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## On the extension of Hermite–Hadamard type inequalities for coordinated convex mappings

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**Abstract:** In this paper, we obtain an important inequalities for coordinated convex functions and as a result of these inequalities we give the extension of Hermite–Hadamard type inequalities for Riemann–Liouville fractional integral and logarithmic integral. The inequalities obtained in this study provide generalizations of some result given in earlier works.

**Key words:** Hermite–Hadamard’s inequalities, fractional integral, coordinated convex, integral inequalities

### 1. Introduction

The Hermite–Hadamard inequality discovered by C. Hermite and J. Hadamard see, e.g., [12], [25, p.137]) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that if  $f : I \rightarrow \mathbb{R}$  is a convex function on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ , then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

Both inequalities hold in the reversed direction if  $f$  is concave. We note that Hermite–Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen’s inequality. Hermite–Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied (see, for example, [3], [8]–[15], [16], [23], [24], [32], [37], [38], [42]).

A formal definition for coordinated convex function may be stated as follows: Let us now consider a bidimensional interval  $\Delta =: [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $a < b$  and  $c < d$ . A mapping  $f : \Delta \rightarrow \mathbb{R}$  is said to be convex on  $\Delta$  if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x, y) + (1-t)f(z, w)$$

holds, for all  $(x, y), (z, w) \in \Delta$  and  $t \in [0, 1]$ . A function  $f : \Delta \rightarrow \mathbb{R}$  is said to be on the co-ordinates on  $\Delta$  if the partial mappings  $f_y : [a, b] \rightarrow \mathbb{R}$ ,  $f_y(u) = f(u, y)$  and  $f_x : [c, d] \rightarrow \mathbb{R}$ ,  $f_x(v) = f(x, v)$  are convex where defined for all  $x \in [a, b]$  and  $y \in [c, d]$  (see [11]).

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**Definition 1.1** A function  $f : \Delta \rightarrow \mathbb{R}$  will be called *coordinated convex on  $\Delta$* , for all  $t, s \in [0, 1]$  and  $(x, y), (u, w) \in \Delta$ , if the following inequality holds:

$$\begin{aligned} & f(tx + (1-t)y, su + (1-s)w) \\ & \leq tsf(x, u) + s(1-t)f(y, u) + t(1-s)f(x, w) + (1-t)(1-s)f(y, w). \end{aligned} \quad (1.2)$$

Clearly, every convex function is coordinated convex. Furthermore, there exists coordinated convex function which is not convex (see, [11]). For several recent results concerning Hermite–Hadamard’s inequality for some convex function on the coordinates on a rectangle from the plane  $\mathbb{R}^2$ , we refer the reader to ([1], [2],[4], [11], [17], [18], [19]-[22], [29],[33], [35], [36], [39], [40]).

Now, we give the definitions Riemann–Liouville fractional integrals for two variable functions:

**Definition 1.2** [29] Let  $f \in L_1([a, b] \times [c, d])$ . The Riemann–Liouville fractional integrals  $J_{a+,c+}^{\alpha,\beta}$ ,  $J_{a+,d-}^{\alpha,\beta}$ ,  $J_{b-,c+}^{\alpha,\beta}$  and  $J_{b-,d-}^{\alpha,\beta}$  are defined by

$$J_{a+,c+}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1} (y-s)^{\beta-1} f(t, s) ds dt, \quad x > a, y > c,$$

$$J_{a+,d-}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x-t)^{\alpha-1} (s-y)^{\beta-1} f(t, s) ds dt, \quad x > a, y < d,$$

$$J_{b-,c+}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1} (y-s)^{\beta-1} f(t, s) ds dt, \quad x < b, y > c,$$

and

$$J_{b-,d-}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (t-x)^{\alpha-1} (s-y)^{\beta-1} f(t, s) ds dt, \quad x < b, y < d.$$

Similar the above definitons, we can give the following integrals:

$$J_{a+}^{\alpha} f(x, c) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t, c) dt, \quad x > a$$

$$J_{a+}^{\alpha} f(x, d) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t, d) dt, \quad x > a,$$

$$J_{c+}^{\beta} f(a, y) = \frac{1}{\Gamma(\beta)} \int_c^y (y-s)^{\beta-1} f(a, s) ds, \quad y > c,$$

and

$$J_{d-}^{\beta} f(b, y) = \frac{1}{\Gamma(\beta)} \int_y^d (s - y)^{\beta-1} f(b, s) ds, \quad y < d.$$

One can find some recent Hermite–Hadamard type inequalities for function of one and two variables via Riemann–Liouville fractional integrals in ([1], [5], [6], [7], [26], [27], [28], [30], [31], [34], [36], [39], [41], [43]–[45]).

