tau, x)$ are the Fourier transform of functions $u(t, x)$ and $f(t, x)$ with respect to the variable t. Further, we denote the Fourier transform by $F_{t\to\tau}$ and the Fourier inverse formula by $F^{-1}_{\tau\to t}$.

Hence, it is easy to see that the problem of solving equation (2.12) turns into the problem of solving of the equation (2.13). Therefore, according to Lemma 2.6, we have:

$$\widetilde{u} = (l_\tau + \lambda I)\widetilde{f} = K_\lambda (I - B\lambda)^{-1}\widetilde{f}.$$

Now, using the inverse operator $F^{-1}_{t\to\tau}$, we find

$$u(t, x) = F^{-1}_{\tau\to t}\widetilde{u} = F^{-1}_{\tau\to t}(l_\tau + \lambda I)^{-1}\widetilde{f}.$$

Due to the continuity of the operator $(l_\tau + \lambda I)^{-1}$ and the Fourier transform, the last equality holds for all $f(t, x) \in L_2(R^2)$. The uniqueness follows from Lemma 2.9. Theorem 1.1 is proved.

3. Separability of operator

In this section, we prove a number of lemmas and estimates that reduce the separability of the operator $L + \lambda I$ with unbounded coefficients to the case of an operator with periodic coefficients.

Lemma 3.1 *Let the conditions i)- ii) be fulfilled. Then the estimates*

$$\left\| |\tau| \cdot (l_\tau + \lambda I)^{-1} \right\|_{2\to 2} \le 1; \tag{3.1}$$

$$\left\| (l_{\tau,j} + \lambda I)^{-1} \right\|_{2\to 2} \le \frac{1}{q(x_j) + \lambda} \tag{3.2}$$

hold, where $q(x_j) = \min\limits_{x\in\overline{\Delta}_j} q(x)$.

Proof The proof of the estimate (3.1) follows from the inequality (2.4).

Let us prove the inequality (3.2). From the inequality (2.1) and using the Cauchy-Bunyakovsky inequality, we obtain

$$\|(l_{\tau,j} + \lambda I)u\|_2 \cdot \|u\|_2 \ge \int\limits_R (q_j(x) + \lambda)|u|^2 dx.$$

Hence,

$$\|(l_{\tau,j}+\lambda I)\|_2 \geq (q_j(x_j)+\lambda)\|u\|_2,$$

where $q_j(x_j) = \min\limits_{x\in\overline{\Delta}_j} q_j(x)$.

Since $q_j(x) = q(x)$ on the segment Δ_j then $q(x_j) = \min\limits_{x\in\overline{\Delta}_j} q(x)$. Therefore, the last inequality implies that

$$\|(l_{\tau,j}+\lambda I)\|_2 \geq (q(x_j)+\lambda)\|u\|_2.$$

The last inequality proves the estimate (3.2). Lemma 3.1 is proved. □

Lemma 3.2 *Let the conditions i)- ii) be fulfilled and $\lambda > 0$ be a number such that $\|B_\lambda\|_{2\to 2} < 1$. Then the following estimates*

$$\left\|q(x)(l_\tau+\lambda I)^{-1}\right\|_{2\to 2}^2 \leq 9\cdot C(\lambda)\sup_{\{j\}}\left\|q(x)\varphi_j(l_{\tau,j}+\lambda I)^{-1}\right\|_{2\to 2}^2, \tag{3.3}$$

$$\left\|q(x)(l_\tau+\lambda I)^{-1}\right\|_{2\to 2}^2 \leq C < \infty \tag{3.4}$$

hold, where $C(\lambda)$ depends only on λ, $C = 9\cdot C(\lambda)\cdot\mu$, the number μ is from the condition ii).

Proof Lemma 2.6 implies that to prove the boundedness of the operator $q(x)(l_\tau+\lambda I)^{-1}$, it suffices to prove the boundedness of the operator $q(x)K_\lambda(I-B_\lambda)^{-1}$. Therefore, we will study the norm of the operator $q(x)K_\lambda(I-B_\lambda)^{-1}$.

Let $f \in L_2(R)$. Then, using Lemma 2.6 and the equality (2.8), we have

$$\left\|q(x)(l_\tau+\lambda I)^{-1}f\right\|_2^2 = \left\|q(x)\sum_{\{j\}}\varphi_j(l_{\tau,j}+\lambda I)^{-1}\varphi_j(I-B_\lambda)^{-1}f\right\|_2^2 \leq$$

$$\leq \sum_{\{j\}}\int_{j-1}^{j+1}|\sum_j q(x)\varphi_j(l_{\tau,j}+\lambda I)^{-1}\varphi_j(I-B_\lambda)^{-1}f|^2dx.$$

