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On the paper "A study on (strong) order-congruences in ordered semihypergroups"

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To the memory of my teachers Professor Nazım Terzioğlu and Professor Suzan Kahramaner

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Abstract: Throughout the paper in the title by Jian Tang, Yanfeng Luo and Xiangyun Xie in Turk J Math 42 (2018) the following lemma has been used. **Lemma:** *Let* $(S,*)$ *be a semihypergroup and* ρ *an equivalence relation on S. Then* (i) *If* ρ *is a congruence, then* $(S/\rho,\otimes)$ *is a semihypergroup with respect to the hyperoperation* $(a)_\rho\otimes(b)_\rho=\bigcup\limits_{c\in a*b}(c)_\rho$. (ii) *If* ρ *is a strong congruence, then* $(S/\rho,\otimes)$ *is a semigroup with respect to the operation* $(a)_\rho\otimes(b)_\rho=(c)_\rho$ *for all* $c\in a*b$. The property (i) of the paper is certainly wrong as $\bigcup\limits_{c\in a*b}(c)_\rho$ is a subset of S and not a nonempty subset of S/ρ as it should be. Property (ii) has no sense in the way is written. In addition, according to the authors, as an application of the results of this paper they solved the open problem on ordered semihypergroups given by Davvaz, Corsini and Changphas in European J Combin 44 (2015). The problem is that the above mentioned problem has not been solved in the above mentioned article; we point out the reason, and we solve it in the present paper. Some further related results; also results necessarily for the completeness of the paper are given. Examples illustrate the results.

Key words: Ordered hypersemigroup, congruence, pseudoorder, quasi pseudoorder, strong congruence

1. Introduction

"Congruence relation" is the standard term in semigroups or ordered semigroups, and this is what Tang, Luo and Xie use for ordered hypersemigroups in Turk J Math [13]. However, in the next paper by Tang and Xie joint with Feng and Davvaz in Open Math [12] they use the terminology "regular relation" and this is the terminology has been used by Davvaz, Corsini and Changphas in European J Combin [2] as well. In the present paper we refer to [2, 12, 13] so, to avoid any misunderstanding, we will use the standard notion of semigroups "congruence" (we try to extend); we will also write "hypersemigroup" instead of "semihypergroup".

A *congruence* on a groupoid or an ordered groupoid S is an equivalence relation σ on S such that $(a,b)\in\sigma$ implies $(ac,bc)\in\sigma$ and $(ca,cb)\in\sigma$ for every $c\in S$ [1, 5, 11]. When we pass from a groupoid $(S,\cdot)$ to an hypergroupoid $(S,\circ)$ or from an ordered groupoid $(S,\cdot,\leq)$ to an ordered hypergroupoid $(S,\circ,\leq)$, the congruence should, naturally, defined by any one of the following two definitions:

Definition 1.1. *Let* S *be an hypergroupoid or an ordered hypergroupoid. An equivalence relation* ρ *on* S *is called congruence if* $(a,b)\in\rho$ *implies* $(a\circ c,b\circ c)\in\rho$ *and* $(c\circ a,c\circ b)\in\rho$ *for every* $c\in S$*; in the sense that*

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for every $u \in a \circ c$ *and every* $v \in b \circ c$ *we have* $(u,v) \in \rho$ *and for every* $u \in c \circ a$ *and every* $v \in c \circ b$ *we have* $(u,v) \in \rho$.

Definition 1.2. *Let* S *be an hypergroupoid or an ordered hypergroupoid. An equivalence relation* ρ *on* S *is called congruence if* $(a,b) \in \rho$ *implies* $(a \circ c, b \circ c) \in \rho$ *and* $(c \circ a, c \circ b) \in \rho$ *for every* $c \in S$*; in the sense that for every* $u \in a \circ c$ *there exists* $v \in b \circ c$ *such that* $(u,v) \in \rho$ *and for every* $u \in c \circ a$ *there exists* $v \in c \circ b$ *such that* $(u,v) \in \rho$.

Each of the above two definitions, being of equal importance, can be considered as the definition of congruence. According to the problem we want to solve, we can select any of them and call it "congruence" at the beginning of the paper. We do not use both of them in the same paper to distinguish them by different names. However, since Definition 1.1 implies Definition 1.2, when is convenient, we can call "strong congruence" the notion defined by Definition 1.1 and "congruence" that one defined by Definition 1.2 [2, 12, 13].

Let us come now to the concept of pseudoorder introduced in [9] (see also [10]). A relation σ on an ordered groupoid $(S,\cdot,\leq)$ is called *pseudoorder* if we have the following: (1) $\leq \subseteq \sigma$; (2) if $(a,b) \in \sigma$ and $(b,c) \in \sigma$, then $(a,c) \in \sigma$; (3) if $(a,b) \in \sigma$ and $c \in S$, then $(ac,bc) \in \sigma$ and $(ca,cb) \in \sigma$. When we pass from an ordered groupoid $(S,\cdot,\leq)$ to an ordered hypergroupoid $(S,\circ,\leq)$, the pseudoorder can be defined by any one of the following two definitions (and not only -see, for example, Definition 3.1 below).

Definition 1.3. *Let* $(S,\circ,\leq)$ *be an ordered hypergroupoid. A relation* σ *on* S *is called pseudoorder if we have the following:*

(1) $\leq \subseteq \sigma$

(2) if $(a,b) \in \sigma$ *and* $(b,c) \in \sigma$*, then* $(a,c) \in \sigma$

(3) if $(a,b) \in \sigma$ *and* $c \in S$*, then* $(a \circ c, b \circ c) \in \sigma$ *and* $(c \circ a, c \circ b) \in \sigma$*; in the sense that for every* $u \in a \circ c$ *and every* $v \in b \circ c$ *we have* $(u,v) \in \sigma$ *and for every* $u \in c \circ a$ *and every* $v \in c \circ b$ *we have* $(u,v) \in \sigma$.

Definition 1.4. *Let* $(S,\circ,\leq)$ *be an ordered hypergroupoid. A relation* σ *on* S *is called pseudoorder if we have the following:*

(1) $\leq \subseteq \sigma$

(2) if $(a,b) \in \sigma$ *and* $(b,c) \in \sigma$*, then* $(a,c) \in \sigma$

(3) if $(a,b) \in \sigma$ *and* $c \in S$*, then* $(a \circ c, b \circ c) \in \sigma$ *and* $(c \circ a, c \circ b) \in \sigma$*; in the sense that for every* $u \in a \circ c$ *there exists* $v \in b \circ c$ *such that* $(u,v) \in \sigma$ *and for every* $u \in c \circ a$ *there exists* $v \in c \circ b$ *such that* $(u,v) \in \sigma$.

Each of the above two definitions, being of equal importance, can be considered as the definition of pseudoorder.

Remark 1.5. *Definition 1.1 implies Definition 1.2. In fact: Assuming* ρ *is a congruence in the sense of Definition 1.1, let* $(a,b) \in \rho$, $c \in S$*, and* $u \in a \circ c$*. Take an element* $v \in b \circ c$ $(b \circ c \neq \emptyset)$*. By Definition 1.1, we have* $(u,v) \in \rho$*. Thus we have* $(a \circ c, b \circ c) \in \rho$*. Similarly* $(c \circ a, c \circ b) \in \rho$.

