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(Co)Limits of Hom-Lie crossed module

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Abstract: In this paper, we give categorical properties such as pullback, finite product, finite limit, coproduct, colimit and pushout in $\boldsymbol{XHom-Lie/A}$ of the category of Hom-Lie crossed modules.

Key words: Hom-Lie algebra, pullback, pushout, crossed module

1. Introduction

Hom-Lie algebras were initially defined by Hartwig et al. in [5]. In that study, Hom-Lie algebras were handled with a part of the deformations of the Witt and Virasoro algebras. Also, the authors of that paper showed that their approach can be used as a new method for deformations of Lie algebras and their central extensions. A Hom-Lie algebra is a noncommutative algebra, and the category of Hom-Lie algebras is an example of semi-abelian categories [6]. In a Hom-Lie algebra, Hom-Jacobi identity replaced the Jacobi identity twisted by a linear map. If this linear map is the identity, then Hom-Lie algebra becomes a Lie algebra. Thus, Lie algebras can be considered as an example of Hom-Lie algebras.

Crossed modules on groups were defined by Whitehead [13]. After that, they are defined on many algebraic structures such as algebra, commutative algebra, Lie algebra, Leibniz algebra, Lie-Rinehart algebra and their properties, and their relationship with other structures has been investigated [1–3, 12]. Recently, crossed modules over Hom-Lie algebras were introduced by Shen and Chen in [11] in order to reveal relationship between Hom-Lie 2-algebras and crossed modules on Hom-Lie algebras. After then, many authors studied this topic [4, 9, 10, 14]. Especially, Casas and Garcia-Martinez investigated crossed modules of Hom-Lie algebras in [4], and they studied on low dimensional cohomology groups of Hom-Lie algebra and their relation with crossed modules. Also, they defined cat^1-Hom-Lie algebras and showed naturally equivalence between category of cat^1-Hom-Lie algebras and category of crossed modules over Hom-Lie algebras.

As is known, studying the properties of a given category is very important in terms of itself and its relationship with other categories. Also, the relationship between some features makes it easy for us to study. In this context, it is enough to prove that it has finite products and equalizers to show that a given category is finitely complete. Similar condition applies to dual versions of these properties. The goal of this article is to investigate the categorical properties of crossed A-modules and prove that the category of crossed A-modules is finitely complete and cocomplete.

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2. Preliminaries

In this section, we will give the basic concepts that we will use in the paper given earlier in the literature and throughout the article, see [4] for details. We will consider vector spaces on field k.

2.1. Hom-Lie algebras

Definition 2.1 *[5] Let A be a vector space with bilinear map $[-,-]:A\times A\longrightarrow A$, and $\alpha_A:A\longrightarrow A$ be a homomorphism of $k-$vector space. Then triple $(A,[-,-],\alpha_A)$ called Hom-Lie algebra (or shortly called H-L algebra) denoted by (A,α_A) if*

$$\begin{array}{lll} (i) & [a,a'] = -[a',a] & \textit{(Skew-symmetry)} \\ (ii) & [\alpha_A(a),[a',a'']] + [\alpha_A(a'),[a'',a]] + [\alpha_A(a''),[a,a']] = 0 & \textit{(Hom-Jacobi identity)} \end{array}$$

for all $a,a',a''\in A$.

Example 2.2 *a) Let (A,α_A) be an H-L algebra. If $\alpha_A = Id$, then (A,α_A) is a Lie algebra.*

b) Let W be a k-vector space, which has trivial bracket and linear map $\alpha:W\longrightarrow W$. In this case, (W,α_W) is called an abelian H-L algebra, [5].

Definition 2.3 *Let (A,α_A) and $(A',\alpha_{A'})$ be two H-L algebras. Then a $k-$linear map $f:A\longrightarrow A'$ is called a homomorphism of H-L algebra if*

$$\begin{array}{ll} (i) & f([a,a']) = [f(a),f(a')] \\ (ii) & f\alpha_A = \alpha_{A'}f \end{array}$$

for all $a,a'\in L$.

Thus, we can obtain the category of H-L algebras denoted by $\boldsymbol{Hom-Lie}$.

Definition 2.4 *Let (A,α_A) be an H-L algebra and H be a vector subspace of A. Then, (H,α_H) is called an H-L subalgebra of (A,α_A) if*

$$(i)\ [h,h']\in H \qquad (ii)\ \alpha_A(h)\in H$$

for all $h,h'\in H$.

