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Uniform convergent modified weak Galerkin method for convection-dominated two-point boundary value problems

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Abstract: We propose and analyze a modified weak Galerkin finite element method (MWG-FEM) for solving singularly perturbed problems of convection-dominated type. The proposed method is constructed over piecewise polynomials of degree $k \geq 1$ on interior of each element and piecewise constant on the boundary of each element. The present method is parameter-free and has less degrees of freedom compared to the classical weak Galerkin finite element method. The method is shown uniformly convergent for small perturbation parameters. An uniform convergence rate of $\mathcal{O}((N^{-1}\ln N)^k)$ in the energy-like norm is established on the piecewise uniform Shishkin mesh, where N is the number of elements. Various numerical examples are presented to confirm the theoretical results. Moreover, we numerically confirm that the proposed method has the optimal order error estimates of $\mathcal{O}(N^{-(k+1)})$ in a discrete L^2-norm and converges at superconvergence order of $\mathcal{O}((N^{-1}\ln N)^{2k})$ in the discrete L^∞-norm.

Key words: Singularly perturbed problem, modified weak Galerkin method, Shishkin mesh, uniformly convergence

1. Introduction

In this paper we consider the following singularly perturbed convection-diffusion two-point boundary value problems: Find $u \in C^2(0,1) \cap C[0,1]$ such that

$$\begin{aligned} &-\varepsilon u''(x) + \beta(x)u'(x) + \gamma(x)u(x) = g(x) \text{ in } \Omega = (0,1), \\ &u(0) = 0, \quad u(1) = 0, \end{aligned} \tag{1.1}$$

where $0 < \varepsilon << 1$ and we assume that $\beta(x), \gamma(x)$ and $g(x)$ are sufficiently smooth functions with $\beta(x) \geq \alpha > 0$, $\gamma(x) \geq 0$, and

$$\gamma(x) - \frac{1}{2}\beta'(x) \geq a > 0 \quad \forall x \in \Omega, \tag{1.2}$$

where a is a constant. Under these assumptions, the problem (1.1) has a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$. For small perturbation parameter ε, the problem is singularly perturbed [19], [18]. If ε is small enough, with the help of the change of variable $w(x) = \exp(-\eta x)u(x)$ for a suitable η, the condition $\beta \geq \alpha > 0$ implies the condition (1.2) and $\gamma \geq 0$.

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The solution $u(x)$ to the problem (1.1) has an exponential boundary layer of width $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ at $x=1$. This boundary layer makes the conventional numerical methods such as the standard finite difference and finite element methods useless unless prohibitively large number of mesh points or smaller than the parameter ε are used.

The numerical solutions of singularly perturbed problems (SPPs) often suffer from layers which are very thin regions near the boundaries where the solution and its derivatives are very large. Some numerical approaches have been proposed and analyzed for solving such kinds of problems, such as fitted mesh B-spline [12], uniformly convergent nonstandard finite difference methods [21], variable mesh finite difference method [13], implicit-explicit local differential method [31]. A popular approach for uniformly convergent solutions for these problems is the use of layer-adapted meshes such as Bakhvalov-type mesh [1] and Shishkin mesh [19] which have gained much attention and they are still popular. Some nonphysical oscillations occur in the approximation even if the layer adapted meshes are employed in two dimensions [36]. We refer the readers to the books [19, 22, 27] and references therein for further details.

The other efficient technique for solving singularly perturbed problem is fitted operator methods. Upwind-type schemes are example of the commonly used fitted operator methods in the literature. The streamline diffusion finite element (SDFE) method [4, 11] and its variants [7] have successfully been used for solving singularly perturbed convection dominated problems. These methods add residuals with weights for the stability of the conforming Galerkin method. However, there are some disadvantages of these methods because of adding too much diffusion and they also have oscillatory solutions [6].

Weak Galerkin finite element method (WG-FEM) has been developed for solving second order elliptic partial differential equations [32]. The key idea of WG-FEM is the introduction of weak function spaces and weak derivatives in the corresponding variational formulation. The weak functions are defined as $u=\{u_0,u_b\}$ where $u=u_0$ inside of the element and $u=u_b$ on the boundary of the element. The weak Galerkin method has been studied and applied to a variety of problems including Stokes equations [33], interface problem [24], Maxwell equation [23], singularly perturbed elliptic equations [15] and time fractional convection-dominated problems [29]. Although the WG-FEM allows the use of totally discontinuous piecewise polynomials and it is is highly flexible, it has extra degrees of freedom associated with basis functions u_b. As an alternative, a modified weak Galerkin finite element method (MWG-FEM) has been introduced in [34] to reduce the degrees of freedom, i.e. the number of unknown in the discrete system. The WG-FEM has some features over the standard finite element method. For example, there is no continuity condition on the approximation spaces and we do not need to have H^1 conforming in the WG-FEM. The MWG-FEM has less degree of freedom than the local discontinuous Galerkin methods which introduce auxiliary variables (e.g., fluxes) in the formulation and has the same degrees of freedom with the discontinuous Galerkin (DG) methods in primal formulation (see [5]). However MWG-FEM is absolutely stable and it is a parameter-free method, that is, we do not need to choose a sufficiently large stabilization parameters in the MWG-FEM unlike DG methods. Recently, the MWG-FEMs have been successfully applied to parabolic problems [8], convection-diffusion equations [9], Stokes equations [25] and convection-dominated diffusion with weakly imposed boundary condition [10].

The main concern of this paper is to study and analyze a class of uniform convergent MWG-FEM for SPPs of diffusion-convection type on the uniform Shishkin mesh. The uniform convergent weak Galerkin method has been proposed in [37]. Compared to [37], the proposed method here has reduced numbers of unknown and has shorter and simplified error analysis. The obtained results in this paper are the first uniform convergence results of MWG-FEM for singularly elliptic problems in one dimension. We prove a uniform convergence order of $\mathcal{O}(N^{-1}\ln N)^k$ in the discrete energy norm and the optimal order error estimates with order of $\mathcal{O}(N^{-(k+1)})$ in a discrete L^2- norm for the strongly convection-dominated cases (e.g., $\varepsilon = 10^{-8}$) and order of $\mathcal{O}(N^{-(k+1/2)})$ in a discrete L^2- norm for the intermediate cases (e.g., $\varepsilon = 10^{-3}$) under some conditions.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries and notations. Additionally, the bounds for the regular and layer components of the solution and their derivatives are established and the piecewise uniform Shishkin mesh is given in Section 2. The MWG-FEM scheme for the singularly perturbed convection-diffusion-reaction problems is introduced in Section 3. The stability and the error analysis of the method are studied in Sections 4 and 5, respectively. Various numerical examples are given to confirm the theoretical findings in Section 6. In Section 7, conclusion and some future direction are summarized.

Throughout this article, we use C for generic constants independent of ε, N and the mesh size h which may be different at each inequality.

2. Preliminaries

In this section, we first give the decomposition of the analytical solution of the problem (1.1). Then we will derive the bounds for the regular and layer parts of the solution and their derivatives. Next, we provide the piecewise-uniform Shishkin mesh which is layer adapted mesh to deal with layers. Sobolev spaces with the related norms and some basic notations are introduced at the end of this section.

2.1. Properties of the solution

We decompose the solution u of the problem (1.1) into a sum of a regular and layer components in the following lemma. This solution decomposition is necessary for the uniform convergence of numerical methods on Shishkin mesh for SPPs [17].

Lemma 2.1 *(Regularity of the solution) [27] Let m be a positive integer. The exact solution u of the problem (1.1) can be decomposed into $u = u_R + u_L$ where u_R and u_L are regular and singular parts, respectively and they satisfy the following bounds for $0 \le l \le m$*

$$|u_R^{(l)}(x)| \le C \tag{2.1}$$

$$|u_L^{(l)}(x)| \le C\varepsilon^{-l} \exp((-\alpha(1-x))/\varepsilon). \tag{2.2}$$

Here, the constant C is independent of ε and m depends on the smoothness of the solution u.

2.2. The piecewise uniform Shishkin mesh

Let N be an even integer. Define the transition point τ_ε by

$$\tau_\varepsilon = \min\big(\frac{1}{2}, \frac{\varepsilon(k+1)}{\alpha}\ln N\big),$$

where k is the order of polynomials used in the approximation space. In practise, we assume that $\tau_\varepsilon = \dfrac{(k+1)\varepsilon}{\alpha}\ln N$ for otherwise either ε is not small or $N^{-1} < \varepsilon$ (when N is sufficiently large) which can be handled by using the uniform mesh and throughout this paper we assume $\varepsilon < CN^{-1}$ which is not a restriction in the singularly perturbed problems. We divide the computational domain Ω into two intervals $\Omega_1 = [0, 1-\tau_\varepsilon]$ and $\Omega_2 = [1-\tau_\varepsilon, 1]$. Then divide each of the subintervals Ω_1 and Ω_2 into $N/2$ equal subintervals.

We define the nodes of mesh recursively as follows:

$$x_0 = 0, \quad x_n = x_N + h_n, \quad h_n = \begin{cases} H & \text{for } n = 1, \dots, N/2, \\ h & \text{for } n = N/2+1, \dots, N. \end{cases} \tag{2.3}$$

where

$$H = \frac{2(1-\tau_\varepsilon)}{N} \quad \textit{and} \quad h = \frac{2\tau_\varepsilon}{N}.$$

Note that $H = \mathcal{O}(N^{-1})$ and $h = \mathcal{O}(N^{-1}\ln N)$. Observe that $x_N = 1-\tau_\varepsilon$ is the transition point.

We denote the mesh and a partition of the domain Ω by $I_n = [x_{n-1}, x_n], \quad n = 1, \dots, N$ and $\mathcal{T}_N = \{I_n : n = 1, \dots, N\}$, respectively. For $I_n \in \mathcal{T}_N$, the outward unit normal $\mathbf{n}_{I_n}$ on I_n is defined as $\mathbf{n}_{I_n}(x_n) = 1$ and $\mathbf{n}_{I_n}(x_{n-1}) = -1$; for simplicity, we use $\mathbf{n}$ instead of $\mathbf{n}_{I_n}$.

3. MWG-FEM

In this section, we first introduce the notions of weak functions and weak derivatives. Based on these concepts, we will construct the MWG-FEM for the system (1.1).

We define the space of weak functions $\mathcal{W}(I_n)$ on the interval I_n by

$$\mathcal{W}(I_n) = \{u = \{u_0, u_b\} : u_0 \in L^2(I_n), v_b \in L^\infty(\partial I_n)\}.$$

Here, $u = \{u_0, u_b\}$ is called a weak function such that u_0 is the value of u inside of the interval (x_N, x_n) and u_b is the value of u on the boundary of the interval $\partial I_n = \{x_{n-1}, x_n\}$. The inclusion map

$$\mathcal{I}_{\mathcal{W}}(u) = \{u|_{I_n}, u|_{\partial I_n}\}, \quad \forall u \in H^1(I_n)$$

embeds the local Sobolev space $H^1(I_n)$ into the weak function space $\mathcal{W}(I_n)$.