As in the case of congruences, Definition 1.3 implies Definition 1.4.

To be precise with the corresponding definitions of congruences we already gave, when is convenient, we will call "strong pseudoorder" the pseudoorder defined by Definition 1.3 and "pseudoorder" that one defined by Definition 1.4. However, to keep the terminology given by the authors in [2], and avoid any misunderstanding, in particular for this paper, we will call "pseudoorder" the pseudoorder defined by Definition 1.3.

It is well known that if S is a semigroup and σ a congruence on S, then the set $S/\sigma := \{(a)_\sigma \mid a \in S\}$ of all σ-classes of S is a semigroup [1, 11]. It was interesting to know if a similar result holds in case of ordered semigroups as well. So, in the paper "On subdirectly irreducible ordered semigroups" in Semigroup Forum [9], the author of the present paper put the following problem: If $(S, \cdot, \leq)$ is an ordered semigroup and σ is a congruence on S, then is the set S/σ an ordered semigroup? An example has been first given in [9] to show that the probably order $\preceq$ on S/σ defined by means of the order "$\leq$" of S as

$$(x)_\sigma \preceq (y)_\sigma \text{ if and only if there exist } a \in (x)_\sigma \text{ and } b \in (y)_\sigma \text{ such that } a \leq b$$

is not an order in general. Then, searching for conditions under which there exists a congruence σ on S such that S/σ is an ordered semigroup, the concept of *pseudoorder* has been introduced and the following lemma has been proved.

The Lemma 1 in [9]: *Let $(S, \cdot, \leq)$ be an ordered semigroup and σ a pseudoorder on S. Then the relation ρ on S defined by*

$$\rho := \{(a, b) \mid (a, b) \in \sigma \text{ and } (b, a) \in \sigma\}$$

is a congruence on S and the set S/ρ with the usual multiplication $(x)_\rho \cdot (y)_\rho = (xy)_\rho$ and the order "$\preceq$" on S/ρ defined by

$$\preceq := \{((x)_\rho, (y)_\rho \mid \exists\, a \in (x)_\rho,\, b \in (y)_\rho \text{ such that } (a, b) \in \sigma\}$$

equivalently by any one of the following two equivalent definitions:

$$\preceq := \{((x)_\rho, (y)_\rho) \mid \forall\, a \in (x)_\rho \forall\, b \in (y)_\rho,\, (a, b) \in \sigma\}$$

or

$$(x)_\rho \preceq (y)_\rho \Leftrightarrow (x, y) \in \sigma$$

is an ordered semigroup.

If we want to generalize this result to ordered hypersemigroups (as in [2, 12, 13]), then this is the corresponding problem: If $(S, \circ, \leq)$ is an ordered hypersemigroup and σ a pseudoorder on S, is there a congruence relation ρ on S for which the set S/ρ is an ordered hypersemigroup? But of course we have to define the concept of congruence and the concept of pseudoorder in case of ordered hypersemigroups before we face the problem.

The Lemma 2.2 in [13] the authors use throughout the paper (sections $[2, 3, 4]$) is the following: Let $(S, \circ)$ be an hypersemigroup and ρ an equivalence relation on S. Then

(1) if ρ is a congruence in the sense of Definition 1.2, then $(S/\rho, \otimes)$ is an hypersemigroup with respect to the hyperoperation $(a)_\rho \otimes (b)_\rho = \bigcup\limits_{c \in a \circ b} (c)_\rho$.

(2) if ρ is a congruence in the sense of Definition 1.1, then $(S/\rho, \otimes)$ is a semigroup with respect to the hyperoperation $(a)_\rho \otimes (b)_\rho = (c)_\rho$ for all $c \in a \circ b$.

The property (1) of the above lemma is certainly wrong as $\bigcup_{c\in a\circ b}(c)_\rho$ is a subset of S and not an element of $\mathcal{P}^*(S/\rho)$ as it should be; so condition (1) of Lemma 2.2 in [13] cannot define an hyperoperation. Indeed, an hypersemigroup is an hypergroupoid having the "associativity property" that makes it hypersemigroup; and an hypergroupoid is a nonempty set S with an hyperoperation "$\circ$" that assigns to each couple a, b of elements of S a nonempty subset of S. Look at Table 6 of the Example 3.3 below, for example. We have $(b)_\rho \odot (d)_\rho = \{(b)_\rho, (d)_\rho\} \subseteq S/\rho$, while according to [12, 13],

$$(b)_\rho \odot (d)_\rho = \bigcup_{u\in b\circ d}(u)_\rho = \bigcup_{u\in\{b,d\}}(u)_\rho = (b)_\rho \cup (d)_\rho = \{b, c, d\} \subseteq S.$$

Property (2) has no sense in the way is written (it is not well defined).

Davvaz, Corsini and Changphas considered the Lemma 1 in Semigroup Forum [9] to extend it to ordered hypersemigroups [2, Theorem 4.2]. And as to a congruence of an ordered semigroup correspond two concepts of congruences in case of an ordered hypersemigroup, they dealt with one of them –the strong congruence– and put the case of a congruence (Definition 1.2) as an open problem. In fact, they proved that if $(S, \circ, \leq)$ is an ordered hypersemigroup and σ a pseudoorder on S (Definition 1.3), then the relation ρ on S defined by

$$\rho := \{(a, b) \mid (a, b) \in \sigma \text{ and } (b, a) \in \sigma\}$$

is a strong congruence (Definition 1.1) on S and S/ρ is an *ordered semigroup*. Their proof is on the line of Lemma 1 in [9], but they use the second property of the above mentioned lemma in it, in which the definition $(a)_\rho \otimes (b)_\rho = (c)_\rho$ *"for all* $c \in a \circ b$*"* has no sense in the way is written. In the present paper we show that S/ρ is not only an ordered semigroup, but it is an ordered hypersemigroup as it was expected to be. The open problem by Davvaz et al. in [2] is the following: If $(S, \circ, \leq)$ is an ordered hypersemigroup, is there a congruence relation (Definition 1.2) ρ on S such that S/ρ is an ordered hypersemigroup? (as we see, they ask if S/σ is an "ordered hypersemigroup" and not an "ordered semigroup" as it is proved in [2].)