It is easy to make sure that only φ_{j-1}, φ_j, $\varphi_{j+1} \neq 0$ on the segment $\Delta_j = [j-1, j+1]$. Taking this into account and using the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we find

$$\left\|q(x)(l_\tau+\lambda I)^{-1}f\right\|_2^2 \leq \sum_{\{j\}}\int_{j-1}^{j+1}|\sum_j q(x)\varphi_j(l_{\tau,j}+\lambda I)^{-1}\varphi_j(I-B_\lambda)^{-1}f|^2dx \leq$$

$$\leq 3\cdot\sum_{\{j\}}\int_{\Delta_j}\sum_{k=j-1}^{j+1}|q(x)\varphi_k(l_{\tau,k}+\lambda I)^{-1}\varphi_k(I-B_\lambda)^{-1}f|^2dx \leq 9\cdot\sum_{\{j\}}\left\|q(x)\varphi_j(l_{\tau,j}+\lambda I)^{-1}\varphi_j(I-B_\lambda)^{-1}f\right\|_2^2 \leq$$

$$\leq 9\cdot\sup_{\{j\}}\left\|q(x)\varphi_j(l_{\tau,j}+\lambda I)^{-1}\right\|_{2\to 2}^2\cdot\int_{-\infty}^{\infty}(\sum_{\{j\}}\varphi_j^2)\cdot|(I-B_\lambda)^{-1}f|^2dx \leq 9\cdot\sup_{\{j\}}\left\|q(x)\varphi_j(l_{\tau,j}+\lambda I)^{-1}\right\|_{2\to 2}^2\cdot$$

$$\cdot \left\|(I-B_\lambda)^{-1}f\right\|_2^2 \le 9\cdot C(\lambda)\sup_{\{j\}}\left\|q(x)\varphi_j(l_{\tau,j}+\lambda I)^{-1}\right\|_2^2\cdot\|f\|_2^2,$$

where $C(\lambda)=\left\|(I-B_\lambda)^{-1}\right\|_{2\to 2}^2$. Here we used the fact that $\sum\limits_{\{j\}}\varphi_j^2\equiv 1$.

Hence:

$$\left\|q(x)(l_\tau+\lambda I)^{-1}f\right\|_2^2 \le 9\cdot C(\lambda)\sup_{\{j\}}\left\|q(x)\varphi_j(l_{\tau,j}+\lambda I)^{-1}\right\|_2^2\cdot\|f\|_2^2.$$

From the last inequality, according to the definition of the operator norm, we obtain the proof of the inequality (3.3) of Lemma 3.2.

Further, according to Lemma 3.1 and the inequality (3.3), we obtain

$$\left\|q(x)(l_\tau+\lambda I)^{-1}\right\|_{2\to 2}^2 \le C(\lambda)\sup_{\{j\}}\left\|q(x)\varphi_j(l_{\tau,j}+\lambda I)^{-1}\right\|_{2\to 2}^2 \le 9C(\lambda)\frac{\max\limits_{x\in\overline{\Delta}_j}|q(x)\varphi_j|}{q(x_j)+\lambda} \le$$

$$\le 9C(\lambda)\sup_{|x-t|\le 2}\frac{q(x)}{q(t)} < 9C(\lambda)\mu < \infty.$$

Lemma 3.2 is proved. □

Lemma 3.3 *Let the conditions i)- ii) be fulfilled. Then the estimate*

$$\|u''(x)\|_2+\|i\tau u\|_2+\|q(x)u\|_2 \le C(\lambda)\,\|(l_\tau+\lambda I)u\|_2,$$

holds, where $C(\lambda)>0$ is a number independent of $u(x)$.

Proof From inequality (2.4) we obtain that

$$\|i\tau u\|_2 \le \|(l_\tau+\lambda I)u\|_2,\quad -\infty<\tau<\infty. \tag{3.5}$$

Further, using Lemma 3.2, we find

$$\|q(x)u\|_2 \le C\,\|(l_\tau+\lambda I)u\|_2, \tag{3.6}$$

where $C>0$ is a constant.

Now, the inequalities (3.5), (3.6), and (2.3) imply that

$$\|u''(x)\|_2 = \|(l_\tau+\lambda I)u - i\tau u - q(x)u - \lambda u\|_2 \le$$

$$\le \|(l_\tau+\lambda I)u\|_2+\|i\tau u\|_2+\|q(x)u\|_2+\lambda\,\|u\|_2 \le C(\lambda)\,\|(l_\tau+\lambda I)u\|_2, \tag{3.7}$$

where $C(\lambda)>0$ is a number independent of $u(x)$.