For a given integer $k \geq 1$, we define a local weak Galerkin (WG) finite element space $S_N(I_n)$ as follows:

$$S_N(I_n) = \{u = \{u_0, u_b\} : u_0|_{I_n} \in \mathbb{P}_k(I_n), u_b|_{\partial I_n} \in \mathbb{P}_0(\partial I_n) \quad \forall I_n \in \mathcal{T}_N\}, \tag{3.1}$$

where $\mathbb{P}_k(I_n)$ is the set of polynomials on I_n of degree at most k and $\mathbb{P}_0(\partial I_n)$ is the set of constant polynomials on ∂I_n.

A global WG finite element space S_N consists of $u = \{u_0, u_b\}$ such that $u_0|_{I_n} \in \mathbb{P}_k(I_n)$ and u_b is the constant at the nodes x_n for $n = 1, \dots, N$.

The weak derivative of a weak function $u = \{u_0, u_b\} \in S_N$ denoted by $d_{w,I_n}u \in \mathbb{P}_{k-1}(I_n)$ is defined on I_n as the unique polynomial satisfying the following equation,

$$(d_{w,I_n}u, v)_{I_n} = -(u_0, v')_{I_n} + \langle u_b, v\mathbf{n}\rangle_{\partial I_n} \qquad \forall v \in \mathbb{P}_{k-1}(I_n), \tag{3.2}$$

where

$$(w, z)_{I_n} = \int_{I_n} w(x)z(x)\,dx$$

and

$$\langle w, z\mathbf{n}\rangle_{\partial I_n} = w(x_n)z(x_n) - w(x_N)z(x_N).$$

The weak convection derivative of a weak function $u = \{u_0, u_b\} \in S_N$ denoted by $d^{\beta}_{w,I_n}u \in \mathbb{P}_k(I_n)$ is defined on I_n as the unique polynomial satisfying the following equation,

$$(d^{\beta}_{w,I_n}u, v)_{I_n} = -(u_0, (\beta v)')_{I_n} + \langle u_b, \beta v\mathbf{n}\rangle_{\partial I_n} \qquad \forall v \in \mathbb{P}_k(I_n). \tag{3.3}$$

Then the weak derivatives $d_w u$ and $d^{\beta}_w u$ of a weak function $u = \{u_0, u_b\}$ on S_N is given by

$$(d_w u)|_{I_n} = d_{w,I_n}(u|_{I_n}), \quad (d^{\beta}_w u)|_{I_n} = d^{\beta}_{w,I_n}(u|_{I_n}) \forall u \in S_N.$$

We define the average $\{u\}$ and jump $[u]$ of a function $u \in S_N$ at the interelement boundaries

$$\{u(x_n)\} = \frac{1}{2}(u(x_n^+) + u(x_n^-)), \tag{3.4}$$

$$[u(x_n)] = u(x_n^+) - u(x_n^-), \quad \textit{for} \quad n = 1, 2 \ldots, N. \tag{3.5}$$

where $u(x_n^{\pm}) = \lim_{s\to 0} u(x_n \pm s)$. We extend the definition of average and jump at the boundary points of the domain as follows:

$$\{u(x_0)\} = u(x_0^+), \quad \{u(x_N)\} = u(x_N^-),$$
$$[u(x_0)] = u(x_0^+), \quad [u(x_N)] = -u(x_N^-).$$

In the MWG-FEM, the boundary value u_b is replaced by the average $\{u\}$ of the function u in S_N. Thus the finite element space in the MWG-FEM approximation is defined as

$$V_N = \big\{v \in L^2(\Omega) : v|_{I_n} \in \mathbb{P}_k(I_n), I_n \in \mathcal{T}_N \text{ and } v(0^+) = v(1^-) = 0\big\}.$$

The following useful identity will be used repeatedly in our later analysis. For $v, w \in V_N$ we have

$$\sum_{I_n \in \mathcal{T}_N} \langle v - \{v\}, \mathbf{n}w\rangle_{\partial I_n} = \sum_{n=1}^{N} \langle [v], \{w\}\rangle_{\partial I_n}. \tag{3.6}$$

For any function $v \in V_N$, we define a weak function $v = \{v, \{v\}\} \in S_N$, which is also denoted by v if there is no confusion.

Based on (3.2) and (3.3), for a function $u \in V_N$, the modified weak derivative $d^m_w u \in \mathbb{P}_{k-1}(I_n)$ and modified weak convection derivative $d^{\beta,m}_w u \in \mathbb{P}_k(I_n)$ defined on I_n as the unique polynomial satisfying the following equation

$$\big(d^m_w u, v\big)_{I_n} = -\big(u, v'\big)_{I_n} + \big\langle \{u\}, v\mathbf{n}\big\rangle_{\partial I_n} \qquad \forall v \in \mathbb{P}_{k-1}(I_n), \tag{3.7}$$

and

$$\big(d^{\beta,m}_w u, v\big)_{I_n} = -\big(u, (\beta v)'\big)_{I_n} + \big\langle \{u\}, \beta v\mathbf{n}\big\rangle_{\partial I_n} \qquad \forall v \in \mathbb{P}_k(I_n), \tag{3.8}$$

respectively.

Remark 3.1 *This newly defined modified weak derivative is different from the weak derivative operator defined in [35]. This modified definition replaces the values u_b of u by the average operator $\{u\}$ on the boundary points of I_n. This reduces the degree of freedom for the problem, that is, the unknown coefficients in the system are reduced.*

Remark 3.2 *If u is continuous in Ω, then we have $\{u\} = u$. Using integration by parts, we see that from the definition of weak derivative (3.2)*

$$\begin{aligned}\int_{I_n} d_w^m u(x)v(x)\,dx &= -\int_{I_n} u(x)v'(x)\,dx + \langle \{u\}, v\boldsymbol{n}\rangle_{\partial I_n} \\ &= \int_{I_n} u'(x)v(x)\,dx \qquad \forall v \in \mathbb{P}_{k-1}(I_n),\end{aligned} \tag{3.9}$$

which implies the modified weak derivative in fact is the L^2 projection of the standard differential operator on the space of polynomials. Thus, we have $d_w^m u(x) = u'(x)$ when $u \in \mathbb{P}_k(\Omega)$.

Similarly, when u is continuous in Ω, the integration by parts and the definition of modified weak convection derivative (3.8) lead to

$$\begin{aligned}\int_{I_n} d_w^{\beta,m} u(x)w(x)\,dx &= -\int_{I_n} u(x)(\beta w)'(x)\,dx + \langle \beta\{u\}, w\boldsymbol{n}\rangle_{\partial I_n} \\ &= \int_{I_n} \beta(x)u'(x)w(x)\,dx \qquad \forall v \in \mathbb{P}_k(T).\end{aligned} \tag{3.10}$$

showing that the modified weak divergence is the L^2 projection of the classical differential operator related to $\beta(x)u'(x)$ on the space $\mathbb{P}_k(\Omega)$. Thus we have $d_w^{\beta,m}u(x) = \beta(x)u'(x)$ when $u \in \mathbb{P}_k(\Omega)$ and $\beta(x)$ is a constant function.

We use the following basic notations. $L^2(\Omega)$ denotes the space of square integrable functions on Ω with the norm $\|u\|^2_{L^2(\Omega)} = \int_\Omega u^2(x)\,dx$ which sometimes denoted by $\|u\|^2$. The standard Sobolev space is denoted by $H^k(\Omega)$ with the norm $\|\cdot\|_{k,\Omega}$ and seminorm $|\cdot|_{k,\Omega}$ given as

$$\|u\|^2_{k,\Omega} = \sum_{j=0}^{k} \|u^{(j)}\|^2_{L^2(\Omega)}, \quad |u|^2_{k,\Omega} = \|u^{(k)}\|^2_{L^2(\Omega)}.$$

For each interval I_n, the broken Sobolev space is defined by

$$H^k_N(\Omega) = \{u \in L^2(\Omega) : u|_{I_n} \in H^k(I_n), \quad \forall I_n \in \mathcal{T}_h\},$$

and the corresponding norm and seminorm

$$\|u\|^2_{H^k_N(\Omega)} = \sum_{n=1}^{N} \|u\|^2_{k,I_n}, \quad |u|^2_{H^k_N(\Omega)} = \sum_{n=1}^{N} |u|^2_{k,I_n}.$$

For the future reference, we adopt the following notations

$$(u,v)_{I_n} = \int_{I_n} u(x)v(x)\,dx, \quad (u,v) = \sum_{I_n\in\mathcal{T}_N} (u,v)_{I_n}$$

$$\|u\|_{I_n}^2 = (u,u)_{I_n}, \quad \|u\|^2 = \sum_{n=1}^{N} \|u\|_{I_n}^2,$$

$$\langle u,v\rangle_{\partial I_n} = u(x_n^-)v(x_n^-) + u(x_{n-1}^+)v(x_{n-1}^+),$$

$$\langle u,v\rangle = \sum_{I_n\in\mathcal{T}_N} \langle u,v\rangle_{\partial I_n}, \quad \|u\|_{\partial I_n}^2 = \langle u,u\rangle_{\partial I_n}.$$

The variational formulation of the problem (1.1), after multiplying the equation (1.1) by the test functions $v \in H_0^1(\Omega)$ is to seek $u \in H_0^1(\Omega)$ such that

$$\varepsilon(u',v') + (\beta u',v) + (\gamma u,v) = (g,v), \quad \forall v \in H_0^1(\Omega). \tag{3.11}$$

We now formulate the MWG-FEM for the problem (1.1) based on the variational formulation (3.11) as follows:

Algorithm 1 The modified weak Galerkin scheme for convection-diffusion-reaction problem.

The MWG-FEM for the problem (1.1) is to find $u_N \in V_N$ satisfying the following equation:

$$a(u_N,v_N) = L(v_N) \quad \forall v_N \in V_N. \tag{3.12}$$

where the bilinear form $a(v,z) = a_d(v,z) + a_c(v,z) + s_d(v,z) + s_c(v,z)$ and the linear functional $L(\cdot)$ on V_N are given by:

$$a_d(v,z) = \varepsilon\big(d_w^m v, d_w^m z\big), \quad \forall v,z \in V_N, \tag{3.13}$$

$$a_c(v,z) = \big(d_w^{\beta,m} v, z\big) + \big(\gamma v, z\big), \quad \forall v,z \in V_N, \tag{3.14}$$

$$s_d(v,z) = \sum_{n=1}^{N} \sigma_n \langle [v],[z]\rangle_{\partial I_n}, \quad \forall v,z \in V_N, \tag{3.15}$$

$$s_c(v,z) = \sum_{n=1}^{N} \big\langle \beta \mathbf{n}_{I_n}(v - \{v\}), z - \{z\}\big\rangle_{\partial_+ I_n}, \quad \forall v,z \in V_N, \tag{3.16}$$

$$L(v) = \big(g,v\big), \quad \forall v \in V_N, \tag{3.17}$$

where $\partial_+ I_n = \{x \in \partial I_n : \beta(x)\mathbf{n}_{I_n}(x) \geq 0\}$, and $\sigma_n \geq 0$ is a penalization parameter associated with the node x_n. The penalization parameter σ_n is very sensitive for the uniform convergence analysis and will be determined exactly in the error analysis below.