Definition 2.5 *Let (A,α_A) be an H-L algebra and (H,α_H) be an H-L subalgebra of (A,α_A). Then, (H,α_H) is called an H-L ideal of (A,α_A) if $[h,a]\in H$, for all $h\in H$ and $a\in A$.*

Definition 2.6 *Let (A,α_A) be an H-L algebra and (H,α_H) be an H-L ideal of (A,α_A). Then, $(A/H,\overline{\alpha_A})$ is called quotient H-L algebra if the map $\overline{\alpha_A}:A/H\longrightarrow A/H$ induced by α_A, satisfies $[\overline{a},\overline{a'}] = \overline{[a,a']}$, for all $\overline{a},\overline{a'}\in A/H$.*

Definition 2.7 *Let (A,α_A) be an H-L algebra. The centre of (A,α_A) is H-L subalgebra*

$$Z(A) = \{a\in A : [\alpha^n(a),a'] = 0, \forall a'\in A, n\geq 0\}.$$

If α_A is epic, then $Z(A)$ is an H-L ideal of A.

2.2. Hom-Lie crossed modules

2.2.1. Actions and crossed modules

Definition 2.8 *Let (A,α_A) and (B,α_B) be two H-L algebras. An H-L action of (A,α_A) on (B,α_B) consists of a bilinear map $A\otimes B \longrightarrow B$,$a\otimes b\mapsto a\diamond b$ that satisfies following conditions.*

$$
\begin{array}{ll}
A_1) & [a,a']\diamond\alpha_B(b)=\alpha_A(a)\diamond(a'\diamond b)-\alpha_A(a')\diamond(a\diamond b)\\
A_2) & \alpha_A(a)\diamond[b,b']=[a\diamond b,\alpha_B(b')]+[\alpha_B(b),a\diamond b']\\
A_3) & \alpha_B(a\diamond b)=\alpha_A(a)\diamond\alpha_B(b)
\end{array}
$$

for all $a,a'\in A,\ b,b'\in B$.

Let (C,α_C) be an abelian H-L algebra and (A,α_A) be an H-L algebra. If there is an H-L action of (A,α_A) on (C,α_C), then (C,α_C) is called a Hom-A module, [15].

Example 2.9 *a) Let C be a Lie A-algebra, which has the action $\alpha : A\longrightarrow Der(C)$. Then, there is an H-L action of (A,Id_A) on (C,Id_C) via α.*

b) (A,α_A) be an H-L algebra, (B,α_B) be a Hom-subalgebra of (A,α_A) and (C,α_C) be a Hom-ideal of (A,α_A). Then there is H-L action of (B,α_B) on (C,α_C) given by the bracket in (A,α_A).

See [4, 15] for more examples.

Let (A,α_A) and (B,α_B) be two H-L algebras with H-L action of (A,α_A) on (B,α_B). In this situation, the semi-direct product $(B\rtimes A,\widetilde{\alpha})$ is defined on vector space $B\oplus A$ where

$$
\begin{array}{rcl}
\widetilde{\alpha}:B\oplus A & \longrightarrow & B\oplus A\\
(b,a) & \longmapsto & \widetilde{\alpha}(b,a)=(\alpha_B(b),\alpha_A(a))
\end{array}
$$

and the bracket operation is

$$[(b_1,a_1),(b_2,a_2)]=([b_1,b_2]+a_1\diamond b_2-a_2\diamond b_1,[a_1,a_2])$$

for all $a,a_1,a_2\in A,\ b,b_1,b_2\in B$.

Definition 2.10 *Let (A,α_A) and (B,α_B) be two H-L algebras with H-L action of (A,α_A) on (B,α_B). The triple $((B,\alpha_B),(A,\alpha_A),\mu)$ is called a crossed module of H-L algebras if the homomorphism of H-L algebra $\mu:(B,\alpha_B)\longrightarrow(A,\alpha_A)$ has the following identities:*

$$
\begin{array}{ll}
CM_1) & \mu(a\diamond b)=[a,\mu(b)]\\
CM_2) & \mu(b)\diamond b'=[b,b']
\end{array}
$$

for all $a,a'\in A,\ b,b'\in B$.

We will denote such a crossed module with (B,A,μ).