The aim of this paper is to establish an important inequalities for coordinated convex functions and as a result of these inequalities we give the extension of Hermite–Hadamard type inequalities for Riemann–Liouville fractional integral and logarithmic integral. The results presented in this paper provide extensions of those given in [11] and [29].

In [42], Zabandan gave the following important inequalities associated with the Hermite–Hadamard inequalities and he gave a few inequalities regarding the special cases of these inequalities.

**Theorem 1.3** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a positive convex function on  $[a, b]$  and  $h : [0, 1] \rightarrow \mathbb{R}$  be a positive function such that  $h \in L([0, 1])$ . Then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2I_h(b-a)} \int_a^b \left[ h\left(\frac{x-a}{b-a}\right) + h\left(\frac{b-x}{b-a}\right) \right] f(x) dx \leq \frac{f(a) + f(b)}{2} \quad (1.3)$$

where  $I_h = \int_0^1 h(t) dt$ .

## 2. Main results

Throughout this section, we will use the following symbols

$$H_1(x) = h_1\left(\frac{b-x}{b-a}\right) + h_1\left(\frac{x-a}{b-a}\right),$$

$$H_2(y) = h_2\left(\frac{d-y}{d-c}\right) + h_2\left(\frac{y-c}{d-c}\right)$$

and

$$I_{h_1} = \int_0^1 h_1(t) dt \quad \text{and} \quad I_{h_2} = \int_0^1 h_2(s) ds.$$

In this part, we will give the following inequalities by using convex functions of 2-variables on the coordinates.

**Theorem 2.1** *Let  $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  is coordinated convex on  $\Delta := [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $0 \leq a < b$ ,  $0 \leq c < d$  and  $f \in L_1(\Delta)$  and  $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}$  be two positive functions such that  $h_1, h_2 \in L([0, 1])$ . Then, one has the inequalities:*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{4I_{h_1}I_{h_2}(b-a)(d-c)} \int_a^b \int_c^d H_1(x) H_2(y) f(x, y) dy dx \quad (2.1)$$

$$\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.$$

**Proof** According to (1.2) with  $x = ta + (1 - t)b$ ,  $y = (1 - t)a + tb$ ,  $u = sc + (1 - s)d$ ,  $w = (1 - s)c + sd$  and  $t = s = \frac{1}{2}$ , we find that

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{4} [f(ta + (1-t)b, sc + (1-s)d) + f(ta + (1-t)b, (1-s)c + sd) \\ & \quad + f((1-t)a + tb, sc + (1-s)d) + f((1-t)a + tb, (1-s)c + sd)]. \end{aligned} \quad (2.2)$$

Thus, multiplying both sides of (2.2) by  $h_1(t)h_2(s)$ , then by integrating with respect to  $(t, s)$  on  $[0, 1] \times [0, 1]$ , we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_0^1 \int_0^1 h_1(t)h_2(s) dsdt \\ & \leq \frac{1}{4} \left[ \int_0^1 \int_0^1 h_1(t)h_2(s) [f(ta + (1-t)b, sc + (1-s)d) + f(ta + (1-t)b, (1-s)c + sd)] dsdt \right. \\ & \quad \left. + \int_0^1 \int_0^1 h_1(t)h_2(s) [f((1-t)a + tb, sc + (1-s)d) + f((1-t)a + tb, (1-s)c + sd)] dsdt \right]. \end{aligned}$$