According to J Tang, Luo and Xie, as an application of their results in Turk J Math [13] they solved the open problem by Davvaz, Corsini and Changphas. This is the Theorem 4.10 in [13] (there is a typing mistake in [13], it should actually be Theorem 4.11); the same with the Theorem 4.1 by Gu and X. Tang in J. Algebra [3]. There is no proof of their theorem in Turk J Math, they just say that its proof is similar to the proof of Theorem 4.8 of the same paper. But the proof of Theorem 4.8 is wrong as the property (1) of the above mentioned Lemma 2.2 in [13] has been used in it. Later, in Open Math [12, Theorem 3.8] J. Tang and Xie –two of the authors in Turk J Math, joint with Feng and Davvaz claimed that they completely solved the problem this time as the proof by Gu and X. Tang in J Algebra [3] provided only a partial solution while an hypersemigroup does not contain proper ideals in general. The proof in [3] mimics the proof of the corresponding result on ordered semigroups by Xie in Semigroup Forum [14, Proposition 2.7]. But Gu and X. Tang have not assumed that the ideal in their theorem is proper. If the ideal in [3] is not proper, then the relation $\rho_S = S \times S$ is still a congruence relation on S. In addition, the proof of Theorem 3.8 by J. Tang, Feng, Davvaz and Xie in Open Math [12], also based on the same Lemma 2.2 in [13] is again wrong, and the problem is still open. Besides, the proof of the problem being not unique in general, we cannot say "we completely solved the problem", we can just say "we gave a solution (or the best possible solution) to the problem".

As far as the notion of congruence relation is concerned, this is defined in [2, 12, 13] as follows: "An equivalence relation ρ on an hypersemigroup S is called *congruence* if $x\rho y$ implies $a \circ x\overline{\rho} a \circ y$ and $x \circ a\overline{\rho} y \circ a$

for every $a \in S$, while for nonempty subsets A, B of S, $A\overline{\rho}B$ means that for every $a \in A$ there exists $b \in B$ such that $a\rho b$ *and* for every $b \in B$ there exists $a \in A$ such that $a\rho b$". So the authors define the congruence relation as an equivalence relation such that :

$x\rho y$ implies (1) for every $u \in a \circ x$ there exists $v \in a \circ y$ such that $u\rho v$;
(2) for every $u \in a \circ y$ there exists $v \in a \circ x$ such that $u\rho v$;
(3) for every $u \in x \circ a$ there exists $v \in y \circ a$ such that $u\rho v$ and
(4) for every $u \in y \circ a$ there exists $v \in x \circ a$ such that $u\rho v$.

But, as ρ is an equivalence relation, it is clearly symmetric and so $x\rho y$ and (1) imply (2); and $x\rho y$ and (3) imply (4); so unnecessary properties should be omitted from [2, 12, 13].

The present paper is also written to correct the Lemma 2.2 in [12, 13] which plays an essential role in the investigation of hypersemigroups, in general. This is done in Theorem 2.1 and Proposition 2.2 of the second section. Using Theorem 2.1 and Definitions 1.2 and 3.1, we get the Theorem 3.2 that gives the best possible answer to the open problem by Davvaz, Corsini and Changphas [2] (a natural problem corresponding to that one on ordered semigroups [9]). Again using Theorem 2.1, Proposition 2.2 and Definitions 1.1 and 1.3 we obtain Theorem 3.4 that shows that in European J Compin [2, Theorem 4.2], S/ρ is not only an ordered semigroup but an ordered hypersemigroup as it was expected to be. Moreover we prove that the Lemma 1 in Semigroup Forum [9] can be also obtained as application of Theorem 3.4; though after using the correct definitions and the results given in the prerequisites' section, both Theorem 3.2 and Theorem 3.4 go on the line of the corresponding result on ordered semigroups –the Lemma 1 in [9]. Illustrative examples to section 2 as well as to the main Theorems 3.2 and 3.4 are given.

2. Prerequisites

For the sake of completeness, let us begin with the definition of an ordered hypersemigroup.

An *hypergroupoid* is a nonempty set S with an hyperoperation "$\circ$" (to each $a, b \in S$ corresponds a unique nonempty subset $a \circ b$ of S) and an operation "$*$" between the nonempty subsets of S defined by

$$A * B = \bigcup_{a \in A, b \in B} a \circ b \tag{2.1}$$

We clearly have $x \in A * B$ if and only if $x \in a \circ b$ for some $a \in A$, $b \in B$ which we often use. We also have $\{a\} * \{b\} = a \circ b$ for any $a, b \in S$. For nonempty subsets A, B, C of S, $A \subseteq B$ implies $A * C \subseteq B * C$ and $C * A \subseteq C * B$. An hypergroupoid S is called *hypersemigroup* if, for any $a, b, c \in S$, we have

$$(a \circ b) * \{c\} = \{a\} * (b \circ c) \tag{2.2}$$

Taking into account the equality in (2.1), condition (2.2) can be also written as follows: $x \in u \circ c$ for some $u \in a \circ b$ if and only if $x \in a \circ v$ for some $v \in b \circ c$; in other words,

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

However, we do not use this last equality in the investigation in which we always use condition (2.2). Identifying the singletons by the element they contain we can write, for short, $(a \circ b) * c = a * (b \circ c)$. Recall that if S is an hypergroupoid, A, B, C, D nonempty subsets of S and ρ a congruence on S either in the sense of Definition 1.1 or in the sense of Definition 1.2, then $(A,B) \in \rho$ and $(B,C) \in \rho \Rightarrow (A,C) \in \rho$. As a natural extension of the concept of an ordered groupoid, an *ordered hypergroupoid* is an hypergroupoid $(S,\circ)$ with an order relation "$\le$"on S such that $a \le b$ implies $a \circ c \preceq b \circ c$ and $c \circ a \preceq c \circ b$ for all $c \in S$, in the sense that if $u \in a \circ c$ then there exists $v \in b \circ c$ such that $u \le v$ and if $u \in c \circ a$ then there exists $v \in c \circ b$ such that $u \le v$ [4]. An ordered hypergroupoid satisfying the relation (2.2) is called an *ordered hypersemigroup*.

In this section, unless otherwise is stated, when we say "ρ is a congruence on S" we mean that ρ is a congruence either in the sense of Definition 1.1 or in the sense of Definition 1.2.

In the next theorem, $\mathcal{P}^*(S/\rho)$ denotes the set of nonempty subsets of S/ρ.

Theorem 2.1. *Let $(S,\circ)$ be an hypergroupoid and ρ a congruence on S. Let $\odot$ be the hyperoperation on S/ρ defined by*

$$\odot : S/\rho \times S/\rho \to \mathcal{P}^*(S/\rho) \mid \Big((a)_\rho, (b)_\rho\Big) \to (a)_\rho \odot (b)_\rho, \text{ where}$$

$$(a)_\rho \odot (b)_\rho := \{(c)_\rho \mid c \in a \circ b\}.$$

Then $(S/\rho, \odot)$ is an hypergroupoid. If now $(S,\circ)$ is an hypersemigroup, then the set $(S/\rho, \odot)$ is an hypersemigroup as well.