4. Stability of the MWG-FEM

The following multiplicative trace inequality will be useful in proving the error estimates.

Lemma 4.1 *[38] If $\varphi \in H^1(I_n)$, we have*

$$\|\varphi\|^2_{\partial I_n} \le C\big(h_n^{-1}\|\varphi\|^2_{I_n} + \|\varphi\|_{I_n}\|\varphi'\|_{I_n}\big). \tag{4.1}$$

We define an energy norm $|||\cdot|||_\varepsilon$ in V_N: for $v \in V_N$,

$$|||v|||^2 = \varepsilon\|v\|^2_w + \|v\|^2_a, \tag{4.2}$$

where

$$\|v\|^2_w = \sum_{n=1}^{N} ||d^m_w v\|^2_{I_n} + s^2_d(v,v), \tag{4.3}$$

$$\|v\|^2_a = \sum_{n=1}^{N} c_n|\sqrt{\beta(x_n)}(v - \{v\})(x_n^-)|^2 + \sum_{n=1}^{N}\|v\|^2_{I_n}, \tag{4.4}$$

with $c_n = \begin{cases} \frac{1}{2}, & \text{for } n = N \\ 1, & \text{for } n = 1,\dots,N-1. \end{cases}$

We also introduce the discrete H^1 energy norms $|||\cdot|||_\varepsilon$ in $V_N + H^1_0(\Omega)$ defined as

$$|||v|||^2_\varepsilon = \varepsilon\|v\|^2_{1,\varepsilon} + \|v\|^2_a \tag{4.5}$$

where

$$\|v\|^2_{1,\varepsilon} = \sum_{n=1}^{N} ||v'\|^2_{I_n} + s^2_d(v,v). \tag{4.6}$$

We show that the norms $|||\cdot|||$ and $|||\cdot|||_\varepsilon$ are equivalent in the MWG finite element space V_N.

Lemma 4.2 *If $v_N \in V_N$, then there are two positive constant C_l and C_s such that*

$$C_l|||v_N||| \le |||v_N|||_\varepsilon \le C_s|||v_N|||. \tag{4.7}$$

Proof For $v_N \in V_N$, by using the definition of weak derivative (3.7) and integration by parts we arrive at

$$\big(d^m_w v_N, w)_{I_n} = \big(v'_N, w\big)_{I_n} + \big\langle \{v_N\} - v_N, w\mathbf{n}\big\rangle_{\partial I_n}, \quad \forall w \in \mathbb{P}_{k-1}(I_n). \tag{4.8}$$

Choosing $w = d^m_w v_N$ in the above equation (4.8) yields

$$\|d^m_w v_N\|^2_{I_n} = \big(v'_N, d^m_w v_N\big)_{I_n} + \big\langle \{v_N\} - v_N, d^m_w v_N\mathbf{n}\big\rangle_{\partial I_n}.$$

Summing up the above equation over all interval I_n and using the identity (3.6) and the trace inequality, we get

$$\begin{aligned}\|d^m_w v_N\|^2 &= \big(v'_N, d^m_w v_N\big) + \big\langle [v_N], \{d^m_w v_N\}\big\rangle \\ &\le C\big(\sum_{n=1}^{N} ||v'_N\|^2_{I_n} + \sum_{n=1}^{N} h_n^{-1}\|[v_N]\|^2_{\partial I_n}\big)^{1/2}\|d^m_w v_N\|.\end{aligned}$$

This shows that

$$\varepsilon\|d_w^m v_N\|^2 \le C\big(\sum_{n=1}^{N}\varepsilon||v_N'\|_{I_n}^2 + \sum_{n=1}^{N}\varepsilon h_n^{-1}\|[v_N]\|_{\partial I_n}^2\big). \tag{4.9}$$

We choose the penalty parameter σ_n (see (5.37)) such that

$$\frac{\varepsilon h_n^{-1}}{\sigma_n} \le C \quad n = 1, 2, \ldots, N.$$

Then we have

$$\sum_{n=1}^{N}\varepsilon h_n^{-1}\|[v_N]\|_{\partial I_n}^2 = \sum_{n=1}^{N}\frac{\varepsilon h_n^{-1}}{\sigma_n}\sigma_n\|[v_N]\|_{\partial I_n}^2 \le Cs_d(v_N, v_N).$$

Therefore, we have

$$\varepsilon\|d_w^m v_N\|^2 \le C\big(\sum_{n=1}^{N}\varepsilon||v_N'\|_{I_n}^2 + s_d^2(v_N, v_N)\big). \tag{4.10}$$

Taking $w = v_N'$ in the equation (4.8) yields

$$\|v_N'\|_{I_n}^2 = \big(v_N', d_w^m v_N\big)_{I_n} - \big\langle\{v_N\} - v_N, v_N'\mathbf{n}\big\rangle_{\partial I_n}.$$

We sum up the above equation over all interval I_n and use the identity (3.6) along with the trace inequality to get

$$\begin{aligned}\|v_N'\|^2 &= \big(v_N', d_w^m v_N\big) - \big\langle[v_N], \{v_N'\}\big\rangle \\ &\le C\big(\sum_{n=1}^{N}||d_w^m v_N\|_{I_n}^2 + \sum_{n=1}^{N}h_n^{-1}\|[v_N]\|_{\partial I_n}^2\big)^{1/2}\|v_N'\|.\end{aligned}$$

This shows that

$$\varepsilon\|v_N'\|^2 \le C\big(\sum_{n=1}^{N}\varepsilon||d_w^m v_N\|_{I_n}^2 + s_d^2(v_N, v_N)\big). \tag{4.11}$$

We obtain the desired result (4.7) in view of the inequalities (4.10) and (4.11) and the definition of the norms $|||\cdot||$ and $|||\cdot|||_\varepsilon$. Thus we complete the proof. □

We next show the coercivity property of the bilinear form $a(\cdot,\cdot)$ given in (3.12).

Lemma 4.3 *There is a positive constant C such that*

$$a(v_N, v_N) \ge C|||v_N|||^2, \quad \forall v_N \in V_N. \tag{4.12}$$

Proof If $v_N, z_N \in V_N$, we obtain from the definition of the modified weak convection derivative (3.8) and integration by parts that

$$\begin{aligned}\big(d_w^{\beta,m}v_N, z_N\big) &= -\big(v_N, (\beta z_N)'\big) + \big\langle\{v_N\}, \beta z_N\mathbf{n}\big\rangle \\ &= \big(\beta v_N', z_N\big) - \big\langle\beta(v_N - \{v_N\}), z_N\mathbf{n}\big\rangle,\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\left(d_w^{\beta,m} z_N, v_N\right) &= -\left(z_N, (\beta v_N)'\right) + \left\langle \{z_N\}, \beta v_N \mathbf{n}\right\rangle \\ &= -\left(z_N, (\beta v_N)'\right) + \left\langle \{z_N\}, \beta \mathbf{n}(v_N - \{v_N\})\right\rangle,\end{aligned} \tag{4.14}$$

where we use the facts that $\sum_{n=1}^{N} \left\langle \beta \mathbf{n}\{v_N\}, \{z_N\}\right\rangle_{\partial I_n} = 0$ in the last equality. Taking $v_N = z_N$ and summing up the equations (4.13) and (4.14), we arrive at

$$\left(d_w^{\beta,m} v_N, v_N\right) = -\frac{1}{2}\left(\beta' v_N, v_N\right) - \frac{1}{2}\left\langle \beta \mathbf{n}(v_N - \{v_N\}), v_N - \{v_N\}\right\rangle. \tag{4.15}$$

A simple calculation reveals that

$$s_c(v_N, v_N) - \frac{1}{2}\left\langle \beta \mathbf{n}(v_N - \{v_N\}), v_N - \{v_N\}\right\rangle = \sum_{n=1}^{N} c_n |\sqrt{\beta(x_n)}(v - \{v\})(x_n^-)|^2.$$

Thus, we have

$$\begin{aligned} a_c(v_N, v_N) + s_c(v_N, v_N) &= \left((\gamma - \frac{1}{2}\beta')v_N, v_N\right) + \sum_{n=1}^{N} c_n |\sqrt{\beta(x_n)}(v - \{v\})(x_n^-)|^2 \\ &\geq \left(a v_N, v_N\right) + \sum_{n=1}^{N} c_n |\sqrt{\beta(x_n)}(v - \{v\})(x_n^-)|^2 \\ &\geq C\|v_N\|_a^2. \end{aligned}$$

Combining this with the trivial result that $\varepsilon^2\left(d_w^m v_N, d_w^m v_N\right) + \left(\gamma v_N, v_N\right) + s_d(v_N, v_N) \geq C|||v_N|||^2$, we have

$$a(v_N, v_N) \geq C|||v_N|||^2,$$

with $C = \min\{a, 1\}$. The proof is now completed.

□

Lemma 4.3 implies that

$$|||u_N||| \leq ||g||,$$

which in turn implies that the problem (3.12) has a unique solution. The existence follows from the uniqueness.

From Lemma 4.2 and Lemma 4.3, we have the following coercivity property in $|||\cdot|||_\varepsilon-$ norm.

Lemma 4.4 *There is a positive constant C such that*

$$a(v_N, v_N) \geq C|||v_N|||_\varepsilon^2, \quad \forall v_N \in V_N. \tag{4.16}$$

5. Error analysis

In this section, we derive the error estimates for the MWG-FEM for the problem (1.1). We will establish an optimal order of convergence for the MWG-FEM. We adapt the idea given in [30]. On each interval I_n, we introduce the set of $k+1$ nodal functional N_ℓ defined as follows: for any $v \in C(I_n)$

$$N_0(v) = v(x_{n-1}), \quad N_k(v) = v(x_n),$$

$$N_m(v) = \frac{1}{h_n^m}\int_{x_{n-1}}^{x_n} (x - x_{n-1})^{m-1} v(x)\, dx, \quad m = 1, \ldots, k-1.$$

A local interpolation $\mathcal{I} : H^1(I_n) \to P_k(I_n)$ is now defined by

$$N_m(\mathcal{I}v - v) = 0, \quad m = 0, 1, \ldots, k. \tag{5.1}$$

A continuous global interpolation can be constructed from the local interpolation operator $\mathcal{I}$.