Example 2.11 *Let (C,α_C) be a Hom-L module and (A,α_A) be an H-L algebra. Then $(C,A,0)$ is a crossed module.*

Definition 2.12 *Let (B, A, μ) and (B', A', μ') be two crossed modules of H-L algebras. A homomorphism of crossed modules is a pair (f, g) where $f : B \longrightarrow B'$ and $g : A \longrightarrow A'$ are homomorphisms of H-L algebras such that:*

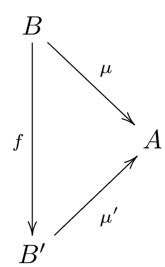
$$\begin{array}{ll} (i) & g\mu = \mu' f \\ (ii) & f(a \diamond b) = g(a) \diamond f(b) \end{array}$$

for all $a \in A,\ b \in B$.

Consequently, one can construct the category of crossed modules of H-L algebras denoted by $\boldsymbol{XHom - Lie}$, whose objects are crossed modules and morphisms are homomorphisms of crossed modules.

3. Categorical properties of $\boldsymbol{XHom - Lie}$

In this section, we will use the basic structures available in [7, 8]. Also, we will define the category $\boldsymbol{XH - L/A}$ crossed modules of H-L algebras with fixed base A. In this category, the objects are crossed modules of H-L algebras with fixed base A. An object (B, A, μ) of $\boldsymbol{XHom - Lie/A}$ is called crossed A-module and denoted by (B, μ) for short. A morphism between crossed A-modules (B, μ) and (B', μ') is a homomorphism of H-L algebras $f : B \longrightarrow B'$ such that the diagram



commutes.

Theorem 3.1 *In the category $\boldsymbol{XHom - Lie/A}$, every pair of parallel morphisms has equalizer.*

Proof Let $f, g : (B, \mu) \longrightarrow (B', \mu')$ be two morphisms between crossed A-modules. Define

$$C = \{b \in B : f(b) = g(b)\} \text{ and } \eta = \mu|_C : C \longrightarrow A.$$

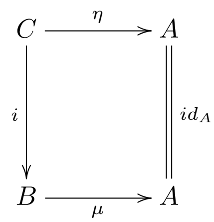
Since

$$\begin{array}{rcl} f(a \diamond c) & = & a \diamond f(c) \\ & = & a \diamond g(c) \\ & = & g(a \diamond c) \end{array}$$

for all $a \in A,\ c \in C$, we have $a \diamond c \in C$. On the other hand,

$$\begin{array}{ll} CM_1) & \eta(a \diamond c) = \mu(a \diamond c) = [a, \mu(c)] = [a, \eta(c)] \\ CM_2) & \eta(c) \diamond c' = \mu(c) \diamond c' = [c, c'] \end{array}$$

for all $a \in A$, $c, c' \in C$. Thus, (C, η) is a crossed A-module. Additionally, inclusion map $i : (C, \eta) \longrightarrow (B, \mu)$ is a morphism in $\boldsymbol{XHom - Lie/A}$, since the diagram



is commutative. Furthermore, let (C', η') be a crossed A-module and $j : (C', \eta') \longrightarrow (B, \mu)$ be a morphism of crossed A-modules such that $fj = gj$. Define the morphism

$$\begin{array}{rcl} k : C' & \longrightarrow & C \\ c' & \longmapsto & k(c') = j(c'). \end{array}$$

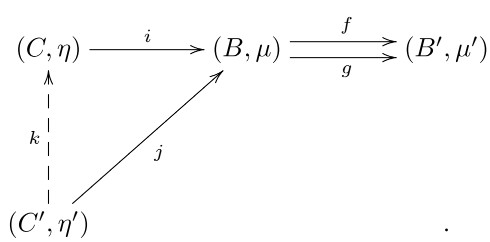
We have $ik = j$, since

$$\begin{array}{rcl} (ik)(c') & = & i(k(c')) \\ & = & k(c') \\ & = & j(c') \end{array}$$

for all $c' \in C'$. Now, we will show that k is unique. Suppose that there exist two morphisms $k_1, k_2 : C' \to C$ in $\boldsymbol{XHom - Lie/A}$ such that $ik_1 = j$ and $ik_2 = j$, Because of

$$\begin{array}{rcl} j(c') = (ik_1)(c') = (ik_2)(c') & \Longrightarrow & i(k_1(c')) = i(k_2(c')) \\ & \Longrightarrow & k_1(c') = k_2(c') \\ & \Longrightarrow & k_1 = k_2 \end{array}$$

for all $c' \in C'$, k is unique, as required. These morphisms can be pictured as follows;