Proof. The operation $\odot$ is well defined. In fact: First of all, we have $\emptyset \ne (a)_\rho \odot (b)_\rho \subseteq S/\rho$. Indeed: Take an element $c \in a \circ b$ $(a \circ b \ne \emptyset)$. Then $(c)_\rho \in (a)_\rho \odot (b)_\rho$, so $(a)_\rho \odot (b)_\rho$ is a nonempty set. If $x \in (a)_\rho \odot (b)_\rho$, then $x = (d)_\rho$ for some $d \in a \circ b (\subseteq S)$. Since $d \in S$, we have $(d)_\rho \in S/\rho$, so $x \in S/\rho$. Let now $(a)_\rho = (c)_\rho$ and $(b)_\rho = (d)_\rho$. We have

$$(a)_\rho \odot (b)_\rho = \{(z)_\rho \mid z \in a \circ b\} \text{ and } (c)_\rho \odot (d)_\rho = \{(t)_\rho \mid t \in c \circ d\}.$$

Since $(a,c) \in \rho$, $(b,d) \in \rho$ and ρ is a congruence on S, we have $(a \circ b, c \circ b) \in \rho$ and $(c \circ b, c \circ d) \in \rho$ and so $(a \circ b, c \circ d) \in \rho$. On the other hand,

$$\{(z)_\rho \mid z \in a \circ b\} = \{(t)_\rho \mid t \in c \circ d\} \tag{2.3}$$

Indeed: Suppose ρ is a congruence on S in the sense of Definition 1.2.
Let now $z \in a \circ b$. Then $(z)_\rho \in \{(t)_\rho \mid t \in c \circ d\}$. Indeed: Since $(a \circ b, c \circ d) \in \rho$ and $z \in a \circ b$, there exists $u \in c \circ d$ such that $(z,u) \in \rho$. Then $(z)_\rho = (u)_\rho$, where $u \in c \circ d$, and so $(z)_\rho \in \{(t)_\rho \mid t \in c \circ d\}$, and $\{(z)_\rho \mid z \in a \circ b\} \subseteq \{(t)_\rho \mid t \in c \circ d\}$. Similarly $\{(t)_\rho \mid t \in c \circ d\} \subseteq \{(z)_\rho \mid z \in a \circ b\}$ and property (2.3) holds.

Since Definition 1.1 implies Definition 1.2, property (2.3) in case of Definition 1.1 also holds.

Denote by $\otimes$ the operation on $\mathcal{P}^*(S/\rho)$ defined in the way indicated by (2.1).

Let now $(S,\circ)$ be an hypersemigroup, ρ a congruence on S and $(a)_\rho, (b)_\rho, (c)_\rho$ in S/ρ. Then

$$\Big((a)_\rho \odot (b)_\rho\Big) \otimes \{(c)_\rho\} = \{(a)_\rho\} \otimes \Big((b)_\rho \odot (c)_\rho\Big) \tag{2.4}$$

Indeed: Let $x \in \Big((a)_\rho \odot (b)_\rho\Big) \otimes \{(c)_\rho\}$. Then $x \in u \odot (c)_\rho$ for some $u \in (a)_\rho \odot (b)_\rho$ (see (2.1)). Then $u = (d)_\rho$ for some $d \in a \circ b$; and $x \in (d)_\rho \odot (c)_\rho$ from which $x = (z)_\rho$ for some $z \in d \circ c$. Then we have

$z \in d \circ c = \{d\} * \{c\} \subseteq (a \circ b) * \{c\} = \{a\} * (b \circ c)$. By (2.1), $z \in a \circ e$ for some $e \in b \circ c$; and so $(z)_\rho \in (a)_\rho \odot (e)_\rho$ and $(e)_\rho \in (b)_\rho \odot (c)_\rho$. Hence we obtain $x = (z)_\rho \in \{(a)_\rho\} \otimes \Big((b)_\rho \odot (c)_\rho\Big)$ (see (2.1)). The proof of the rest is similar; property (2.4) is satisfied and the proof of the theorem is complete. □

Proposition 2.2. *If S is an hypergroupoid, ρ a congruence on S in the sense of Definition 1.1, and $d \in a \circ b$, then $\{(c)_\rho \mid c \in a \circ b\} = \{(d)_\rho\}$.*

Proof. Let $x = (c)_\rho$ for some $c \in a \circ b$. Since $b \in S$ and ρ is an equivalence relation on S, we have $(b, b) \in \rho$. Since ρ is a congruence on S, we have $(a \circ b, a \circ b) \in \rho$. Since $(a \circ b, a \circ b) \in \rho$, $c \in a \circ b$ and $d \in a \circ b$, by Definition 1.1, we have $(c, d) \in \rho$, so $(c)_\rho = (d)_\rho$, and $x = (d)_\rho$. Thus we have $\{(c)_\rho \mid c \in a \circ b\} \subseteq \{(d)_\rho\}$. On the other hand, since $d \in a \circ b$, we have $(d)_\rho \in \{(c)_\rho \mid c \in a \circ b\}$ and so $\{(d)_\rho\} \subseteq \{(c)_\rho \mid c \in a \circ b\}$. □

We apply Theorem 2.1 to the following example (see Proposition 2.2) as well.

Example 2.3. We consider the ordered semigroup given by Table 1 and Figure 1.

Table 1. The multiplication of the ordered semigroup of the Example 2.3.

$\cdot$	a	b	c	d
a	a	b	b	a
b	a	b	b	a
c	a	b	b	a
d	a	b	b	a

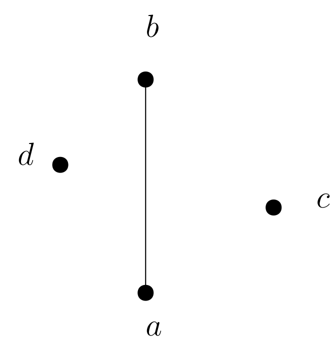

Figure 1. Figure that shows the order of the ordered semigroup of Example 2.3.

For a very easy way to check that this is an ordered semigroup see [6].

Defining $x \circ y := \{t \in S \mid t \leq xy\}$ for each $x, y \in S$, according to [7], from this ordered semigroup the hypersemigroup defined by Table 2 can be obtained.

The relation ρ on S defined by

$$\rho = \{(a,a), (b,b), (c,c), (d,d), (a,b), (b,a)\}$$

is a congruence on $(S, \circ)$ in the sense of Definition 1.1 as a consequence, by Remark 1.5, it is a congruence in the sense of Definition 1.2 as well.

We have $(a)_\rho = (b)_\rho = \{a, b\}$, $(c)_\rho = \{c\}$, $(d)_\rho = \{d\}$.

Table 2. The hypersemigroup of the Example 2.3.

$\circ$	a	b	c	d
a	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a\}$
c	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a\}$
d	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a\}$

The set S/ρ of the ρ-classes of S with the operation defined by Table 3 is an hypersemigroup. Independently, one can check that this is true.

Table 3. The hyperoperation on S/ρ in Example 2.3.

$\odot$	$(a)_\rho$	$(c)_\rho$	$(d)_\rho$
$(a)_\rho$	$\{(a)_\rho\}$	$\{(a)_\rho\}$	$\{(a)_\rho\}$
$(c)_\rho$	$\{(a)_\rho\}$	$\{(a)_\rho\}$	$\{(a)_\rho\}$
$(d)_\rho$	$\{(a)_\rho\}$	$\{(a)_\rho\}$	$\{(a)_\rho\}$

Conclusion 2.4. *If $(S,\circ)$ is an hypergroupoid and ρ a congruence either in the sense of Definition 1.1 or in the sense of Definition 1.2, then the set S/ρ with the hyperoperation $\odot$ defined by:*

$$(a)_\rho \odot (b)_\rho = \{(c)_\rho \mid c \in a \circ b\}$$

is an hypergroupoid. If ρ is a congruence in the sense of Definition 1.1, then the set $\{(c)_\rho \mid c \in a \circ b\}$ is equal to $\{(d)_\rho\}$, where d is an arbitrary element of $a \circ b$. So, in case of Definition 1.1, the $(a)_\rho \odot (b)_\rho$ is a singleton and can be also defined as

$$(a)_\rho \odot (b)_\rho = \{(d)_\rho\},$$

where d is an arbitrary element of $a \circ b$.