Since $\mathcal{I}v|_{I_n}$ is continuous on I_n and is in the $H^1(I_n)$ space, we denote $\mathcal{I}v|_{\partial I_n}$ by $\mathcal{I}v|_{I_n}$ for simplicity. Form this fact we observe that for any $v \in H^1(I_n)$ we have

$$d_w^m(\mathcal{I}v) = (\mathcal{I}v)'. \tag{5.2}$$

Lemma 5.1 *[30][37] Let the exact solution $u = u_R + u_L$ of the problem (1.1) can be decomposed into a regular and layer component, respectively. If $\mathcal{I}u_R$ and $\mathcal{I}u_L$ are the interpolations u_R and u_L on a layer adapted uniform Shishkin mesh, respectively. Then, we have $\mathcal{I}u = \mathcal{I}u_R + \mathcal{I}u_L$ and the following interpolation estimates*

$$\|u - \mathcal{I}u\|_{L^\infty(\Omega_1)} \le CN^{-(k+1)}, \tag{5.3}$$

$$\|u - \mathcal{I}u\|_{L^\infty(\Omega_2)} \le C(N^{-1}\ln N)^{k+1}, \tag{5.4}$$

$$\|(u_R - \mathcal{I}u_R)^{(l)}\|_{L^2(\Omega)} \le CN^{l-(k+1)},\ l = 0, \ldots, k, \tag{5.5}$$

$$\|u_L - \mathcal{I}u_L\|_{L^2(\Omega_2)} \le C\varepsilon^{1/2}(N^{-1}\ln N)^{k+1}, \tag{5.6}$$

$$N^{-1}\|(\mathcal{I}u_L)'\|_{L^2(\Omega_1)} + \|\mathcal{I}u_L\|_{L^2(\Omega_1)} \le C(\varepsilon^{1/2} + N^{-1/2})N^{-(k+1)}, \tag{5.7}$$

$$\|u_L\|_{L^\infty(\Omega_1)} + \varepsilon^{-1/2}\|u_L\|_{L^2(\Omega_1)} \le CN^{-(k+1)}, \tag{5.8}$$

$$\|u_L'\|_{L^2(\Omega_1)} \le C\varepsilon^{-1/2}N^{-(k+1)}. \tag{5.9}$$

If $u \in H^{k+1}(\Omega)$ we also have

$$\|(u_L - \mathcal{I}u_L)^{(l)}\|_{L^2(\Omega_1)} \quad \le C\varepsilon^{1/2-l}N^{-(k+1)}, \tag{5.10}$$

$$\|(u_L - \mathcal{I}u_L)^{(l)}\|_{L^2(\Omega_2)} \quad \le C\varepsilon^{1/2-l}(N^{-1}\ln N)^{k+1-l} \tag{5.11}$$

when $l = 1, 2$.

In order to perform the error analysis, the following error equations will be needed.

Lemma 5.2 *Let u be the solution of the problem (1.1). Then for any $v_N \in V_N$, we have*

$$-\varepsilon\big(u'', v_N\big) = \varepsilon\big(d_w^m(\mathcal{I}u), d_w^m v_N\big) - T_1(u, v_N), \tag{5.12}$$

$$\big(\gamma u, v_N\big) = \big(\gamma\mathcal{I}u, v_N\big) - T_2(u, v_N), \tag{5.13}$$

where

$$T_1(u, v) = \varepsilon\big\langle\{(u - \mathcal{I}u)'\}, [v_N]\big\rangle, \tag{5.14}$$

$$T_2(u, v) = \big(\gamma(\mathcal{I}u - u), v_N\big). \tag{5.15}$$

Proof For any $v_N \in V_N$, we know from the commutativity of the interpolation operator (5.2) that $d_w^m(\mathcal{I}u) = (\mathcal{I}u)'$. Then we have

$$\big(d_w^m(\mathcal{I}u), d_w^m v_N\big)_{I_n} = \big((\mathcal{I}u)', d_w^m v_N\big)_{I_n}, \quad \forall I_n \in \mathcal{T}_N. \tag{5.16}$$

By using the definition of the weak derivative (3.7) and integration by parts, one can show that

$$\begin{aligned}\big(d_w^m v_N, (\mathcal{I}u)'\big)_{I_n} &= -\big(v_N, (\mathcal{I}u)''\big)_{I_n} + \big\langle(\mathcal{I}u)', \{v_N\}\mathbf{n}\big\rangle_{\partial I_n} \\ &= \big(v_N', (\mathcal{I}u)'\big)_{I_n} - \big\langle(\mathcal{I}u)', (v_N - \{v_N\})\mathbf{n}\big\rangle_{\partial I_n}.\end{aligned} \tag{5.17}$$

From the property of the interpolation (5.1), we have

$$\big(v_N', (\mathcal{I}u)'\big)_{I_n} = \big(v_N', u'\big)_{I_n}, \quad \forall v_N \in V_N. \tag{5.18}$$

We infer from Equations (5.16),(5.17) and (5.18) that

$$\big(d_w^m(\mathcal{I}u), d_w^m v_N\big)_{I_n} = \big(v_N', u'\big)_{I_n} - \big\langle(\mathcal{I}u)', (v_N - \{v_N\})\mathbf{n}\big\rangle_{\partial I_n}. \tag{5.19}$$

Summing up Equation (5.19) over all interval $I_n \in \mathcal{T}_h$, we find

$$\big(d_w^m(\mathcal{I}u), d_w^m v_N\big) = \big(v_N', u'\big) - \big\langle(\mathcal{I}u)', (v_N - \{v_N\})\mathbf{n}\big\rangle. \tag{5.20}$$

Using integration by parts, we have

$$-\big(u'', v_N\big)_{I_n} = \big(u', v_N'\big)_{I_n} - \big\langle u', v_N\mathbf{n}\big\rangle_{\partial I_n}.$$

Summing up the above equation over all interval $I_n \in \mathcal{T}_h$, we get

$$\big(u', v_N'\big) = -\big(u'', v_N\big) + \big\langle u', (v_N - \{v_N\})\mathbf{n}\big\rangle, \tag{5.21}$$

where we used the fact that $\big\langle u', \{v_N\}\mathbf{n}\big\rangle = 0$. Finally, combining the equation (5.21) and (5.20) and making use of the identity (3.6) yield the desired result (5.12).

Finally, Equation (5.13) is obvious. Thus, we complete the proof. □

We proceed with establishing an error equation related to modified weak convection derivative.

Lemma 5.3 *Let u solve the problem (1.1). For any $v_N \in V_N$, we have the following*

$$\big(\beta u', v_N\big) = \big(d_w^{\beta,m}(\mathcal{I}u), v_N\big) - T_3(u, v_N), \tag{5.22}$$

where

$$T_3(u, v) = \big(u - \mathcal{I}u, (\beta v_N)'\big). \tag{5.23}$$

Proof From the definition of the modified weak convection derivative (3.8) we have

$$\big(d_w^{\beta,m}(\mathcal{I}u), v_N\big) = -\big(\mathcal{I}u, (\beta v_n)'\big) + \big\langle\mathcal{I}u, \beta v_N\mathbf{n}\big\rangle. \tag{5.24}$$

On the other hand, by using integration by parts one can show

$$\big(\beta u', v_N\big) = -\big(u, (\beta v_N)'\big) + \big\langle u, \beta v_N \mathbf{n}\big\rangle. \tag{5.25}$$

Note that $\mathcal{I}u = u$ on ∂I_n for each $n = 1, 2, \ldots, N$, thus we have the desired result by combining the equation (5.24) with the equation (5.25). □

We split the error $u - u_N$ into the interpolation error $\theta := u - \mathcal{I}u$ and the discretization error $\rho := \mathcal{I}u - u_N$ so that $u - u_N = \theta + \rho$. To establish the error bound for the error $u - u_N$, we obtain the interpolation and discretization errors separately, as the triangle inequality implies the result

$$|||u - u_n|||_\varepsilon \le |||\theta|||_\varepsilon + |||\rho|||_\varepsilon.$$

Lemma 5.4 *Let u and u_N be the exact solution and the numerical approximation of the problem (1.1) and (3.12), respectively. Then we have the following error equation for the discretization error ρ*

$$a(\rho, v_N) = T(u, v_N), \quad \forall v_N \in V_N, \tag{5.26}$$

where $T(u, v) = \sum_{j=1}^{3} T_j(u, v)$ and $T_j(u, v), j = 1, 2, 3$ are defined by (5.14), (5.15) and (5.23), respectively.

Proof Multiplying the equation (1.1) by $v_N \in V_N$, we obtain

$$-\varepsilon\big(u'', v_N\big) + \big(\beta u', v_N\big) + \big(\gamma u, v_N\big) = \big(g, v_N\big). \tag{5.27}$$

We infer from Equations (5.12) and (5.22) that Equation (5.27) becomes

$$a_d(\mathcal{I}u, v_N) + a_c(\mathcal{I}u, v_N) = \big(g, v_N\big) + T(u, v_N).$$

The continuity of $\mathcal{I}u$ implies that $S_d(\mathcal{I}u, v_N) = S_c(\mathcal{I}u, v_N) = 0$. Therefore, we have

$$a(\mathcal{I}u, v_N) = \big(g, v_N\big) + T(u, v_N). \tag{5.28}$$

Finally we obtain the desired result by subtracting Equation (3.12) from Equation (5.28). □

Lemma 5.5 *The average of the derivative of interpolation error $\{\theta'\}$ satisfies the following bounds*

$$\sum_{n=1}^{N/2} \|\{\theta'\}\|^2_{\partial I_n} \le C\varepsilon^{-2} N^{-(2k+1)},$$

$$\sum_{n=N/2+1}^{N} \|\{\theta'\}\|^2_{\partial I_n} \le C\varepsilon^{-2} (N^{-1}\ln N)^{2k-1}.$$

Proof From the definition of the average operator and the trace inequality, Lemma 4.1, we have

$$\begin{aligned}\{\theta'(x_n)\}^2 &= \frac{1}{4}\big(\theta'(x_n^+) + \theta'(x_n^-)\big)^2 \le \frac{1}{2}\big(\theta'(x_n^+)^2 + \theta'(x_n^-)^2\big)\\ &\le h_n^{-1}||\theta'||^2_{I_n} + ||\theta'||_{I_n}||\theta''||_{I_n} + h_{n+1}^{-1}||\theta'||^2_{I_{n+1}} + ||\theta'||_{I_{n+1}}||\theta''||_{I_{n+1}}.\end{aligned} \tag{5.29}$$

We now find the bounds for the terms $||\theta'||_{I_n}$ and $||\theta''||_{I_n}$. The interpolation errors $(u_R - \mathcal{I}u_R)'$ and $(u_R - \mathcal{I}u_R)''$ of the regular part of the solution can be bounded using the estimate (5.5) as

$$||(u_R - \mathcal{I}u_R)'||_{I_n} \leq CN^{-k},$$
$$||(u_R - \mathcal{I}u_R)''||_{I_n} \leq CN^{-k+1}.$$