□

Theorem 3.2 *The category* $\boldsymbol{XHom - Lie/A}$ *has pullbacks.*

Proof Let $f : (B, \mu) \longrightarrow (B', \mu')$ and $g : (C, \eta) \longrightarrow (B', \mu')$ be two morphisms in $\boldsymbol{XHom - Lie/A}$. Define

$$D = B \times_{B'} C = \{(b, c) : f(b) = g(c)\}$$

with bracket

$$[(b,c),(b',c')] = ([b,b'],[c,c'])$$

for all $(b,c),(b',c') \in D$, and

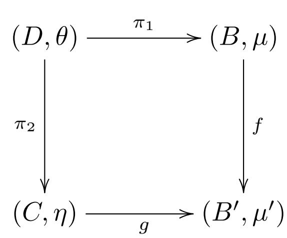
$$\begin{array}{rcl} \theta : D & \longrightarrow & A \\ (b,c) & \longmapsto & \theta(b,c) = \mu(b) = \eta(c) \end{array}$$

with the H-L action of A on D defined by $a \diamond (b,c) = (a \diamond b, a \diamond c)$ for all $a \in A$, $(b,c) \in D$. Moreover, (D,θ) is a crossed A-module, since

$$\begin{array}{lrcl} CM_1) & \theta(a \diamond (b,c)) & = & \theta(a \diamond b, a \diamond c) \\ & & = & \mu(a \diamond b) \\ & & = & [a, \mu(b)] \\ & & = & [a, \theta(b,c)] \end{array}$$

$$\begin{array}{lrcl} CM_2) & \theta(b,c) \diamond (b',c') & = & \mu(b) \diamond (b',c') \\ & & = & (\mu(b) \diamond b', \mu(b) \diamond c') \\ & & = & (\mu(b) \diamond b', \eta(c) \diamond n') \\ & & = & ([b,b'],[c,c']) \\ & & = & [(b,c),(b',c')] \end{array}$$

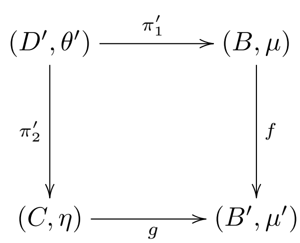
for all $(b,c),(b',c') \in D$. We get the commutative diagram of crossed A-modules where π_1, π_2 are projection maps because of $f(b) = g(c)$.



Here π_1 is a homomorphism of crossed A-module, since

$$\begin{array}{rcl} (\mu\pi_1)(b,c) & = & \mu(\pi_1(b,c)) \\ & = & \mu(b) \\ & = & \theta(b,c) \\ & = & (1_A\theta)(b,c) \end{array} \qquad \begin{array}{rcl} \pi_1(a \diamond (b,c)) & = & \pi_1(a \diamond b, a \diamond c) \\ & = & a \diamond b \\ & = & 1_A(a) \diamond \pi_1(b,c) \end{array}$$

for all $a \in A$, $(b,c) \in D$. With the same calculations, it can be show that π_2 is a homomorphism of crossed A-module, too. Now, for the crossed A-module (D',θ'), we have the following commutative diagram because of $f\pi_1' = g\pi_2'$.



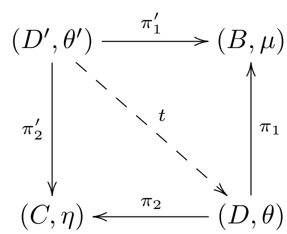
In this case, we have $(\pi_1'(b',c'),\pi_2'(b',c'))\in D$, since

$$\begin{aligned}(f\pi_1')(b',c') &= f(\pi_1'(b',c'))\\ &= f(b')\\ &= g(c')\\ &= g(\pi_2'(b',c'))\\ &= (g\pi_2')(b',c')\end{aligned}$$

for all $(b',c')\in D$. Define the morphism of A-module t as

$$\begin{aligned} t: D' &\longrightarrow D\\ u &\longmapsto t(u)=(\pi_1'(u),\pi_2'(u))\end{aligned}$$

such that in the diagram



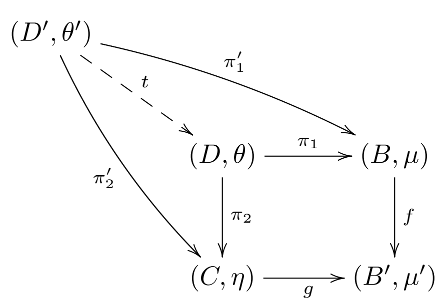
upper triangle and lower triangle diagrams are commutative. Now, we will show that t is unique. Suppose that t_1 and t_2 have same properties. In this case, due to

$$\pi_1 t_1=\pi_1' \text{ and } \pi_1 t_2=\pi_1'$$

and

$$\begin{aligned} t_1(u) &= (\pi_1'(u),\pi_2'(u))\\ &= t_2(u)\end{aligned}$$

for all $u\in D'$, we get $t_1=t_2$, i.e. t is unique. These morphisms can be pictured as follows:



□

Theorem 3.3 *The category* $\boldsymbol{XHom-Lie/A}$ *has zero object.*

Proof Let (A,α) be a crossed A-module where $\alpha : A \longrightarrow A,\ \alpha(a) = a$, for all $a \in A$. We will show that (A,α) is initial object in this category. Assume that (B,μ) be any crossed A-module, and $f : A \longrightarrow B$ be a homomorphism of H-L algebra. Define the action

$$\begin{array}{rcl} A \times B & \longrightarrow & B \\ (a,b) & \longmapsto & a \diamond b = f(a)b. \end{array}$$

So, (f, Id) is a morphism of crossed module, i.e. (A,α) is initial object. Similarly, it can be shown that (A,α) is terminal object, as required. □

Theorem 3.4 *The category* $\boldsymbol{XHom-Lie/A}$ *has finite products.*

Proof Let (B,μ) and (C,η) be two crossed A-module. The cartesian product of B and C is the set

$$B \times C = \{(b,c) : b \in B,\ c \in C\}.$$

Define

$$\begin{array}{rcl} \theta : B \times_A C & \longrightarrow & A \\ (b,c) & \longmapsto & \theta(b,c) = \mu(b) = \eta(c) \end{array}$$

with H-L action

$$\begin{array}{rcl} A \times (B \times_A C) & \longrightarrow & A \\ (a,(b,c)) & \longmapsto & a \diamond (b,c) = (a \diamond b, a \diamond c). \end{array}$$

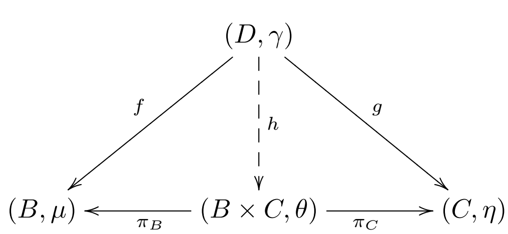
It is clear that $(B \times C, \theta)$ is a crossed A-module. So, we have

$$(B,\mu) \xleftarrow{\ \pi_B\ } (B \times C, \theta) \xrightarrow{\ \pi_C\ } (C,\eta)$$

where

$$\begin{array}{rcl} \pi_B : B \times C & \longrightarrow & B \\ (b,c) & \longmapsto & b \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi_C : B \times C & \longrightarrow & C \\ (b,c) & \longmapsto & c \end{array}$$

are canonical projections. It is clear that π_B and π_C are homomorphisms of crossed A-module. Let (D,γ) be an arbitrary crossed A-module such that the diagram



is commutative where f and g are arbitrary homomorphisms of crossed A-module and we define h as

$$\begin{array}{rcl} h : D & \longrightarrow & B \times C \\ d & \longmapsto & h(d) = (f(d), g(d)). \end{array}$$

Since,

$$\begin{aligned} (\pi_B h)(d) &= \pi_B(h(d)) \\ &= \pi_B(f(d), g(d)) \\ &= f(d) \end{aligned} \quad \text{and} \quad \begin{aligned} (\pi_C h)(d) &= \pi_C(h(d)) \\ &= \pi_C(f(d), g(d)) \\ &= g(d) \end{aligned}$$

we have $\pi_B h = f$ and $\pi_C h = g$. At the same time, h is a homomorphism of crossed A-module. Because of

$$\begin{aligned} h(a \diamond d) &= (f(a \diamond d), g(a \diamond d)) \\ &= (a \diamond f(d), a \diamond g(d)) \\ &= a \diamond (f(d), g(d)) \\ &= a \diamond h(d) \end{aligned}$$

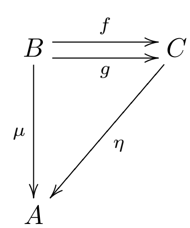
for all $a \in A$, $d \in D$. Also, h is unique. Thus, $(B \times C, \theta)$ is product of (B, μ) and (C, η). Then, by induction $\boldsymbol{XHom - Lie/A}$ has finite products. □