In particular, if $(S,\circ)$ is an hypersemigroup and ρ a congruence either in the sense of Definition 1.1 or in the sense of Definition 1.2, then the set S/ρ is an hypersemigroup as well.

As a consequence, the property (2) of Lemma 2.2 in [12, 13], the property (1) of Theorem 2.2 in [2] and the same property given in section 2 in [3] should be read as follows:

$$(a)_\rho \odot (b)_\rho = \{(d)_\rho\} \text{ where } d \text{ is an arbitrary element of } a \circ b.$$

In other words, we fix an arbitrary element of $a \circ b$, say d, and define $(a)_\rho \odot (b)_\rho = \{(d)_\rho\}$.

In Theorem 2.1, if ρ is a congruence in the sense of Definition 1.1, a second interesting proof of property (2.4), based on Proposition 2.2 is as follows:

We have

$$\begin{aligned}\Big((a)_\rho \odot (b)_\rho\Big) \otimes \{(c)_\rho\} &= \{(d)_\rho\} \otimes \{(c)_\rho\}, \text{ where } d \in a \circ b \\ &= (d)_\rho \odot (c)_\rho = \{(e)_\rho\}, \text{ where } e \in d \circ c,\end{aligned}$$

$$\begin{aligned}\{(a)_\rho\} \otimes \Big((b)_\rho \odot (c)_\rho\Big) &= \{(a)_\rho\} \otimes \{(u)_\rho\}, \text{ where } u \in b \circ c \\ &= (a)_\rho \odot (u)_\rho = \{(f)_\rho\}, \text{ where } f \in a \circ u.\end{aligned}$$

Since $d \in a \circ b$, $e \in d \circ c$, $u \in b \circ c$ and $f \in a \circ u$, we have $(e)_\rho = (f)_\rho$. Indeed:

We have $e \in d \circ c = \{d\} * \{c\} \subseteq (a \circ b) * \{c\}$, so $e \in w \circ c$ for some $w \in a \circ b$ (see (2.1)). Since $(a \circ b, a \circ b) \in \rho$, $w \in a \circ b$ and $d \in a \circ b$, we have $(w, d) \in \rho$. We also have $e \in \{a\} * (b \circ c)$ from which $e \in a \circ t$ for some $t \in b \circ c$. Since $(b \circ c, b \circ c) \in \rho$, $t \in b \circ c$ and $u \in b \circ c$, we have $(t, u) \in \rho$. Then $(a \circ t, a \circ u) \in \rho$ and, since $e \in a \circ t$ and $f \in a \circ u$, we have $(e, f) \in \rho$ that is $(e)_\rho = (f)_\rho$.

3. Main results

Theorem 3.2 of this section gives an answer to the problem by Davvaz et al. in [2] using the concept of quasi pseudoorder introduced in Definition 3.1 below. According to Theorem 3.4, the set S/ρ in [2, Theorem 4.2] is not only an ordered semigroup, but an ordered hypersemigroup as it was expected to be. We prove that from Theorem 3.4, the Lemma 1 in [9] can be obtained. We use the terms "congruence" and "strong congruence" as we deal with both of them in this section. Examples to illustrate the results have been also given.

Definition 3.1. *Let $(S, \circ, \le)$ be an ordered hypergroupoid. A relation σ on S is called quasi pseudoorder if we have the following:*

(1) $\le \subseteq \sigma$

(2) if $(a, b) \in \sigma$ and $(b, c) \in \sigma$, then $(a, c) \in \sigma$

(3) if $(a, b) \in \sigma$ and $c \in S$, then $(a \circ c, b \circ c) \in \overline{\sigma}$ and $(c \circ a, c \circ b) \in \overline{\sigma}$ in the sense that for every $u \in a \circ c$ there exists $v \in b \circ c$ such that $(u, v) \in \sigma$ and $(v, u) \in \sigma$ and for every $u \in c \circ a$ there exists $v \in c \circ b$ such that $(u, v) \in \sigma$ and $(v, u) \in \sigma$.

Theorem 3.2. *Let $(S, \circ, \le)$ be an ordered hypersemigroup and σ a quasi pseudoorder on S. Then there exists a congruence ρ on S (Definition 1.2) such that the set S/ρ is an ordered hypersemigroup.*

Proof. We consider the relation ρ on S defined by

$$\rho := \{(a, b) \mid (a, b) \in \sigma \text{ and } (b, a) \in \sigma\}.$$

This is a congruence on $(S, \circ)$. In fact:

First of all the relation ρ is an equivalence relation on S: If $a \in S$, then $(a, a) \in \le \subseteq \sigma$, and so $(a, a) \in \rho$. If $(a, b) \in \rho$, then clearly $(b, a) \in \rho$. If $(a, b) \in \rho$ and $(b, c) \in \rho$, then $(a, b) \in \sigma$, $(b, a) \in \sigma$, $(b, c) \in \sigma$, $(c, b) \in \sigma$, then $(a, c) \in \sigma$ and $(c, a) \in \sigma$, and so $(a, c) \in \rho$.

Let now $(a, b) \in \rho$ and $c \in S$. Then $(a \circ c, b \circ c) \in \rho$, that is for every $u \in a \circ c$ there exists $v \in b \circ c$ such that $(u, v) \in \rho$. Indeed: Let $u \in a \circ c$. Since $(a, b) \in \rho$, we have $(a, b) \in \sigma$ and $(b, a) \in \sigma$. Since $(a, b) \in \sigma$ and σ is a quasi pseudoorder on S, we have $(a \circ c, b \circ c) \in \overline{\sigma}$. Since $u \in a \circ c$, there exists $v \in b \circ c$ such that $(u, v) \in \sigma$ and $(v, u) \in \sigma$, and so $(u, v) \in \rho$. Let now $(a, b) \in \rho$ and $c \in S$. Then $(c \circ a, c \circ b) \in \rho$. Indeed: Let $u \in c \circ a$. We have $(a, b) \in \sigma$ and $(b, a) \in \sigma$. Since $(a, b) \in \sigma$ and σ is a quasi pseudoorder, we have

$(c\circ a, c\circ b)\in\overline{\sigma}$. Since $u\in c\circ a$, there exists $v\in c\circ b$ such that $(u,v)\in\sigma$ and $(v,u)\in\sigma$, and so $(u,v)\in\rho$. Hence ρ is a congruence on S.

Since ρ is a congruence on $(S,\circ)$, by Theorem 2.1, the set S/ρ with the hyperoperation $\odot$ on S/ρ defined by

$$(a)_\rho\odot(b)_\rho := \{(c)_\rho \mid c\in a\circ b\}$$

is an hypersemigroup.