We also deduce from the estimates (5.10) and (5.11) that

$$||(u_L - \mathcal{I}u_L)'||_{I_n} \leq C\varepsilon^{\frac{-1}{2}}N^{-k-1}, \qquad I_n \subset \Omega_1,$$
$$||(u_L - \mathcal{I}u_L)''||_{I_n} \leq C\varepsilon^{\frac{-3}{2}}N^{-k-1}, \qquad I_n \subset \Omega_1,$$
$$||(u_L - \mathcal{I}u_L)'||_{I_n} \leq C\varepsilon^{\frac{-1}{2}}(N^{-1}\ln N)^k, \quad I_n \subset \Omega_2,$$
$$||(u_L - \mathcal{I}u_L)''||_{I_n} \leq C\varepsilon^{\frac{-3}{2}}(N^{-1}\ln N)^{k-1}, I_n \subset \Omega_2.$$

Combining the above error estimates and using the triangle inequality, we arrive at

$$\begin{aligned} \|\theta'\|_{I_n} &\leq C\varepsilon^{-1/2}N^{-k}(\varepsilon^{1/2} + N^{-1}), \quad I_n \subset \Omega_1 \\ \|\theta'\|_{I_n} &\leq C\varepsilon^{-1/2}(N^{-1}\ln N)^k, \qquad I_n \subset \Omega_2. \end{aligned} \tag{5.30}$$

and

$$\begin{aligned} \|\theta''\|_{I_n} &\leq C\varepsilon^{-3/2}N^{-k+1}(\varepsilon^{3/2} + N^{-2}), \quad I_n \subset \Omega_1, \\ \|\theta''\|_{I_n} &\leq C\varepsilon^{-3/2}(N^{-1}\ln N)^{k-1}, \qquad I_n \subset \Omega_2. \end{aligned} \tag{5.31}$$

Plugging the above estimates (5.30) and (5.31) into the inequality (5.29) and summing on Ω_1 and Ω_2 respectively conclude the desired estimate. Thus the proof of Lemma 5.5 is completed. □

Theorem 5.6 *Let u be the solution of the problem (1.1) and $\mathcal{I}u$ be the interpolation defined by (5.1) of the solution u, then we have the following interpolation error estimate*

$$|||\theta|||_\varepsilon \leq C(N^{-1}\ln N)^k.$$

Proof Since u and $\mathcal{I}u$ are continuous in Ω, we have $[\theta(x_n)] = \theta(x_n) - \{\theta(x_n)\} = 0$ for $n = 1, 2, \ldots, N$. Thus,

$$|||\theta|||_\varepsilon^2 = \varepsilon\sum_{n=1}^N \|\theta'\|_{I_n}^2 + \sum_{n=1}^N \|\theta\|_{I_n}^2.$$

The interpolation estimates (5.3) and (5.4) imply that

$$\begin{aligned} \sum_{n=1}^N \|\theta\|_{I_n}^2 &\leq (1-\tau_\varepsilon)\|\theta\|_{L^\infty(\Omega_1)}^2 + \tau_\varepsilon\|\theta\|_{L^\infty(\Omega_2)}^2 \\ &\leq CN^{-(k+1)} + C\ln N(N^{-1}\ln N)^{2(k+1)} \\ &\leq C\big(N^{-(k+1)} + N^{-2}\ln^3 N(N^{-1}\ln N)^{2k}\big) \\ &\leq C(N^{-1}\ln N)^{2k}, \end{aligned} \tag{5.32}$$

where we used the fact $N^{-2}\ln^3 N < 1$.

From the estimate (5.30), one can show that

$$\begin{aligned}\varepsilon\sum_{n=1}^{N}\|\theta'\|_{I_n}^2 &\leq \varepsilon\sum_{n=1}^{N}\|\theta'\|_{I_n}^2+\varepsilon\sum_{n=N}^{N}\|\theta'\|_{I_n}^2\\ &\leq C\varepsilon\varepsilon^{-1}N^{-2k}(\varepsilon+N^{-2})+C\varepsilon\varepsilon^{-1}(N^{-1}\ln N)^{2k}\\ &\leq C(N^{-2k-1}+(N^{-1}\ln N)^{2k})\\ &\leq C(N^{-1}\ln N)^{2k}.\end{aligned}$$

Therefore, we get

$$|||\theta|||_\varepsilon \leq C(N^{-1}\ln N)^k,$$

which is the desired result. Hence, the proof is completed. □

Next, we derive the error estimate for the discretization error $\rho = \mathcal{I}u - u_N$ for the MWG-FEM (3.12) in the $|||\cdot|||_\varepsilon$.

Theorem 5.7 *Let u be the solution of the problem (1.1) and $u_N \in V_N$ be the MWG-FEM approximation of (3.12) on the layer adapted piecewise uniform Shishkin mesh. Then, there is a positive constant C independent of ε, N and h_n such that*

$$|||\rho|||_\varepsilon \leq C(N^{-1}\ln N)^k. \tag{5.33}$$

Proof From the coercivity of the bilinear form (4.16), we have

$$C|||\rho|||_\varepsilon^2 \leq a(\rho,\rho). \tag{5.34}$$

Taking $v_N = \rho$ in (5.26), we get

$$a(\rho,\rho) = T(u,\rho). \tag{5.35}$$

It remains to estimate the term $T(u,\rho)$. We begin with the first term $T_1(u,\rho)$. Using the Cauchy–Schwarz inequality and Lemma 5.5, we have

$$\begin{aligned}T_1(u,\rho) &= \sum_{n=1}^{N}\big\langle\varepsilon\{\theta'\},[\rho]\big\rangle_{\partial I_n} \leq \sum_{n=1}^{N}\varepsilon\|\{\theta'\}\|_{\partial I_n}\|[\rho]\|_{\partial I_n}\\ &\leq \Big(\sum_{n=1}^{N}\frac{\varepsilon^2}{\sigma_n}\|\{\theta'\}\|_{\partial I_n}^2\Big)^{1/2}\Big(\sum_{n=1}^{N}\sigma_n\|[\rho]\|_{\partial I_n}^2\Big)^{1/2}\\ &\leq C(N^{-1}\ln N)^k|||\rho|||_\varepsilon,\end{aligned} \tag{5.36}$$

where σ_n is defined as

$$\sigma_n = \begin{cases}1 & \text{for } n = 1\ldots,N/2\\ N(\ln N)^{-1} & \text{for } n = N/2+1,\ldots,N.\end{cases} \tag{5.37}$$

Next, we estimate the terms $T_2(u,\rho)$ and $T_3(u,\rho)$ as follows. We infer from (5.15) and (5.23)

$$T_2(u,\rho) + T_3(u,\rho) = \big(\theta, (\beta' - \gamma)\rho\big) + \big(\theta, \beta\rho'\big) := Z_1(\theta,\rho) + Z_2(\theta,\rho).$$

For $Z_1(\theta,\rho)$, we infer from Theorem 5.6 that

$$|Z_1(\theta,\rho)| \le C\|\theta\|\|\rho\| \le C(N^{-1}\ln N)^k |||\rho|||_\varepsilon. \tag{5.38}$$

We estimate $Z_2(\theta,\rho)$ by making use of the Cauchy–Schwartz inequality and the estimate (5.32)

$$\begin{aligned} |Z_2(\theta,\rho)| &\le C\big(\|\theta\|_{L^2(\Omega)}\|\rho'\|_{L^2(\Omega)}\big) \\ &\le C(N^{-1}\ln N)^k |||\rho|||_\varepsilon. \end{aligned} \tag{5.39}$$

Combining the inequalities above (5.36), (5.38) and (5.39), we obtain

$$|T(u,\rho)| \le C(N^{-1}\ln N)^k |||\rho|||_{\varepsilon,N}. \tag{5.40}$$

Plugging (5.40) into (5.35), we get the desired result (5.33). □

Remark 5.8 *We see that from Lemma 5.5 and the estimate (5.36), the penalization parameter σ_n is the key ingredient in the uniform convergence. In [38], uniform convergent nonsymmetric interior penalty Galerkin (NIPG) methods have been presented with the penalty parameter chosen as $\sigma_n = N$ in Ω_2 and $\sigma = 1$ in Ω_1 for the problem (1.1). In the weak or modified weak Galerkin methods [15, 24, 34], this penalty parameter is chosen as $\sigma_n = \varepsilon h_n^{-1}$ for the elliptic and singularly perturbed convection-dominated problems, however, the uniform convergence results can not be achieved for this choice.*

The main result of this section is given in the following theorem.

Theorem 5.9 *Let u be the solution of the problem (1.1) and u_N be the MWG-FEM approximation of (3.12) on the layer adapted piecewise uniform Shishkin mesh. Then, we have*

$$|||u - u_N|||_\varepsilon \le C(N^{-1}\ln N)^k.$$

Proof By triangle inequality we know that

$$|||u - u_N|||_\varepsilon \le |||\theta|||_\varepsilon + |||\rho|||_\varepsilon.$$

Then the result follows from Theorem 5.6 and Theorem 5.7. This completes the proof. □

6. Numerical experiment

In this section, we give various numerical examples to verify numerically the theoretical convergence results obtained in this paper.

Example 6.1 *Consider the following singularly perturbed convection-diffusion-reaction problem with homogeneous Dirichlet boundary condition on $\Omega = [0,1]$:*

$$\begin{aligned} -\varepsilon u''(x) + u'(x) + u(x) &= g(x), \quad x \in (0,1), \\ u(0) = u(1) &= 0. \end{aligned} \tag{6.1}$$

The function g is given so that the true solution is

$$u(x) = \sin(x)(1 - \exp(\frac{-(1-x)}{\varepsilon})).$$

The solution u has a boundary layer near $x = 1$ of the width $\mathcal{O}(\varepsilon|\ln\varepsilon|)$. We use the piecewise uniform Shishkin mesh with N number of interval where $N = 2^l, l = 3,4,5,6,7,8,9$. We choose the transition point $1-\tau_\varepsilon$ where $\tau_\varepsilon = \varepsilon(k+1)\ln N$. Then we divide uniformly each interval $(0, 1-\tau_\varepsilon)$ and $(1-\tau_\varepsilon, 1)$ into $N/2$ elements (intervals). We display the numerical results with linear element functions ($k = 1$), quadratic element functions ($k = 2$) and cubic element functions ($k = 3$) in energy-like norm defined in (4.5) in Table 1 for different values of the parameter ε, respectively. The logarithmic order of convergence (LOC) is calculated by the formula $p = \dfrac{\ln(E(N/2)/E(N))}{\ln(2\ln(N/2)/\ln(N))}$ and the order of convergence (OC) is computed by $r = \dfrac{\ln(E(N/2)/E(N))}{\ln(2)}$ where $E(N) = u - u_N$ is the computed error. In Tables 2 and 3, we also provide history of convergence of the MWG-FEM with linear element functions ($k = 1$), quadratic element functions ($k = 2$) and cubic element functions ($k = 3$) in the discrete L^2- norm defined by

$$\|u - u_N\|_{L^2(\mathcal{T}_N)} := \left\{\sum_{n=1}^{N} \|u - u_N\|^2_{L^2(I_n)}\right\}^{1/2},$$

and in the discrete $L^\infty-$ norm defined by

$$\|u - u_N\|_{L^\infty(\mathcal{T}_N)} := \max_{0\le n\le N} |u(x_n) - u_N(x_n)|$$