Using the change of the variable, we get

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_0^1 \int_0^1 h_1(t)h_2(s) dsdt \\ & \leq \frac{1}{4(b-a)(d-c)} \left\{ \int_a^b \int_c^d h_1\left(\frac{b-x}{b-a}\right) h_2\left(\frac{d-y}{d-c}\right) f(x, y) dydx \right. \\ & \quad + \int_a^b \int_c^d h_1\left(\frac{b-x}{b-a}\right) h_2\left(\frac{y-c}{d-c}\right) f(x, y) dydx \\ & \quad + \int_a^b \int_c^d h_1\left(\frac{x-a}{b-a}\right) h_2\left(\frac{d-y}{d-c}\right) f(x, y) dydx \\ & \quad \left. + \int_a^b \int_c^d h_1\left(\frac{x-a}{b-a}\right) h_2\left(\frac{y-c}{d-c}\right) f(x, y) dydx \right\} \end{aligned}$$

which the first inequality is proved. For the proof of the second inequality in (2.1), we first note that if  $f$  is a coordinated convex on  $\Delta$ , then, by using (1.2) with  $x = a$ ,  $y = b$ ,  $u = c$  and  $w = d$ , it yields

$$f(ta + (1-t)b, sc + (1-s)d) \leq tsf(a, c) + s(1-t)f(b, c) + t(1-s)f(a, d) + (1-t)(1-s)f(b, d)$$

$$f(ta + (1-t)b, (1-s)c + sd) \leq t(1-s)f(a, c) + (1-t)(1-s)f(b, c) + tsf(a, d) + (1-t)sf(b, d)$$

$$f((1-t)a + tb, sc + (1-s)d) \leq (1-t)sf(a, c) + stf(b, c) + (1-t)(1-s)f(a, d) + t(1-s)f(b, d)$$

and

$$f((1-t)a + tb, (1-s)c + sd) \leq (1-t)(1-s)f(a, c) + t(1-s)f(b, c) + (1-t)sf(a, d) + tsf(b, d).$$

By adding these inequalities, we have

$$\begin{aligned} & f(ta + (1-t)b, sc + (1-s)d) + f(ta + (1-t)b, (1-s)c + sd) \\ & + f((1-t)a + tb, sc + (1-s)d) + f((1-t)a + tb, (1-s)c + sd) \\ & \leq f(a, c) + f(b, c) + f(a, d) + f(b, d). \end{aligned} \quad (2.3)$$

Then, multiplying both sides of (2.3) by  $h_1(t)h_2(s)$  and integrating with respect to  $(t, s)$  over  $[0, 1] \times [0, 1]$ , we get

$$\begin{aligned} & \int_0^1 \int_0^1 h_1(t)h_2(s) [f(ta + (1-t)b, sc + (1-s)d) + f(ta + (1-t)b, (1-s)c + sd) \\ & + f((1-t)a + tb, sc + (1-s)d) + f((1-t)a + tb, (1-s)c + sd)] dsdt \\ & \leq [f(a, c) + f(b, c) + f(a, d) + f(b, d)] \int_0^1 \int_0^1 h_1(t)h_2(s) dsdt. \end{aligned}$$

Here, using the change of the variable we have

$$\begin{aligned} & \frac{1}{4(b-a)(d-c)} \left\{ \int_a^b \int_c^d h_1\left(\frac{b-x}{b-a}\right) h_2\left(\frac{d-y}{d-c}\right) f(x, y) dydx \right. \\ & + \int_a^b \int_c^d h_1\left(\frac{b-x}{b-a}\right) h_2\left(\frac{y-c}{d-c}\right) f(x, y) dydx \\ & + \int_a^b \int_c^d h_1\left(\frac{x-a}{b-a}\right) h_2\left(\frac{d-y}{d-c}\right) f(x, y) dydx \\ & \left. + \int_a^b \int_c^d h_1\left(\frac{x-a}{b-a}\right) h_2\left(\frac{y-c}{d-c}\right) f(x, y) dydx \right\} \\ & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \int_0^1 \int_0^1 h_1(t)h_2(s) dsdt. \end{aligned}$$

The proof is completed. □

**Remark 2.2** *If in Theorem 2.1,*

i) we choose  $h_1(t) = t$ ,  $h_2(s) = s$  on  $[0, 1]$ , then the inequalities (2.1) become the inequalities

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}, \quad (2.4)$$

which are proved by Sarikaya and Yaldiz in [28].

ii) we choose  $h_1(t) = t^{\alpha-1}$  ( $\alpha > 0$ ),  $h_2(s) = s^{\beta-1}$  ( $\beta > 0$ ) on  $[0, 1]$ , then the inequalities (2.1) become the fractional integral inequalities

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \left[ J_{a+,c+}^{\alpha,\beta} f(b, d) + J_{a+,d-}^{\alpha,\beta} f(b, c) + J_{b-,c+}^{\alpha,\beta} f(a, d) + J_{b-,d-}^{\alpha,\beta} f(a, c) \right] \\ & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \end{aligned}$$

which are proved by Sarikaya in [29].