We define a relation "$\preceq$" on S/ρ as follows:

$$(a)_\rho\preceq(b)_\rho \text{ if and only if } (a,b)\in\sigma.$$

This is well defined, that is $(a)_\rho\preceq(b)_\rho$, $c\in(a)_\rho$ and $d\in(b)_\rho$ imply $(c,d)\in\sigma$. Indeed, we have $(c,a)\in\sigma$, $(a,b)\in\sigma$ and $(b,d)\in\sigma$, and so $(c,d)\in\sigma$. Moreover, the relation $\preceq$ is an order on S/ρ. Indeed: Let $a,b,c\in S$. Since σ is a quasi pseudoorder, we have $\leq\subseteq\sigma$. Since "$\leq$" is an order on S, we have $(a,a)\in\leq$, then $(a,a)\in\sigma$ and $(a)_\rho\preceq(a)_\rho$. If $(a)_\rho\preceq(b)_\rho$ and $(b)_\rho\preceq(a)_\rho$, then $(a,b)\in\sigma$ and $(b,a)\in\sigma$, then $(a,b)\in\rho$ and $(a)_\rho=(b)_\rho$. If $(a)_\rho\preceq(b)_\rho$ and $(b)_\rho\preceq(c)_\rho$, then $(a,b)\in\sigma$ and $(b,c)\in\sigma$, then $(a,c)\in\sigma$ and $(a)_\rho\preceq(c)_\rho$.

$(S/\rho,\odot,\preceq)$ is an ordered hypersemigroup. In fact:

(A) Let $(a)_\rho\preceq(b)_\rho$ and $c\in S$. Then

$$(a)_\rho\odot(c)_\rho\preceq(b)_\rho\odot(c)_\rho$$

that is for every $u\in(a)_\rho\odot(c)_\rho$ there exists $v\in(b)_\rho\odot(c)_\rho$ such that $u\preceq v$.

Indeed: Let $x\in(a)_\rho\odot(c)_\rho$. Then $x=(u)_\rho$ for some $u\in a\circ c$. Since $(a)_\rho\preceq(b)_\rho$, we have $(a,b)\in\sigma$. Since σ is a quasi pseudoorder on S, we have $(a\circ c, b\circ c)\in\overline{\sigma}$. Since $u\in a\circ c$, there exists $v\in b\circ c$ such that $(u,v)\in\sigma$ and $(v,u)\in\sigma$. Then we have $(u,v)\in\rho$ and so $(u)_\rho=(v)_\rho$. On the other hand, $v\in b\circ c$ implies $(v)_\rho\in(b)_\rho\odot(c)_\rho$. For the element $y:=(v)_\rho$, we have $x=y$ and so $x\preceq y$.

(B) Let $(a)_\rho\preceq(b)_\rho$ and $c\in S$. Then

$$(c)_\rho\odot(a)_\rho\preceq(c)_\rho\odot(b)_\rho.$$

Indeed: Let $(u)_\rho\in(c)_\rho\odot(a)_\rho$, where $u\in c\circ a$. Since $(a)_\rho\preceq(b)_\rho$, we have $(a,b)\in\sigma$. Since σ is a quasi pseudoorder on S and $u\in c\circ a$, there exists $v\in c\circ b$ such that $(u,v)\in\sigma$ and $(v,u)\in\sigma$. Then we have $(u,v)\in\rho$, so $(u)_\rho=(v)_\rho$ and so $(u)_\rho\preceq(v)_\rho$. On the other hand, since $v\in c\circ b$, we have $(v)_\rho\in(c)_\rho\odot(b)_\rho$ and the proof of the theorem is complete. □

We apply Theorem 3.2 to the following example.

Example 3.3. We consider the ordered semigroup $(S,\cdot,\leq)$ given by Table 4 and Figure 2.

The order on S is the following:

$$\leq=\{(a,a),(b,b),(c,c),(d,b),(d,c),(d,d)\}.$$

Using the methodology described in [7], from this ordered semigroup we get the ordered hypersemigroup $(S,\circ,\leq)$ defined by Table 5 and the same order.

Table 4. Multiplication of the ordered semigroup of Example 3.3.

$\cdot$	a	b	c	d
a	b	a	a	a
b	a	b	b	b
c	a	b	b	b
d	a	b	b	d

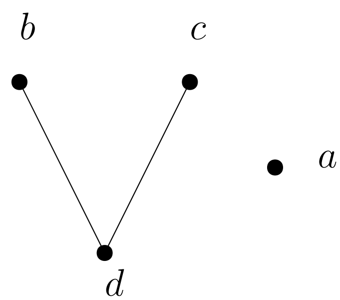

Figure 2. Figure that shows the order of the ordered semigroup of Example 3.3.

Table 5. The hyperoperation of the ordered hypersemigroup of Example 3.3.

$\circ$	a	b	c	d
a	$\{b,d\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$
c	$\{a\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$
d	$\{a\}$	$\{b,d\}$	$\{b,d\}$	$\{d\}$

One can check that the relation

$$\sigma = \{(a,a),(b,b),(b,c),(c,b),(c,c),(d,b),(d,c),(d,d)\}$$

is a quasi pseudoorder on $(S,\circ,\leq)$; thus, by Theorem 3.2, the relation on S defined by

$$\begin{aligned}\rho &= \{(x,y) \mid (x,y)\in\sigma \text{ and } (y,x)\in\sigma\}\\ &= \{(a,a),(b,b),(b,c),(c,b),(c,c),(d,d)\}\end{aligned}$$

is a congruence on $(S,\circ,\leq)$ (Definition 1.2). We have $(a)_\rho=\{a\}$, $(b)_\rho=\{b,c\}$, $(c)_\rho=\{b,c\}$, $(d)_\rho=\{d\}$ and so $S/\rho=\{(a)_\rho,(b)_\rho,(d)_\rho\}$. By Theorem 3.2, the set S/ρ with the hyperoperation "$\odot$" given by Table 6 and the order "$\preceq$" below is an ordered hypersemigroup.

Table 6. The hyperoperation on the set S/ρ in Example 3.3.

$\odot$	$(a)_\rho$	$(b)_\rho$	$(d)_\rho$
$(a)_\rho$	$\{(b)_\rho,(d)_\rho\}$	$\{(a)_\rho\}$	$\{(a)_\rho\}$
$(b)_\rho$	$\{(a)_\rho\}$	$\{(b)_\rho,(d)_\rho\}$	$\{(b)_\rho,(d)_\rho\}$
$(d)_\rho$	$\{(a)_\rho\}$	$\{(b)_\rho,(d)_\rho\}$	$\{(d)_\rho\}$

$$\preceq = \Big\{\Big((a)_\rho,(a)_\rho\Big),\Big((b)_\rho,(b)_\rho\Big),\Big((d)_\rho,(b)_\rho\Big),\Big((d)_\rho,(d)_\rho\Big)\Big\}.$$

If we write, for short, $(a)_\rho := x$, $(b)_\rho := y$, $(d)_\rho := z$ then, we can independently check that the set $S = \{x, y, z\}$ with the hyperoperation "$\circ$" defined by Table 7 and the order "$\preceq$" given by

$$\preceq = \{(x,x), (y,y), (z,y), (z,z)\}$$

is an ordered hypersemigroup.