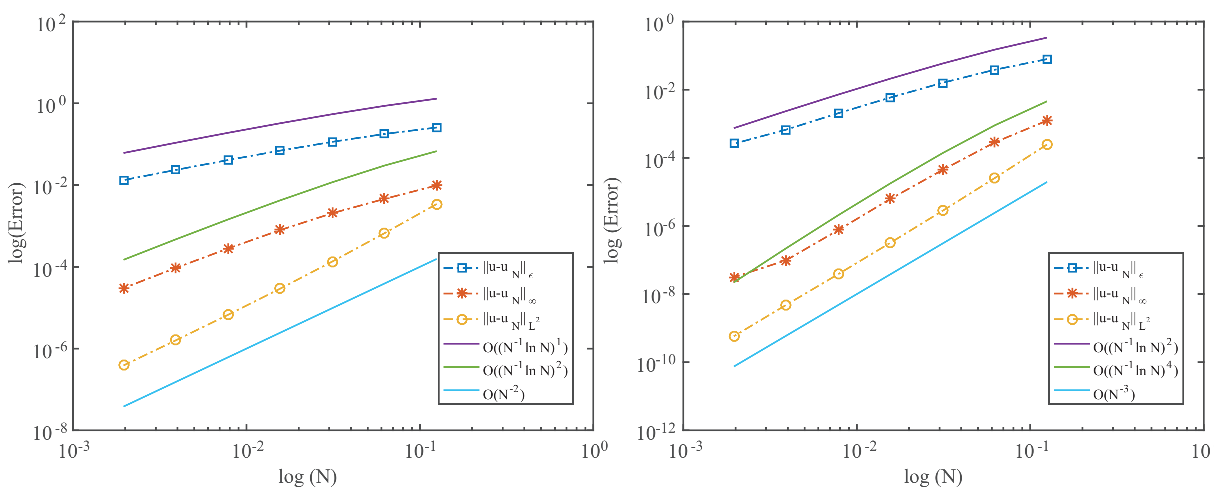
for different values of the parameter ε, respectively. Similar to other upwind scheme, we remark that the MWG-FEM converges poorly for relatively small diffusion parameter (e.g., $\varepsilon = 10^{-3}$) and it has the order of convergence $\mathcal{O}(N^{-(k+1/2)})$ in this case, however it converges very well in the strongly convection-dominated cases and has the the order of convergence $\mathcal{O}(N^{-(k+1)})$ in the discrete L^2- norm. We see that the numerical results for $|||u - u_N|||_\varepsilon$ are in excellent agreement with the theoretical fact of Theorem 5.9. From Table 3, we observe that the MWG-FEM has the superconvergence rate of $\mathcal{O}(N^{-2k}\ln^{2k} N)$ in the discrete $L^\infty-$ norm.

Plotted in Figures 1a and 1b are the errors in the norms $|||u - u_N|||_\varepsilon$, $||u - u_N||$ and $||u - u_N||_\infty$ for Example 6.1 with $\varepsilon = 1.0E{-}10$ on log-log scale using linear and quadratic element functions. It is observed that the order of convergence in the $|||u-u_N|||_\varepsilon$-norm is $\mathcal{O}((N^{-1}\ln N)^k)$ verifying the theoretical results developed in Theorem 5.9. Figure 1 suggests that the proposed MWG-FEM has the order of convergence $\mathcal{O}(N^{-(k+1)})$ in the discrete L^2- norm and the superconvergence rate of $\mathcal{O}(N^{-2k}\ln^{2k} N)$ in the discrete $L^\infty-$ norm.

As stated in [18], the standard FEM on the uniform Shishkin mesh has uniform error estimates with high accuracy. However, these methods lack stability. The discrete systems generated are difficult to solve using an iterative method. They require efficient iterative solver for the discrete systems. Another issue is that the standard FEM lacks local conservation. Due to these drawbacks of FEM, some stabilized finite element methods have been developed for convection-dominated problems. An example of stabilized FEM methods is the discontinuous Galerkin method. The discontinuous Galerkin methods are locally conservative while the DG flux is not continuous [2]. Another drawback of DG methods is choosing a good penalty parameter which

Table 1. The numerical errors in the $|||\cdot|||_\varepsilon$ norm and their orders of convergence for Example 6.1.

N	$\vert\vert\vert u-u_N\vert\vert\vert_\varepsilon$	LOC	$\vert\vert\vert u-u_N\vert\vert\vert_\varepsilon$	LOC	$\vert\vert\vert u-u_N\vert\vert\vert_\varepsilon$	LOC
$k=1$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	2.5634E-01	-	2.5610E-01	-	2.5610E-01	-
16	1.7826E-01	0.8958	1.7810E-01	0.8959	1.7810E-01	0.8959
32	1.1442E-01	0.9433	1.1431E-01	0.9434	1.1431E-01	0.9434
64	6.9638E-02	0.9720	6.9657E-02	0.9721	6.9571E-02	0.9721
128	4.0914E-02	0.9866	4.0875E-02	0.9867	4.0875E-02	0.9866
256	2.3463E-02	0.9936	2.3441E-02	0.9936	2.3441E-02	0.9936
512	1.3222E-02	0.9968	1.3209E-02	0.9969	1.3209E-02	0.9968
$k=2$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	7.9019E-02	-	7.8904E-02	-	7.8904E-02	-
16	3.8608E-02	1.7664	3.8540E-02	1.7665	3.8549E-02	1.7665
32	1.5952E-02	1.8804	1.5927E-02	1.8805	1.5927E-02	1.8805
64	5.9078E-03	1.9446	5.8984E-03	1.9446	5.8983E-03	1.9446
128	2.0372E-03	1.9752	2.0340E-03	1.9752	2.0340E-03	1.9752
256	6.6937E-04	1.9888	6.6830E-04	1.9888	6.6833E-04	1.9888
512	2.1242E-04	1.9948	2.1208E-04	1.9948	2.1266E-04	1.9901
$k=3$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	2.4397E-02	-	2.4349E-02	-	2.4349E-02	-
16	8.4573E-03	2.6128	8.4396E-03	2.6131	8.4396E-03	2.6131
32	2.2653E-03	2.8027	2.2604E-03	2.8029	2.2604E-03	2.8028
64	5.1208E-04	2.9109	5.1095E-04	2.9110	5.1094E-04	2.9110
128	1.0376E-04	2.9618	1.0354E-04	2.9616	1.0374E-04	2.9685
256	1.9538E-05	2.9836	1.9494E-05	2.9835	1.9494E-05	2.9834
512	3.4919E-06	2.9927	3.4846E-06	2.9924	3.4846E-06	2.9924



(a) Linear element (b) Quadratic element.

Figure 1. Convergence rates of three norms using Linear and Quadratic elements for Example 6.1 with $\varepsilon = 10^{-10}$.

Table 2. The numerical errors in the $||\cdot||_{L^2}$ norm and their orders of convergence for Example 6.1.

N	$\|\|u-u_N\|\|$	OC	$\|\|u-u_N\|\|$	OC	$\|\|u-u_N\|\|$	OC
$k=1$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	3.4865E-03	-	3.4458E-03	-	3.4458E-03	-
16	7.5237E-04	2.2122	6.6559E-04	2.3721	6.6559E-04	2.3721
32	2.0253E-04	1.8932	1.3490E-04	2.3027	1.3489E-04	2.3027
64	6.4006E-05	1.6619	2.9184E-05	2.2086	2.9183E-05	2.2086
128	2.0976E-05	1.6094	6.6773E-06	2.1278	6.6770E-06	2.1278
256	6.7687E-06	1.6318	1.5878E-06	2.0722	1.5870E-06	2.0722
512	2.1322E-06	1.6664	3.8645E-07	2.0386	3.8640E-07	2.0387
$k=2$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	3.3302E-04	-	2.4599E-04	-	2.4599E-04	-
16	8.2905E-05	2.0061	2.5566E-05	3.2663	2.5564E-05	3.2663
32	2.1210E-05	1.9666	2.7926E-06	3.1945	2.7916E-06	3.1945
64	4.7720E-06	2.1521	3.2018E-07	3.1246	3.2004E-07	3.1246
128	9.6960E-07	2.2991	3.8122E-08	3.0702	3.7969E-08	3.0702
256	1.8322E-07	2.4038	4.6567E-09	3.0332	4.6404E-09	3.0332
512	3.2833E-08	2.4803	5.7918E-10	3.0072	5.7554E-10	3.0072
$k=3$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	6.8044E-05	-	1.9436E-05	-	1.9436E-05	-
16	1.5612E-05	2.1237	8.6550E-07	4.4890	8.6447E-07	4.4907
32	2.6645E-06	2.5507	3.9305E-08	4.4607	3.9305E-08	4.4819
64	3.6607E-07	2.8637	2.0564E-09	4.2565	2.0543E-09	4.2564
128	4.3612E-08	3.0692	1.1347E-10	4.1797	1.1314E-10	4.1796
256	4.7155E-09	3.2092	6.7147E-12	4.0788	6.7102E-12	4.0784
512	4.7542E-10	3.3101	4.1147E-13	4.0284	4.1103E-13	4.0279

depend on the underlying problems. These disadvantages make our method more efficient over DG methods. For more details on comparison of the WG-FEM over FEM and DG methods, we refer the reader to [16]. In Table 4, we report the convergence results of the nonsymmetric ınterior penalty Galerkin method (NIPG) [38] in the L^2-norm. From Table 4, we see that NIPG has optimal order order convergence when k is odd however it has only suboptimal order of convergence in L^2-norm when k is even. Hence, our method is more robust.

In [26], some classes of $S-type$ meshes have been introduced. Getting from the coarse mesh $[0,1-\tau_\varepsilon]$ to the fine mesh $[1-\tau_\varepsilon,1]$, a mesh-generating function has been used. Assume that $\varphi:[1/2,1]\to[\ln N,0]$ is strictly decreasing mesh-generating function. Let

$$x_n = 1-\frac{(k+1)\varepsilon}{\alpha}\varphi(n/N), \quad n=N/2,\dots,N.$$

Such meshes are called $S-type$ meshes. A mesh characterizing function ψ which is very closely related to φ is defined by

$$\psi := \exp(-\varphi):[1/2,1]\to[1/N,1].$$

In the present paper we have used the piecewise uniform Shishkin mesh and we have $\max|\psi'(x)| = \mathcal{O}(\ln N)$. A

Table 3. The numerical errors in the $||\cdot||_\infty$ norm and their orders of convergence for Example 6.1.