**Corollary 2.3** Under assumption of Theorem 2.1 with  $h_1(t) = (-\ln t)^{\alpha-1}$  ( $\alpha > 0$ ),  $h_2(s) = (-\ln s)^{\beta-1}$  ( $\beta > 0$ ) on  $[0, 1]$ , we get the following logarithmic integral inequalities

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \quad (2.5) \\ & \leq \frac{1}{4\Gamma(\alpha)\Gamma(\beta)(b-a)(d-c)} \\ & \quad \times \int_a^b \int_c^d \left[ \left[ \ln\left(\frac{b-a}{b-x}\right) \right]^{\alpha-1} + \left[ \ln\left(\frac{b-a}{x-a}\right) \right]^{\alpha-1} \right] \\ & \quad \times \left[ \left[ \ln\left(\frac{d-c}{d-y}\right) \right]^{\beta-1} + \left[ \ln\left(\frac{d-c}{y-c}\right) \right]^{\beta-1} \right] f(x, y) dy dx \\ & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned}$$

**Corollary 2.4** Let  $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  is coordinated convex on  $\Delta := [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $0 \leq a < b$ ,  $0 \leq c < d$  and  $f \in L_1(\Delta)$ . Then, one has the inequalities:

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \quad (2.6) \\ & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4(b-a)(d-c)} \\
&\quad \times \int_a^b \int_c^d \left[ \left[ \ln \left( \frac{b-a}{b-x} \right) \right] + \left[ \ln \left( \frac{b-a}{x-a} \right) \right] \right] \\
&\quad \times \left[ \left[ \ln \left( \frac{d-c}{d-y} \right) \right] + \left[ \ln \left( \frac{d-c}{y-c} \right) \right] \right] f(x, y) dy dx \\
&\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\end{aligned}$$

**Proof** From (2.4), we have

$$\begin{aligned}
&f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \tag{2.7} \\
&\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\
&= \frac{1}{(b-a)(d-c)} \left\{ \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x, y) dy dx + \int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d f(x, y) dy dx \right. \\
&\quad \left. + \int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} f(x, y) dy dx + \int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d f(x, y) dy dx \right\}.
\end{aligned}$$

By change of variable  $x = \frac{a+t}{2}$  and  $y = \frac{c+s}{2}$  in the first integral of right side of (2.7), using inequality (2.4) and from Fubini Theorem, we get

$$\begin{aligned}
\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x, y) dy dx &= \frac{1}{4} \int_a^b \int_c^d f\left(\frac{a+t}{2}, \frac{c+s}{2}\right) ds dt \\
&\leq \frac{1}{4} \int_a^b \int_c^d \left( \frac{1}{(t-a)(s-c)} \int_a^t \int_c^s f(x, y) dy dx \right) ds dt \\
&= \frac{1}{4} \int_a^b \int_c^d \left( \int_x^b \int_y^d \frac{1}{(t-a)(s-c)} ds dt \right) f(x, y) dy dx \\
&= \frac{1}{4} \int_a^b \int_c^d \left( \ln \frac{b-a}{x-a} \right) \left( \ln \frac{d-c}{y-c} \right) f(x, y) dy dx.
\end{aligned}$$

By similar way, using change of variable  $x = \frac{a+t}{2}$  and  $y = \frac{d+s}{2}$ ,  $x = \frac{b+t}{2}$  and  $y = \frac{c+s}{2}$ ,  $x = \frac{b+t}{2}$  and  $y = \frac{d+s}{2}$  in the other integrals of right side of (2.7), respectively, we have

$$\int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d f(x, y) dy dx = \frac{1}{4} \int_a^b \int_c^d f\left(\frac{a+t}{2}, \frac{d+s}{2}\right) ds dt$$