Table 7. The Table 6 of the Example 3.3 written in a short way.

$\circ$	x	y	z
x	$\{y,z\}$	$\{x\}$	$\{x\}$
y	$\{x\}$	$\{y,z\}$	$\{y,z\}$
z	$\{x\}$	$\{y,z\}$	$\{z\}$

Indeed, for example, $\{y\} * (x \circ z) = \{y\} * \{x\} = y \circ x = \{x\}$ and $(y \circ x) * \{z\} = \{x\} * \{z\} = x \circ z = \{x\}$ and so on. The relation $z \preceq y$ implies $z \circ t \preceq y \circ t$ and $t \circ z \preceq t \circ y$ for every $t \in S$ which means that for every $u \in z \circ t$ there exists $v \in y \circ t$ such that $u \preceq v$ and for every $u \in t \circ z$ there exists $v \in t \circ y$ such that $u \preceq v$.

Theorem 3.4. *(see also [2, Theorem 4.2]) Let $(S, \circ, \leq)$ be an ordered hypersemigroup and σ a pseudorder on S (Definition 1.3). Then there exists a strong congruence ρ on S (Definition 1.1) such that S/ρ is an ordered hypersemigroup.*

Proof. We consider the relation ρ on S defined by

$$\rho := \{(a,b) \mid (a,b) \in \sigma \text{ and } (b,a) \in \sigma\}.$$

As in the previous theorem, this is an equivalence relation on S. Moreover, this is a strong congruence on S. In fact: Let $(a,b) \in \rho$ and $c \in S$. Then $(a \circ c, b \circ c) \in \rho$. Indeed: Let $u \in a \circ c$ and $v \in b \circ c$. Since $(a,b) \in \rho$, we have $(a,b) \in \sigma$ and $(b,a) \in \sigma$. Since σ is a pseudoorder on S, we have $(a \circ c, b \circ c) \in \sigma$ and $(b \circ c, a \circ c) \in \sigma$. Since $u \in a \circ c$ and $v \in b \circ c$, we have $(u,v) \in \sigma$ and $(v,u) \in \sigma$, and so $(u,v) \in \rho$. Let now $(a,b) \in \rho$, $c \in S$, $u \in c \circ a$ and $v \in c \circ b$. Since $(a,b) \in \sigma$ and $(b,a) \in \sigma$, we have $(c \circ a, c \circ b) \in \sigma$ and $(c \circ b, c \circ a) \in \sigma$, then $(u,v) \in \sigma$ and $(v,u) \in \sigma$, that is $(u,v) \in \rho$.

Since ρ is a strong congruence on S, by Theorem 2.1 and Proposition 2.2, the set S/ρ endowed with the hyperoperation

$$(a)_\rho \odot (b)_\rho = \{(c)_\rho\}, \text{ where } c \text{ is an arbitrary element of } a \circ b$$

is an hypersemigroup. We consider the relation "$\preceq$" on S/ρ defined by:

$$(a)_\rho \preceq (b)_\rho \text{ if and only if } (a,b) \in \sigma.$$

This is an order relation on S/ρ. In fact: First of all, it is well defined, that is $(a)_\rho \preceq (b)_\rho$, $c \in (a)_\rho$ and $d \in (b)_\rho$ imply $(c,d) \in \sigma$. Indeed: We have $(c,a) \in \sigma$, $(a,b) \in \sigma$ and $(b,d) \in \sigma$, and so $(c,d) \in \sigma$. Let $a,b,c \in S$. Since $(a,a) \in \leq \subseteq \sigma$, we have $(a)_\rho \preceq (a)_\rho$. If $(a)_\rho \preceq (b)_\rho$ and $(b)_\rho \preceq (a)_\rho$, then $(a,b) \in \sigma$ and $(b,a) \in \sigma$, then $(a,b) \in \rho$ and so $(a)_\rho = (b)_\rho$. If $(a)_\rho \preceq (b)_\rho$ and $(b)_\rho \preceq (c)_\rho$, then $(a,b) \in \sigma$ and $(b,c) \in \sigma$, then $(a,c) \in \sigma$ and so $(a)_\rho \preceq (c)_\rho$.

The set S/ρ endowed with the hyperoperation "$\odot$" and the order "$\preceq$" is an ordered hypersemigroup.

In fact: First of all, by Theorem 2.1 (see also Proposition 2.2), the $(S/\rho, \odot)$ is an hypersemigroup.

Let now $(a)_\rho \preceq (b)_\rho$ and $c \in S$. Then

$$(a)_\rho \odot (c)_\rho \preceq (b)_\rho \odot (c)_\rho,$$

that is for every $x \in (a)_\rho \odot (c)_\rho$ there exists $y \in (b)_\rho \odot (c)_\rho$ such that $x \preceq y$. Indeed: Let $x \in (a)_\rho \odot (c)_\rho$. Then $x = (u)_\rho$ for some $u \in a \circ c$. Since $(a)_\rho \preceq (b)_\rho$, we have $(a, b) \in \sigma$. Since σ is a pseudoorder, we have $(a \circ c, b \circ c) \in \sigma$. Take an element $v \in b \circ c$ $(b \circ c \neq \emptyset)$. Since $u \in a \circ c$ and $v \in b \circ c$, we have $(u, v) \in \sigma$, then $(u)_\rho \preceq (v)_\rho$. Since $v \in b \circ c$, we have $(v)_\rho \in (b)_\rho \odot (c)_\rho$. For the element $y := (v)_\rho \in (b)_\rho \odot (c)_\rho$, we have $x \preceq y$. Similarly $(a)_\rho \preceq (b)_\rho$ and $c \in S$ implies $(c)_\rho \odot (a)_\rho \preceq (c)_\rho \odot (b)_\rho$ and the proof of the theorem is complete. □

By Theorem 3.4 we get the following corollary that shows that the Lemma 1 in Semigroup Forum [9] has been generalized in case of ordered hypersemigroups.

Corollary 3.5. [9, Lemma 1] *Let $(S, \cdot, \leq)$ be an ordered semigroup. If σ is a pseudoorder on S, then the relation ρ on S defined by $\rho := \{(a, b) \mid (a, b) \in \sigma$ and $(b, a) \in \sigma\}$ is a congruence on S and the set S/ρ with the multiplication $(a)_\rho (b)_\rho := (ab)_\rho$ and the order*

$$(a)_\rho \preceq (b)_\rho \text{ if and only if } (a, b) \in \sigma$$

is an ordered semigroup.

Proof. Let σ be a pseudoorder on $(S, \cdot, \leq)$. The set S with the hyperoperation $a \circ b := \{ab\}$ and the same order "$\leq$" is an ordered hypersemigroup [8]; and σ is a pseudoorder on $(S, \circ, \leq)$ (Definition 1.3). Indeed: If $(a, b) \in \sigma$, $c \in S$, $u \in a \circ c$ and $v \in b \circ c$, then $u = ac$ and $v = bc$; since σ is a pseudoorder on $(S, \cdot, \leq)$ we have $(ac, bc) \in \sigma$ and so $(u, v) \in \sigma$. Similarly $(a, b) \in \sigma$, $c \in S$, implies $(c \circ a, c \circ b) \in \sigma$.