Corollary 3.5 *The category* $\boldsymbol{XHom - Lie/A}$ *is finitely complete, i.e. it has all finite limits.*

Proof As the category $\boldsymbol{XHom - Lie/A}$ has finite products and equalizers, it is finitely complete. □

Theorem 3.6 *In the category* $\boldsymbol{XHom - Lie/A}$*, every pair of parallel morphisms has coequalizer.*

Proof Let $f, g : (B, \mu) \longrightarrow (C, \eta)$ be two morphisms in $\boldsymbol{XHom - Lie/A}$ and J be the ideal generated by $\{f(b) - g(b) : b \in B\}$. As the diagram



commutes and $\eta(f(b) - g(b)) = 0$, for all $b \in B$, we have $J \subseteq \ker \eta$. Consider an H-L algebra C/J with bracket on C/J

$$[c + J, c' + J] = [c, c'] + J$$

for all $c + J, c' + J \in C/J$. Define

$$\begin{aligned} \alpha : C/J &\longrightarrow A \\ c + J &\longmapsto \alpha(c + J) = \eta(c) \end{aligned}$$

with H-L action A on C/J

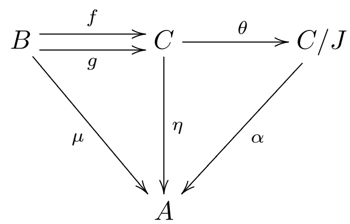
$$\begin{aligned} A \times C/J &\longrightarrow C/J \\ (a, c + J) &\longmapsto a \diamond (c + J) = a \diamond c + J \end{aligned}$$

for all $a \in A$, $c + J \in C/J$. In this case, $(C/J, \alpha)$ is a crossed A-module as

$$\begin{aligned} CM_1) \quad \alpha(a \diamond (c+J)) &= \alpha(a \diamond c + J) \\ &= \eta(a \diamond c) \\ &= [a, \eta(c)] \\ &= [a, \alpha(c+J)] \end{aligned}$$

$$\begin{aligned} CM_2) \quad \alpha(c+J) \diamond (c'+J) &= \eta(c) \diamond (c'+J) \\ &= \eta(c) \diamond c' + J \\ &= [c, c'] + J \\ &= [c+J, c'+J] \end{aligned}$$

for all $c+J, c'+J \in C/J$. Thus, we obtain the following commutative diagram because of the definitions of α and θ where θ is a morphism such that $\theta(c) = c + J$.



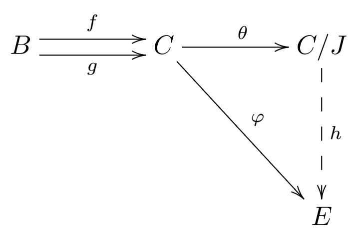
Consider the homomorphism of crossed A-module $\varphi : C \longrightarrow E$ satisfying $\varphi f = \varphi g$ and define the morphism h as

$$\begin{aligned} h : C/J &\longrightarrow E \\ c + J &\longmapsto h(c+J) = \varphi(c). \end{aligned}$$

Since

$$\begin{aligned} (h\theta)(c) &= h(\theta(c)) \\ &= h(c+J) \\ &= \varphi(c) \end{aligned}$$

for all $c \in C$, we have $h\theta = \varphi$. In an other saying, the diagram



commutes. Also, as

$$\begin{aligned} h_1\theta = \varphi = h_2\theta &\implies (h_1\theta)(c) = (h_2\theta)(c) \\ &\implies h_1(c+J) = h_2(c+J) \\ &\implies h_1 = h_2 \end{aligned}$$

for all $c \in C$, h is unique. □

Theorem 3.7 $\boldsymbol{XHom-Lie/A}$ *has finite coproducts.*

Proof Let (B,μ) and (C,η) be two crossed A-modules. So, there is an action of B on C via η such that $c\diamond b=\eta(c)\diamond b$, for all $b\in B$, $c\in C$. With the help of this action, we obtained an H-L algebra

$$B\rtimes C=\{(b,c):b\in B,c\in C\}$$

with the bracket

$$\begin{array}{rcl}[(b,c),(b',c')] & = & ([b,b']+c\diamond b'-c'\diamond b,[c,c']) \\ & = & ([b,b']+\eta(c)\diamond b'-\eta(c')\diamond b,[c,c'])\end{array}$$