From the last estimates (3.5), (3.6), and (3.7), we obtain

$$\|-u''(x)\|_2+\|i\tau u\|_2+\|q(x)u\|_2 \le C(\lambda)\,\|(l_\tau+\lambda I)u\|_2.$$

The last inequality proves Lemma 3.3. □

3.1. Proof of Theorem 1.2

According to Theorem 1.1, we have that the inverse operator $(L+\lambda I)^{-1}$ has the form

$$u(t,x)=(L+\lambda I)^{-1}f=F^{-1}_{\tau\to t}(l_\tau+\lambda I)^{-1}\widetilde{f}(\tau,x),\quad f\in L_2(R^2). \tag{3.8}$$

Taking the equality (3.8) into account, we obtain that

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial t}(L+\lambda I)^{-1}f=\frac{\partial}{\partial t}F^{-1}_{\tau\to t}(l_\tau+\lambda I)^{-1}\widetilde{f}=\frac{\partial}{\partial t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(l_\tau+\lambda I)^{-1}\widetilde{f}(\tau,x)e^{it\tau}d\tau=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}i\tau(l_\tau+\lambda I)^{-1}\widetilde{f}(\tau,x)e^{it\tau}d\tau=F^{-1}_{\tau\to t}i\tau(l_\tau+\lambda I)^{-1}\widetilde{f}(\tau,x).$$

Since the operator $F^{-1}_{\tau\to t}$ is a unitary operator, the last inequality implies that

$$\left\|\frac{\partial u}{\partial t}\right\|_2^2=\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}|\tau(l_\tau+\lambda I)^{-1}\widetilde{f}(\tau,x)|^2dx)d\tau=\int_{-\infty}^{\infty}\left\|\tau(l_\tau+\lambda I)^{-1}\widetilde{f}(\tau,x)|\right\|_2^2d\tau\le$$

$$\le\int_{-\infty}^{\infty}\left\|\tau(l_\tau+\lambda I)^{-1}\right\|_{2\to2}^2\cdot\left\|\widetilde{f}(\tau,x)|\right\|_2^2d\tau.$$

Hence, taking the unitarity of the Fourier transform operator in the space $L_2(R)$ into account, we find

$$\left\|\frac{\partial u}{\partial t}\right\|_2^2\le\sup_{\tau\in R}\left\|\tau(l_\tau+\lambda I)^{-1}\right\|_{2\to2}^2\cdot\int_{-\infty}^{\infty}\left\|\widetilde{f}(\tau,x)\right\|_2^2d\tau\le\sup_{\tau\in R}\left\|\tau(l_\tau+\lambda I)^{-1}\right\|_{2\to2}^2\cdot\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}|\widetilde{f}(\tau,x)|^2dx)d\tau\le$$

$$\le\sup_{\tau\in R}\left\|\tau(l_\tau+\lambda I)^{-1}\right\|_{2\to2}^2\cdot\|f(t,x)\|_2^2.$$

It follows from this and the estimate (3.1) that

$$\left\|\frac{\partial u}{\partial t}\right\|_2^2\le\sup_{\tau\in R}\left\|\tau(l_\tau+\lambda I)^{-1}\right\|_{2\to2}^2\cdot\|f(t,x)\|_2^2\le\|f(t,x)\|_2^2,$$

i.e.

$$\left\|\frac{\partial u}{\partial t}\right\|_2\le\|(L+\lambda I)u\|_2, \tag{3.9}$$

where $(L+\lambda I)u=f$.

Repeating the computations and arguments used in the proof of the estimate (3.9) and using Lemma 3.2, we obtain

$$\|q(x)u\|_2^2\le\int_{-\infty}^{\infty}\left\|q(x)(l_\tau+\lambda I)^{-1}\widetilde{f}(\tau,x)\right\|_2^2d\tau\le\sup_{\tau\in R}\left\|q(x)(l_\tau+\lambda I)^{-1}\right\|_{2\to2}^2\cdot\|f(t,x)\|_2^2\le C\cdot\|f(t,x)\|_2^2.$$

Hence

$$\|q(x)u\|_2\le C\,\|(L+\lambda I)u\|_2, \tag{3.10}$$

where $(L+\lambda I)u=f$.

Using the estimate (3.8) and the properties of the Fourier transform, we find

$$\|u_{xx}\|_2^2 \leq \int_{-\infty}^{\infty} \left\| \frac{d^2}{dx^2}(l_\tau + \lambda I)^{-1} \right\|_{2\to 2}^2 \left\| \widetilde{f}(\tau, x) \right\|_2^2 d\tau \leq \sup_{\tau\in R} \left\| \frac{d^2}{dx^2}(l_\tau + \lambda I)^{-1} \right\|_{2\to 2}^2 \cdot \|f(t,x)\|_2^2 .$$

From this and inequality (3.7) we have

$$\|u_{xx}\|_2 \leq C(\lambda) \cdot \|f(t,x)\| ,$$

i.e.

$$\|u_{xx}\|_2 \leq C(\lambda) \cdot \|(L + \lambda I)u\|_2 . \tag{3.11}$$

The inequalities (3.9) - (3.11) imply that the estimate

$$\left\| \frac{\partial u}{\partial t} \right\|_2 + \left\| \frac{\partial^2 u}{\partial x^2} \right\|_2 + \|q(x)u\|_2 \leq C \, \|(L + \lambda I)u\|_2$$

holds for any $u(x) \in D(L)$. Theorem 1.2 is proved.

Acknowledgment

This work was supported by the grant AP08855802 of the Ministry of Education and Science of the Republic of Kazakhstan.

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