By Theorem 3.4, the relation ρ on S defined by

$$\rho := \{(a, b) \mid (a, b) \in \sigma \text{ and } (b, a) \in \sigma\}$$

is a strong congruence on $(S, \circ, \leq)$ and the set S/ρ with the hyperoperation

$$(a)_\rho \odot (b)_\rho := \{(c)_\rho\},$$

where c is an (arbitrary) element of $a \circ b$ and the order $(a)_\rho \preceq (b)_\rho \Leftrightarrow (a, b) \in \sigma$ is an ordered hypersemigroup.

The relation ρ is a congruence on $(S, \cdot, \leq)$. Indeed: Let $(a, b) \in \rho$, $c \in S$. Since ρ is a strong congruence on $(S, \circ, \leq)$, we have $(a \circ c, b \circ c) \in \rho$ and $(c \circ a, c \circ b) \in \rho$ and, since $ac \in a \circ c$ and $bc \in b \circ c$, we have $(ac, bc) \in \rho$ and $(ca, cb) \in \rho$.

Finally, the set S/ρ with the multiplication $(a)_\rho (b)_\rho = (ab)_\rho$ and the order $(a)_\rho \preceq (b)_\rho \Leftrightarrow (a, b) \in \sigma$ is an ordered semigroup. Indeed: First of all, the multiplication "$\cdot$" on S/ρ is well defined. This is because, if $(a)_\rho = (c)_\rho$ and $(b)_\rho = (d)_\rho$, then $(a, c) \in \rho$, $(b, d) \in \rho$; since ρ is a congruence on $(S, \cdot, \leq)$ we have $(ab, cd) \in \rho$, then $(ab)_\rho = (cd)_\rho$ and so $(a)_\rho (b)_\rho = (c)_\rho (d)_\rho$. Let $a, b, c \in S$. Since $(S/\rho, \odot)$ is an hypersemigroup, we have

$$\{(a)_\rho\} \otimes \Big((b)_\rho \odot (c)_\rho\Big) = \Big((a)_\rho \odot (b)_\rho\Big) \otimes \{(c)_\rho\}. \tag{3.1}$$

On the other hand,

$$\begin{aligned}\{(a)_\rho\} \otimes \Big((b)_\rho \odot (c)_\rho\Big) &= \{(a)_\rho\} \otimes \{(bc)_\rho\} \text{ (since } bc \in b \circ c)\\ &= (a)_\rho \odot (bc)_\rho = \Big(a(bc)\Big)_\rho\end{aligned}$$

and

$$\Big((a)_\rho \odot (b)_\rho\Big) \otimes \{(c)_\rho\} = (ab)_\rho \otimes \{(c)_\rho\} = (ab)_\rho \odot (c)_\rho = \Big((ab)c\Big)_\rho.$$

Thus, by (3.1), we get $\Big(a(bc)\Big)_\rho = \Big((ab)c\Big)_\rho$ and so $(a)_\rho\Big((b)_\rho(c)_\rho\Big) = \Big((a)_\rho(b)_\rho\Big)(c)_\rho$.

Let now $(a)_\rho \preceq (b)_\rho$ and $c \in S$. Then $(a,b) \in \sigma$ and, since σ is a pseudoorder on $(S, \cdot, \leq)$ we have $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$. Then $(ac)_\rho \preceq (bc)_\rho$ and $(ca)_\rho \preceq (cb)_\rho$ from which $(a)_\rho(c)_\rho \preceq (b)_\rho(c)_\rho$ and $(c)_\rho(a)_\rho \preceq (c)_\rho(b)_\rho$. □

We apply Theorem 3.4 to the following example.

Example 3.6. We consider the ordered hypersemigroup $(S, \circ, \leq)$ given by Table 5 and Figure 2 of the previous Example 3.3. The relation

$$\sigma = \{(a,a),(b,b),(b,c),(b,d),(c,c),(d,b),(d,c),(d,d)\}$$

is a pseudoorder on $(S, \circ, \leq)$ in the sense of Definition 1.3. By Theorem 3.4, the relation ρ on S defined by

$$\begin{aligned}\rho &= \{(x,y) \mid (x,y) \in \sigma \text{ and } (y,x) \in \sigma\}\\ &= \{(a,a),(b,b),(b,d),(c,c),(d,b),(d,d)\}\end{aligned}$$

is a strong congruence on $(S, \circ, \leq)$ (Definition 1.1). One can also independently check that this is true. We have $(a)_\rho = \{a\}$, $(b)_\rho = \{b,d\}$, $(c)_\rho = \{c\}$, $(d)_\rho = \{b,d\} = (b)_\rho$ and $S/\rho = \{(a)_\rho, (b)_\rho, (c)_\rho\}$. By Theorem 3.4, the set S/ρ with the hyperoperation "$\odot$" given by Table 8 and the order defined by $(a)_\rho \preceq (b)_\rho$ if and only if $(a,b) \in \sigma$ is an ordered hypersemigroup.

Table 8. The hyperoperation on the set S/ρ of the Example 3.6.

$\odot$	$(a)_\rho$	$(b)_\rho$	$(c)_\rho$
$(a)_\rho$	$\{(b)_\rho\}$	$\{(a)_\rho\}$	$\{(a)_\rho\}$
$(b)_\rho$	$\{(a)_\rho\}$	$\{(b)_\rho\}$	$\{(b)_\rho\}$
$(c)_\rho$	$\{(a)_\rho\}$	$\{(b)_\rho\}$	$\{(b)_\rho\}$

$$\preceq = \Big\{\Big((a)_\rho, (a)_\rho\Big), \Big((b)_\rho, (b)_\rho\Big), \Big((b)_\rho, (c)_\rho\Big), \Big((c)_\rho, (c)_\rho\Big)\Big\}.$$

If we write, for short, $(a)_\rho := x$, $(b)_\rho := y$, $(z)_\rho := z$ then, we can independently check that the set $S = \{x,y,z\}$ with the hyperoperation "$\circ$" defined by Table 9 and the order "$\preceq$" given by $\preceq = \{(x,x),(y,y),(y,z),(z,z)\}$ is an ordered hypersemigroup.

Table 9. The Table 8 of the Example 3.6 written in a short way.

$\circ$	x	y	z
x	$\{y\}$	$\{x\}$	$\{x\}$
y	$\{x\}$	$\{y\}$	$\{y\}$
z	$\{x\}$	$\{y\}$	$\{y\}$

Note Theorem 3.2 provides a direct answer to the open problem by Davvaz, Corsini and Changphas in [2], correcting the existed one in Turk J Math and Open Math. However, an answer to the problem can be also obtained as application to the revised version of [2, Theorem 4.2] given in Theorem 3.4 of the present paper as, regarding to Remark 1.5, every strong congruence on an hypersemigroup S is a congruence on S.

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