$$\begin{aligned}
&\leq \frac{1}{4} \int_a^b \int_c^d \left( \frac{1}{(t-a)(d-s)} \int_a^t \int_s^d f(x,y) dydx \right) dsdt \\
&= \frac{1}{4} \int_a^b \int_c^d \left( \ln \frac{b-a}{x-a} \right) \left( \ln \frac{d-c}{d-y} \right) f(x,y) dydx, \\
\int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} f(x,y) dydx &= \frac{1}{4} \int_a^b \int_c^d f\left(\frac{b+t}{2}, \frac{c+s}{2}\right) dsdt \\
&\leq \frac{1}{4} \int_a^b \int_c^d \left( \frac{1}{(b-t)(s-c)} \int_t^b \int_c^s f(x,y) dydx \right) dsdt \\
&= \frac{1}{4} \int_a^b \int_c^d \left( \ln \frac{b-a}{b-x} \right) \left( \ln \frac{d-c}{y-c} \right) f(x,y) dydx, \\
\int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d f(x,y) dydx &= \frac{1}{4} \int_a^b \int_c^d f\left(\frac{b+t}{2}, \frac{d+s}{2}\right) dsdt \\
&\leq \frac{1}{4} \int_a^b \int_c^d \left( \frac{1}{(b-t)(d-s)} \int_t^b \int_s^d f(x,y) dydx \right) dsdt \\
&= \frac{1}{4} \int_a^b \int_c^d \left( \ln \frac{b-a}{b-x} \right) \left( \ln \frac{d-c}{d-y} \right) f(x,y) dydx.
\end{aligned}$$

By above calculated integrals, writing instead of the (2.7), and using the last inequality of (2.5) for  $\alpha = \beta = 2$ , we get

$$\begin{aligned}
&f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dydx \\
&\leq \frac{1}{4(b-a)(d-c)} \left\{ \int_a^b \int_c^d \left( \ln \frac{b-a}{x-a} \right) \left( \ln \frac{d-c}{y-c} \right) f(x,y) dydx \right. \\
&\quad + \int_a^b \int_c^d \left( \ln \frac{b-a}{x-a} \right) \left( \ln \frac{d-c}{d-y} \right) f(x,y) dydx \\
&\quad \left. + \int_a^b \int_c^d \left( \ln \frac{b-a}{b-x} \right) \left( \ln \frac{d-c}{y-c} \right) f(x,y) dydx \right.
\end{aligned}$$

$$\begin{aligned}
& + \int_a^b \int_c^d \left( \ln \frac{b-a}{b-x} \right) \left( \ln \frac{d-c}{d-y} \right) f(x, y) dy dx \Big\} \\
& \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}
\end{aligned}$$

which is proved the inequalities (2.6).  $\square$

**Theorem 2.5** Let  $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  is coordinated convex on  $\Delta := [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $0 \leq a < b$ ,  $0 \leq c < d$  and  $f \in L_1(\Delta)$  and  $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}$  be two positive functions such that  $h_1, h_2 \in L([0, 1])$ . Then, one has the inequalities:

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \tag{2.8} \\
& \leq \frac{1}{4I_{h_1}(b-a)} \int_a^b H_1(x) f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{4I_{h_2}(d-c)} \int_c^d H_2(y) f\left(\frac{a+b}{2}, y\right) dy \\
& \leq \frac{1}{4I_{h_1}I_{h_2}(b-a)(d-c)} \int_a^b \int_c^d H_1(x) H_2(y) f(x, y) dy dx \\
& \leq \frac{1}{8I_{h_1}(b-a)} \left[ \int_a^b H_1(x) f(x, c) dx + \int_a^b H_1(x) f(x, d) dx \right] \\
& \quad + \frac{1}{8I_{h_2}(d-c)} \left[ \int_c^d H_2(y) f(a, y) dy + \int_c^d H_2(y) f(b, y) dy \right] \\
& \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\end{aligned}$$

**Proof** Since  $f : \Delta \rightarrow \mathbb{R}$  is convex on the coordinates, it follows that the mapping  $g_x : [c, d] \rightarrow \mathbb{R}$ ,  $g_x(y) = f(x, y)$ , is convex on  $[c, d]$  for all  $x \in [a, b]$ . Then by using inequalities (1.3), we can write

$$g_x\left(\frac{c+d}{2}\right) \leq \frac{1}{2I_{h_2}(d-c)} \int_c^d H_2(y) g_x(y) dy \leq \frac{g_x(c) + g_x(d)}{2}, \quad x \in [a, b].$$