for all $(b,c),(b',c')\in B\rtimes C$. Also, action of A on $B\rtimes C$ is

$$a\diamond(b,c)=(a\diamond b,a\diamond c)$$

for all $a\in A$, $(b,c)\in B\rtimes C$. It is clear that this action is a H-L action. Here, there are two morphisms

$$\begin{array}{rclcrcl} i:C & \longrightarrow & B\rtimes C & \qquad & j:B & \longrightarrow & B\rtimes C \\ c & \longmapsto & (0,c) & & b & \longmapsto & (b,0),\end{array}$$

and these morphisms are homomorphisms of H-L algebra, since

$$\begin{array}{rclcrcl} i([c,c']) & = & (0,[c,c']) & \qquad & j([b,b']) & = & ([b,b'],0) \\ & = & ([0,0],[c,c']) & & & = & ([b,b'],[0,0]) \\ & = & [(0,c),(0,c')] & & & = & [(b,0),(b',0)] \\ & = & [i(c),i(c')] & & & = & [j(b),j(b')])\end{array}$$

for all $b,b'\in B$, $c,c'\in C$. Now, consider the map

$$\begin{array}{rcl}\theta':B\rtimes C & \longrightarrow & A \\ (b,c) & \longmapsto & \theta'(b,c)=\mu(b)+\eta(c),\end{array}$$

and the ideal J of $B\rtimes C$ generated by the elements of the form $(\mu(b')\diamond c,\eta(c)\diamond b')$, for all $b'\in B$, $c\in C$. Define

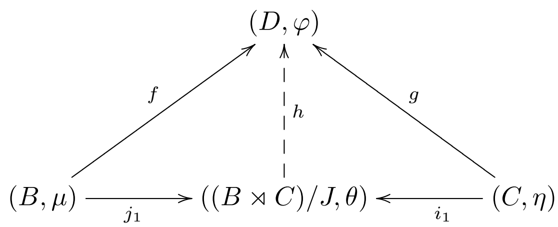
$$\begin{array}{rcl}\theta:(B\rtimes C)/J & \longrightarrow & A \\ (b,c)+J & \longmapsto & \theta((b,c)+J)=\theta'(b,c)=\mu(b)+\eta(c).\end{array}$$

Thus, we obtained crossed A-module $((B\rtimes C)/J,\theta)$. That is,

$$\begin{array}{rcl} CM_1)\ \theta(a\diamond((b,c)+J)) & = & \theta((a\diamond b,a\diamond c)+J)) \\ & = & \theta'(a\diamond b,a\diamond c) \\ & = & \mu(a\diamond b)+\eta(a\diamond c) \\ & = & [a,\mu(b)]+[a,\eta(c)] \\ & = & [a,\mu(b)+\eta(c)] \\ & = & [a,\theta'(b,c)] \\ & = & [a,\theta((b,c)+J)]\end{array}$$

$$
\begin{aligned}
CM_2)\ \theta((b,c)+J))\diamond((b',c')+J) &= \theta'(b,c)\diamond((b',c')+J)\\
&= (\mu(b)+\eta(c))\diamond((b',c')+J)\\
&= ((\mu(b)+\eta(c))\diamond b',(\mu(b)+\eta(c))\diamond c')\\
&= (\mu(b)\diamond b'+\eta(c)\diamond b',\mu(b)\diamond c'+\eta(c)\diamond c')\\
&= ([b,b']+\eta(c)\diamond b',\mu(b)\diamond c'+[c,c'])\\
&= ([b,b'],[c,c'])+(\eta(c)\diamond b',\mu(b)\diamond c')\\
&= ([b,b'],[c,c'])+J\\
&= [(b,c),(b',c')]+J\\
&= [(b,c)+I,(b',c')+J]
\end{aligned}
$$

for all $a\in A$, $(b,c),(b',c')\in B\rtimes C$. On the other hand, consider the diagram



where

$$
\begin{aligned}
i_1: C &\longrightarrow (B\rtimes C)/J & \qquad j_1: B &\longrightarrow (B\rtimes C)/J\\
c &\longmapsto (0,c)+J & b &\longmapsto (b,0)+J,
\end{aligned}
$$

and we define h as

$$
\begin{aligned}
h:(B\rtimes C)/J &\longrightarrow D\\
(b,c)+J &\longmapsto h((b,c)+J)=f(b)+g(c).
\end{aligned}
$$