N	$\|\|u-u_N\|\|_\infty$	LOC	$\|\|u-u_N\|\|_\infty$	LOC	$\|\|u-u_N\|\|_\infty$	LOC
$k=1$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	9.8031E-03	-	9.7987E-03	-	9.7987E-03	-
16	4.5910E-03	1.8708	4.5910E-03	1.8698	4.5910E-03	1.8698
32	2.0510E-03	1.7153	2.0490E-03	1.7164	2.0490E-03	1.7164
64	7.9190E-04	1.8620	7.9157E-04	1.8618	7.9157E-04	1.8618
128	2.7995E-04	1.9292	2.7983E-04	1.9291	2.7983E-04	1.9291
256	9.3782E-05	1.9554	9.3681E-05	1.9554	9.3681E-05	1.9554
512	3.0089E-05	1.9746	3.0076E-05	1.9746	3.0077E-05	1.9746
$k=2$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	1.2360E-03	-	1.2375E-03	-	1.2375E-03	-
16	2.8414E-04	3.6259	2.8425E-04	3.6279	2.8425E-04	3.6279
32	4.4170E-05	3.9604	4.4273E-05	3.9562	4.4273E-05	3.9562
64	6.1150E-06	3.8707	6.1404E-06	3.8672	6.1404E-06	3.8672
128	7.3376E-07	3.9338	7.3838E-07	3.9298	7.3838E-07	3.9298
256	7.9744E-08	3.9658	8.0538E-08	3.9298	8.0538E-08	3.9593
512	8.0363E-09	3.9885	8.2002E-09	3.9706	8.2002E-09	3.9706
$k=3$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	1.0987E-04	-	1.0951E-04	-	1.0951E-04	-
16	1.3301E-05	5.2075	1.3259E-05	5.2072	1.3259E-05	5.2072
32	8.7071E-07	5.8006	8.6903E-07	5.7979	8.6903E-07	5.7979
64	4.2563E-08	5.9087	4.2408E-08	5.9410	4.2408E-08	5.9410
128	1.7169E-09	5.9563	1.7105E-09	5.9569	1.7105E-09	5.9569
256	6.1328E-11	5.9836	6.1328E-11	5.9472	6.1328E-11	5.9472
512	1.9413E-12	6.0012	1.9428E-12	6.0011	1.9428E-12	6.0011

popular and frequently used optimal mesh is the Bakhvalov–Shishkin (B-S) mesh where the mesh-characterizing function

$$\Psi(x) = 1 - 2(1-t)(1-1/N), \quad \max|\psi'(x)| \le 2.$$

The mesh points are defined by

$$x_n = \begin{cases} nH, & for \quad n = 0, 1, \ldots, N/2 - 1 \\ 1 + \dfrac{(k+1)\varepsilon}{\alpha} \ln\left(1 - 2(1-1/N)(1-\frac{n}{N})\right), & for \quad n = N/2, \ldots, N. \end{cases} \tag{6.2}$$

Some examples of $S - type$ meshes can be found in details in [26].

In Table 5, we report the errors in the $|||u-u_N|||_\varepsilon$- norm for the MWG-FEM for the different values of the parameter $\varepsilon \in \{10^{-3}, 10^{-4}, \ldots, 10^{-8}\}$ with linear element functions ($k=1$), quadratic element functions ($k=2$) and cubic element functions ($k=3$) on the piecewise uniform Shishkin mesh defined by (2.3) and B-S mesh defined by (6.2) using $N=256$ elements. We see that the MWG-FEM is stable with higher order elements with respect to the parameter $\varepsilon \to 0$.

Table 4. The numerical errors in the $||\cdot||_{L^2}$-norm and the order of convergence using NIPG method and the present method for Example 6.1.

		NIPG			MWG-FEM	
N/$k=1$	$\varepsilon=10^{-4}$	$\varepsilon=10^{-5}$	$\varepsilon=10^{-6}$	$\varepsilon=10^{-4}$	$\varepsilon=10^{-5}$	$\varepsilon=10^{-6}$
16	8.0600E-04	7.8748E-04	7.8560E-04	6.7473E-04	6.6651E-04	6.6568E-04
32	1.6319E-04	1.4647E-04	1.4469E-04	1.4316E-04	1.3575E-04	1.3498E-04
64	4.0246E-05	3.0188E-05	2.8992E-05	3.4316E-05	2.9737E-05	2.9239E-05
128	1.1484E-05	6.9316E-06	6.2982E-06	9.1703E-06	6.9665E-06	6.7066E-06
256	3.4699E-06	1.7417E-06	1.4606E-06	2.6139E-06	1.7182E-06	1.6012E-06
512	1.0574E-06	3.8645E-07	3.5435E-07	7.6545E-07	4.3931E-07	3.920E-07
OC		1.720		2.0302		
N/$k=2$	$\varepsilon=10^{-3}$	$\varepsilon=10^{-4}$	$\varepsilon=10^{-5}$	$\varepsilon=10^{-6}$	$\varepsilon=10^{-7}$	$\varepsilon=10^{-8}$
16	6.6285E-05	5.0053E-05	4.8131E-05	3.5671E-05	2.6747E-05	2.5685E-05
32	1.5020E-05	7.4116E-06	6.1538E-06	7.1991E-06	3.4922E-06	2.8696E-06
64	4.0530E-06	1.4632E-06	8.4396E-07	1.5363E-06	5.7268E-07	3.5335E-07
128	1.1565E-06	3.7610E-07	1.4775E-07	3.0810E-07	1.0386E-07	4.8772E-08
256	3.3290E-07	1.0580E-07	3.5208E-08	5.7984E-08	1.8847E-08	7.3991E-09
512	9.4948E-08	3.0057E-08	9.6022E-09	1.0375E-08	3.3237E-09	1.1818E-09
OC		1.8744		2.4825		
$k=3$	$\varepsilon=10^{-3}$	$\varepsilon=10^{-4}$	$\varepsilon=10^{-5}$	$\varepsilon=10^{-6}$	$\varepsilon=10^{-7}$	$\varepsilon=10^{-8}$
16	7.1524E-06	2.5053E-06	1.3381E-06	4.9954E-06	1.7794E-06	9.9430E-07
32	1.1443E-06	3.6457E-07	8.4175E-07	5.9594E-08	2.6861E-07	9.2416E-08
64	1.5136E-07	4.7891E-08	1.1554E-07	5.2038E-09	3.6566E-08	1.1675E-08
128	1.7592E-08	5.5623E-09	1.3764E-08	5.6329E-10	4.3523E-09	1.3782E-09
256	1.8701E-09	5.9129E-10	1.4882E-09	5.9260E-11	4.7053E-10	1.4883E-10
512	1.8632E-10	5.8906E-11	1.5004E-10	5.8964E-12	4.7508E-11	1.7094E-11
OC		3.3268		3.3521		

Table 5. Errors in $|||u-u_N|||_{\varepsilon}-$ norms on S- mesh and B-S mesh.

ε	$k=1$		$k=2$		$k=3$	
	S-mesh	B-S mesh	S-mesh	B-S mesh	S-mesh	B-S mesh
10^{-3}	2.3463E-02	9.6537E-03	6.6937E-04	8.2544E-05	1.9538E-05	5.9106E-07
10^{-4}	2.3443E-02	9.6464E-03	6.6655E-04	8.2544E-05	1.9499E-05	5.9110E-07
10^{-5}	2.3441E-02	9.6456E-03	6.6655E-04	8.2533E-05	1.9499E-05	5.9108E-07
10^{-6}	2.3441E-02	9.6456E-03	6.6655E-04	8.2532E-05	1.9499E-05	5.9109E-07
10^{-7}	2.3441E-02	9.6456E-03	6.6655E-04	8.2532E-05	1.9499E-05	5.9109E-07
10^{-8}	2.3441E-02	9.6456E-03	6.6655E-04	8.2551E-05	1.9499E-05	5.9109E-07

Example 6.2 *Consider the following variable convection coefficient convection-diffusion-reaction equation*

$$\begin{aligned} -\varepsilon u''(x)+(3-x)u'(x)+u(x)&=g(x), \quad x\in(0,1),\\ u(0)=u(1)&=0, \end{aligned} \tag{6.3}$$

where the force function g is chosen such that the exact solution is

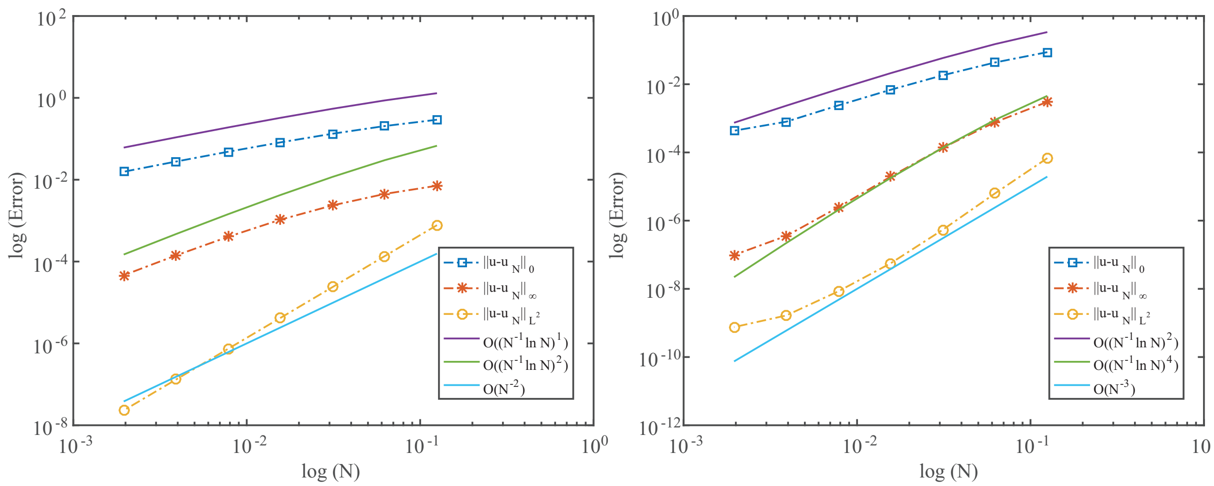
$$u(x) = x - \frac{e^{-(1-x)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}.$$

The history of convergence of MWG-FEM in the $|||u-u_N|||_\varepsilon$-norm for different values of the perturbation parameter is presented in Table 6 for Example 6.2. Plotted in Figures 2a and 2b are the errors in the norms $|||u-u_N|||_\varepsilon$, $||u-u_N||$ and $||u-u_N||_\infty$ for Example 6.2 with $\varepsilon = 1.0E-10$ on log-log scale using linear and quadratic element functions. Again, these results match the theory we have developed in Theorem 5.9.

Table 6. Errors in $|||u-u_N|||_\varepsilon-$ norms and their convergence rate for Example 6.2.