That is,

$$\begin{aligned}
f\left(x, \frac{c+d}{2}\right) & \leq \frac{1}{2I_{h_2}(d-c)} \int_c^d H_2(y) f(x, y) dy \\
& \leq \frac{f(x, c) + f(x, d)}{2}
\end{aligned} \tag{2.9}$$

for all  $x \in [a, b]$ . Then, multiplying both sides of (2.9) by  $\frac{1}{2I_{h_1}(b-a)}H_1(x)$ , and integrating with respect to  $x$  over  $[a, b]$ , we have

$$\begin{aligned} & \frac{1}{2I_{h_1}(b-a)} \int_a^b H_1(x) f\left(x, \frac{c+d}{2}\right) dx \\ & \leq \frac{1}{4I_{h_1}I_{h_2}(b-a)(d-c)} \int_a^b \int_c^d H_1(x) H_2(y) f(x, y) dy dx \\ & \leq \frac{1}{4I_{h_1}(b-a)} \left[ \int_a^b H_1(x) f(x, c) dx + \int_a^b H_1(x) f(x, d) dx \right]. \end{aligned} \quad (2.10)$$

By similar argument applied for the mapping  $g_y : [a, b] \rightarrow \mathbb{R}$ ,  $g_y(x) = f(x, y)$ , we have

$$\begin{aligned} & \frac{1}{2I_{h_2}(d-c)} \int_c^d H_2(y) f\left(\frac{a+b}{2}, y\right) dy \\ & \leq \frac{1}{4I_{h_1}I_{h_2}(b-a)(d-c)} \int_a^b \int_c^d H_1(x) H_2(y) f(x, y) dy dx \\ & \leq \frac{1}{4I_{h_2}(d-c)} \left[ \int_c^d H_2(y) f(a, y) dy + \int_c^d H_2(y) f(b, y) dy \right]. \end{aligned} \quad (2.11)$$

Adding the inequalities (2.10) and (2.11), we have

$$\begin{aligned} & \frac{1}{2I_{h_1}(b-a)} \int_a^b H_1(x) f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{2I_{h_2}(d-c)} \int_c^d H_2(y) f\left(\frac{a+b}{2}, y\right) dy \\ & \leq \frac{1}{2I_{h_1}I_{h_2}(b-a)(d-c)} \int_a^b \int_c^d H_1(x) H_2(y) f(x, y) dy dx \\ & \leq \frac{1}{4I_{h_1}(b-a)} \left[ \int_a^b H_1(x) f(x, c) dx + \int_a^b H_1(x) f(x, d) dx \right] \\ & \quad + \frac{1}{4I_{h_2}(d-c)} \left[ \int_c^d H_2(y) f(a, y) dy + \int_c^d H_2(y) f(b, y) dy \right] \end{aligned}$$

which give the second and the third inequalities in (2.8).

Now, by using the first inequality in (1.3), we also have

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2I_{h_1}(b-a)} \int_a^b H_1(x) f\left(x, \frac{c+d}{2}\right) dx$$

and

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2I_{h_2}(d-c)} \int_c^d H_2(y) f\left(\frac{a+b}{2}, y\right) dy$$

by addition,

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{4I_{h_1}(b-a)} \int_a^b H_1(x) f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{4I_{h_2}(d-c)} \int_c^d H_2(y) f\left(\frac{a+b}{2}, y\right) dy \end{aligned}$$

which give the first inequality in (2.8).

Finally, by using the second inequality in (1.3) we can also state,

$$\begin{aligned} \frac{1}{2I_{h_1}(b-a)} \int_a^b H_1(x) f(x, c) dx & \leq \frac{f(a, c) + f(b, c)}{2} \\ \frac{1}{2I_{h_1}(b-a)} \int_a^b H_1(x) f(x, d) dx & \leq \frac{f(a, d) + f(b, d)}{2} \\ \frac{1}{2I_{h_2}(d-c)} \int_c^d H_2(y) f(a, y) dy & \leq \frac{f(a, c) + f(a, d)}{2} \end{aligned}$$

and

$$\frac{1}{2I_{h_2}(d-c)} \int_c^d H_2(y) f(b, y) dy \leq \frac{f(b, c) + f(b, d)}{2}$$

which give, by addition, the last inequality in (2.8).  $\square$

**Remark 2.6** If in Theorem 2.5 with  $h_1(t) = t$ ,  $h_2(s) = s$  on  $[0, 1]$ , then the inequalities (2.8) become the inequalities

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\ & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ & \leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \\ & \quad \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\ & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned}$$

which are proved by Dragomirin [11].