So, above diagram is commutative, because

$$
\begin{aligned}
(hi_1)(c) &= h(i_1(c)) & \qquad (hj_1)(b) &= h(j_1(b))\\
&= h((0,c)+J) & &= h((b,0)+J)\\
&= f(0)+g(c) & &= f(b)+g(0)\\
&= g(c) & &= f(b)
\end{aligned}
$$

for all $b\in B$, $c\in C$. Also, it is clear that h is unique. All this means that $((B\rtimes C)/J,\theta)$ is coproduct of (B,μ) and (C,η). □

Corollary 3.8 *The category* $\boldsymbol{XHom-Lie/A}$ *is finitely cocomplete.*

Proof Since it has coequalizer and finite coproducts, it is finitely cocomplete. □

Theorem 3.9 *The category* $\boldsymbol{XHom-Lie/A}$ *has pushouts.*

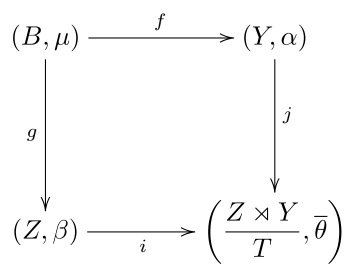
Proof Let $f:(B,\mu)\longrightarrow(Y,\alpha)$ and $g:(B,\mu)\longrightarrow(Z,\beta)$ be two morphisms of crossed A-module and T be an ideal generated by elements of the forms $(\alpha(y)\diamond z,\beta(z)\diamond y)$ and $(g(b),-f(b))$, for all $y\in Y$, $z\in Z$, $b\in B$. Define

$$
\begin{aligned}
\overline{\theta}:\frac{Z\rtimes Y}{T} &\longrightarrow A\\
(z,y)+T &\longmapsto \overline{\theta}((z,y)+T)=\alpha(y)+\beta(z).
\end{aligned}
$$

It is clear that $\left(\dfrac{Z \rtimes Y}{T}, \overline{\theta}\right)$ is a crossed A-module. At the same time, it can be defined two morphisms

$$\begin{array}{rcl} i : Z & \longrightarrow & \dfrac{Z \rtimes Y}{T} \\ z & \longmapsto & (z,0)+T \end{array} \quad \text{and} \quad \begin{array}{rcl} j : Y & \longrightarrow & \dfrac{Z \rtimes Y}{T} \\ y & \longmapsto & (0,y)+T. \end{array}$$

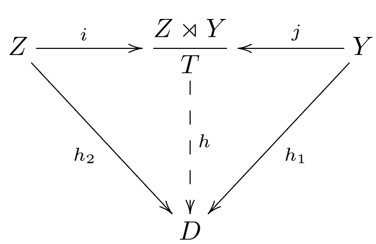
In this case, as $(g(b),-f(b)) = -\left(0,f(b)\right)+(g(b),0) \in T$, for all $b \in B$, we have $jf = ig$, i.e the diagram



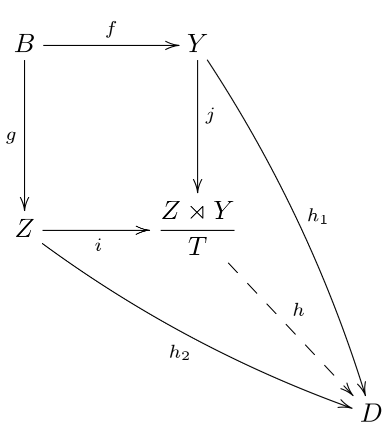
commutes. On the other hand, take the morphisms $h_1 : Y \longrightarrow D$, $h_2 : Z \longrightarrow D$ and with the help of these morphisms, define the morphism of A-module h as

$$\begin{array}{rcl} h : \dfrac{Z \rtimes Y}{T} & \longrightarrow & D \\ (z,y)+T & \longmapsto & h((z,y)+T) = h_1(y) + h_2(z). \end{array}$$

such that, in the diagram,



left triangle and right triangle diagrams are commutative. Thus, h is unique and the diagram



commutes, as required. $\square$

4. Conclusion

In this paper, we investigated the categorical properties of category of crossed A-module $\boldsymbol{XHom-Lie/A}$, which is a full subcategory of category of crossed modules over H-L algebras $\boldsymbol{XHom-Lie}$. On the other hand, it can be investigated whether the category $\boldsymbol{XHom-Lie}$ is cartesian closed category and modified category of interest or not.

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