**Remark 2.7** If in Theorem 2.5 with  $h_1(t) = t^{\alpha-1}$  ( $\alpha > 0$ ),  $h_2(s) = s^{\beta-1}$  ( $\beta > 0$ ) on  $[0, 1]$ , then the inequalities (2.8) become the fractional integral inequalities

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
& \leq \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha} \left[ J_{a+}^\alpha f\left(b, \frac{c+d}{2}\right) + J_{b-}^\alpha f\left(a, \frac{c+d}{2}\right) \right] \\
& \quad + \frac{\Gamma(\beta+1)}{4(d-c)^\beta} \left[ J_{c+}^\beta f\left(\frac{a+b}{2}, d\right) + J_{d-}^\beta f\left(\frac{a+b}{2}, c\right) \right] \\
& \leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \left[ J_{a+,c+}^{\alpha,\beta} f(b, d) + J_{a+,d-}^{\alpha,\beta} f(b, c) + J_{b-,c+}^{\alpha,\beta} f(a, d) + J_{b-,d-}^{\alpha,\beta} f(a, c) \right] \\
& \leq \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha} \left[ J_{a+}^\alpha f(b, c) + J_{a+}^\alpha f(b, d) + J_{b-}^\alpha f(a, c) + J_{b-}^\alpha f(a, d) \right] \\
& \quad + \frac{\Gamma(\beta+1)}{4(d-c)^\beta} \left[ J_{c+}^\beta f(a, d) + J_{c+}^\beta f(b, d) + J_{d-}^\beta f(a, c) + J_{d-}^\beta f(b, c) \right] \\
& \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}
\end{aligned}$$

which are proved by Sarikaya in [29].

**Corollary 2.8** Under assumption of Theorem 2.5 with  $h_1(t) = (-\ln t)^{\alpha-1}$  ( $\alpha > 0$ ),  $h_2(s) = (-\ln s)^{\beta-1}$  ( $\beta > 0$ ) on  $[0, 1]$ , we get the following logarithmic integral inequalities:

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
& \leq \frac{1}{4I_{h_1}(b-a)} \int_a^b \left[ \left[ \ln\left(\frac{b-a}{b-x}\right) \right]^{\alpha-1} + \left[ \ln\left(\frac{b-a}{x-a}\right) \right]^{\alpha-1} \right] f\left(x, \frac{c+d}{2}\right) dx \\
& \quad + \frac{1}{4I_{h_2}(d-c)} \int_c^d \left[ \left[ \ln\left(\frac{d-c}{d-y}\right) \right]^{\beta-1} + \left[ \ln\left(\frac{d-c}{y-c}\right) \right]^{\beta-1} \right] f\left(\frac{a+b}{2}, y\right) dy \\
& \leq \frac{1}{4I_{h_1}I_{h_2}(b-a)(d-c)} \int_a^b \int_c^d \left[ \left[ \ln\left(\frac{b-a}{b-x}\right) \right]^{\alpha-1} + \left[ \ln\left(\frac{b-a}{x-a}\right) \right]^{\alpha-1} \right] \\
& \quad \times \left[ \left[ \ln\left(\frac{d-c}{d-y}\right) \right]^{\beta-1} + \left[ \ln\left(\frac{d-c}{y-c}\right) \right]^{\beta-1} \right] f(x, y) dy dx
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{8I_{h_1}(b-a)} \int_a^b \left[ \left[ \ln \left( \frac{b-a}{b-x} \right) \right]^{\alpha-1} + \left[ \ln \left( \frac{b-a}{x-a} \right) \right]^{\alpha-1} \right] [f(x,c) + f(x,d)] dx \\
&\quad + \frac{1}{8I_{h_2}(d-c)} \int_c^d \left[ \left[ \ln \left( \frac{d-c}{d-y} \right) \right]^{\beta-1} + \left[ \ln \left( \frac{d-c}{y-c} \right) \right]^{\beta-1} \right] [f(a,y) + f(b,y)] dy \\
&\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}.
\end{aligned}$$

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