N	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-8}$	
$k=1$	$\|\|\|u-u_N\|\|\|_\varepsilon$	LOC	$\|\|\|u-u_N\|\|\|_\varepsilon$	LOC	$\|\|\|u-u_N\|\|\|_\varepsilon$	LOC
8	2.9230E-01	-	2.9230E-01	-	2.9230E-01	-
16	2.0494E-01	0.8756	2.0496E-01	0.8754	2.0496E-01	0.8754
32	1.3303E-01	0.9193	1.3307E-01	0.9190	1.3307E-01	0.9190
64	8.1686E-02	0.9548	8.1697E-02	0.9550	8.1697E-02	0.9550
128	4.8256E-02	0.9765	4.8257E-02	0.9767	4.8257E-02	0.9767
256	2.7757E-02	0.9881	2.7758E-02	0.9882	2.7758E-02	0.9882
512	1.5667E-02	0.9939	1.5667E-02	0.9940	1.5667E-02	0.9940
$k=2$	$\|\|\|u-u_N\|\|\|_\varepsilon$	OC	$\|\|\|u-u_N\|\|\|_\varepsilon$	OC	$\|\|\|u-u_N\|\|\|_\varepsilon$	OC
8	8.7170E-02	-	8.7173E-02	-	8.7173E-02	-
16	4.3465E-02	1.7162	4.3465E-02	1.7163	4.3465E-02	1.7163
32	1.8324E-02	1.8377	1.8322E-02	1.8378	1.8322E-02	1.8378
64	6.8831E-03	1.9168	6.8826E-03	1.9167	6.8826E-03	1.9167
128	2.3935E-03	1.9597	2.3934E-03	1.9596	2.3934E-03	1.9597
256	7.9003E-04	1.9807	7.9002E-04	1.9807	7.9003E-04	1.9807
512	2.5132E-04	1.9906	2.5132E-04	1.9905	2.5132E-04	1.9905
$k=3$	$\|\|\|u-u_N\|\|\|_\varepsilon$	OC	$\|\|\|u-u_N\|\|\|_\varepsilon$	OC	$\|\|\|u-u_N\|\|\|_\varepsilon$	OC
8	2.6503E-02	-	2.6504E-02	-	2.6504E-02	-
16	9.4167E-03	2.5521	9.4169E-03	2.5521	9.4169E-03	2.5522
32	2.5824E-03	2.7526	2.5824E-03	2.7526	2.5824E-03	2.7527
64	5.9377E-04	2.8777	5.9377E-04	2.8777	5.9378E-04	2.8777
128	1.2154E-04	2.9429	1.2154E-04	2.9429	1.2156E-04	2.9426
256	2.3016E-05	2.9735	2.3016E-05	2.9735	2.3016E-05	2.9730
512	4.1260E-06	2.9874	4.1260E-06	2.9872	4.1260E-06	2.9874

The MWG-FEM can be applied to nonlinear SPPs of reaction-diffusion type as well. The existence of a unique solution of the nonlinear problem follows from monotonicity methods presented in [3] or [20]. Further details on analysis of FEM for the nonlinear SPPs of reaction-diffusion type can be found in [28]. Different from the preceding analysis and examples, SPPs of reaction-diffusion type have in general two boundary layers at the end points of the domain. Furthermore, we extend our analysis to nonlinear problem. We apply a Newton iteration to solve the nonlinear problem numerically. As an example, we will consider the following nonlinear SPPs.



(a) Linear element (b) Quadratic element.

Figure 2. Convergence rates of there norms of the Linear and Quadratic elements for Example 6.2 with $\varepsilon = 10^{-10}$.

Example 6.3 *[31] Consider*

$$\begin{aligned} -\varepsilon u''(x) + u(x)(1+u(x)) &= \exp(-2x/\sqrt{\varepsilon}), \quad x \in (0,1), \\ u(0) = 1, \quad u(1) &= \exp(-1/\sqrt{\varepsilon}), \end{aligned} \tag{6.4}$$

which has the solution $u(x) = \exp(-x/\sqrt{\varepsilon})$.

The solution u has an exponential boundary layer at $x = 0$ since the reduced problem when $\varepsilon \to 0$ satisfies the boundary condition at $x = 1$. We use the piecewise uniform Shishkin mesh with N number of interval where $N = 2^l, l = 3,4,5,6,7,8,9$. We choose the transition point τ_ε where $\tau_\varepsilon = \sqrt{\varepsilon}(k+1)\ln N$. Then we divide uniformly each interval $(0, \tau_\varepsilon)$ and $(1-\tau_\varepsilon, 1)$ into $N/4$ elements (intervals) while $(\tau_\varepsilon, 1-\tau_\varepsilon)$ into $N/2$ elements . In Table 7, we display the numerical results with linear element functions ($k = 1$), quadratic element functions ($k = 2$) and cubic element functions ($k = 3$) in energy-like norm defined by

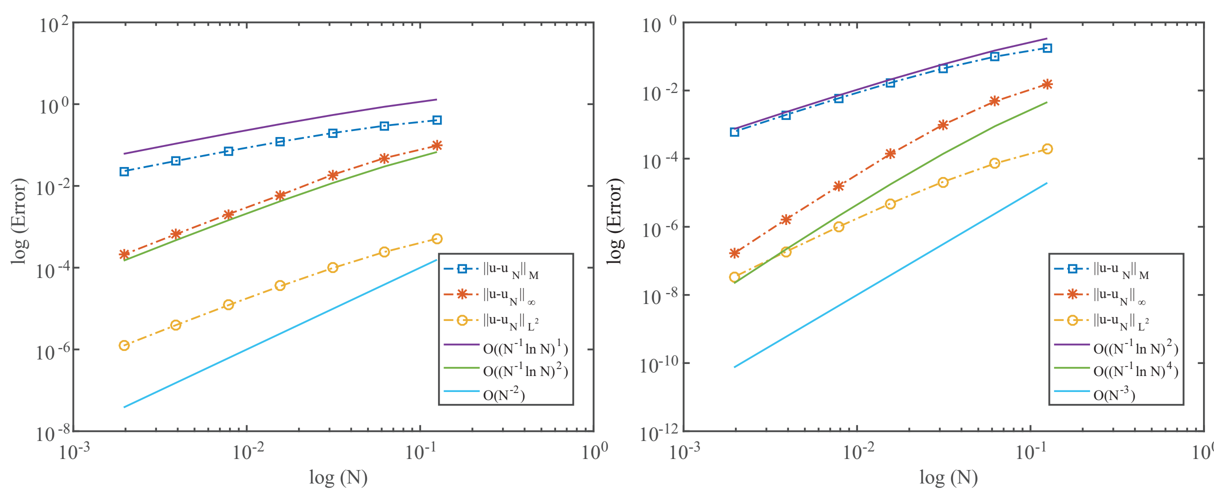
$$\|u - u_N\|_M := \Big(\|u-u_N\|_w^2 + \sum_{n=1}^{N} \|u-u_N\|_{I_n}^2 \Big)^{1/2},$$

where $\|\cdot\|_w$ is defined by (4.3). We require 5 Newton iterations in the following implementation. In Table 8, we also provide history of convergence of the MWG-FEM with linear element functions ($k = 1$), quadratic element functions ($k = 2$) and cubic element functions ($k = 3$) in the discrete L^∞-norm.

Plotted in Figures 3a and 3b are the errors in the norms $||u-u_N||_M$, $||u-u_N||_{L^2}$ and $||u-u_N||_\infty$ for Example 6.3 with $\varepsilon = 1.0E-10$ on log-log scale using linear and quadratic element functions. We again theoretically observe that the order of convergence in the $||u-u_N||_M$-norm is $\mathcal{O}((N^{-1}\ln N)^k)$ for nonlinear SPPs as in the case of the linear problems. Once again, Figure 3 shows that the proposed MWG-FEM has the order of convergence $\mathcal{O}(N^{-(k+1)})$ in the discrete L^2-norm and the super-convergence rate of $\mathcal{O}(N^{-2k}\ln^{2k} N)$ in the discrete L^∞-norm for the problem (6.4) too. The numerical experiments show that the MWG-FEM is quite efficient and stable for nonlinear SPPs as well.

Table 7. The numerical errors in the $||\cdot||_M$ norm and their orders of convergence for Example 6.3.

N	$\|u-u_N\|_M$	LOC	$\|u-u_N\|_M$	LOC	$\|u-u_N\|_M$	LOC
$k=1$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	3.9067E-01	-	4.0440E-01	-	4.0443E-01	-
16	2.8897E-01	0.74	2.9743E-01	0.76	2.9745E-01	0.76
32	1.9143E-01	0.88	1.9646E-01	0.88	1.9648E-01	0.88
64	1.0921E-01	1.10	1.2079E-01	0.95	1.2080E-01	0.95
128	5.0853E-02	1.42	7.1124E-02	0.98	7.1127E-02	0.98
256	2.3744E-02	1.36	4.0776E-02	0.99	4.0778E-02	0.99
512	1.1152E-02	1.31	2.2961E-02	1.00	2.2963E-02	1.00
$k=2$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	1.7356E-01	-	1.7904E-01	-	1.7905E-01	-
16	8.6354E-02	1.72	9.8961E-02	1.46	9.8967E-02	1.46
32	2.2417E-02	2.87	4.4148E-02	1.72	4.4150E-02	1.72
64	5.3153E-03	2.82	1.6910E-02	1.88	1.6911E-02	1.88
128	1.2366E-03	2.71	5.8961E-03	1.95	5.8964E-03	1.95
256	2.8812E-04	2.60	1.9420E-03	1.98	1.9421E-03	1.98
512	6.7508E-05	2.52	6.1624E-04	1.99	6.1627E-04	1.99
$k=3$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	6.8659E-02	-	8.0127E-02	-	8.0132E-02	-
16	1.1897E-02	4.32	3.3658E-02	2.14	3.3659E-02	2.14
32	1.5628E-03	4.32	1.0317E-02	2.52	1.0318E-02	2.52
64	1.8562E-04	4.17	2.4940E-03	2.78	2.4941E-03	2.78
128	2.1574E-05	3.99	5.1824E-04	2.92	5.1827E-04	2.92
256	2.5093E-06	3.84	9.8306E-05	2.97	9.8311E-05	2.97
512	2.9344E-07	3.73	1.7594E-05	2.99	1.7594E-05	2.99



(a) Linear element (b) Quadratic element.

Figure 3. Convergence rates of three norms using Linear and Quadratic elements for Example 6.3 with $\varepsilon=10^{-10}$.

Table 8. The numerical errors in the $||\cdot||_\infty$ norm and their orders of convergence for Example 6.3.

N	$\|\|u-u_N\|\|_\infty$	LOC	$\|\|u-u_N\|\|_\infty$	LOC	$\|\|u-u_N\|\|_\infty$	LOC
$k=1$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	9.3261E-02	-	9.6528E-02	-	9.6536E-02	-
16	4.6754E-02	1.70	4.8101E-02	1.72	4.8104E-02	1.72
32	1.8271E-02	2.00	1.8745E-02	2.01	1.8746E-02	2.01
64	5.2889E-03	2.43	5.9507E-03	2.25	5.9510E-03	2.25
128	1.3112E-03	2.59	2.0255E-03	2.00	2.0256E-03	2.00
256	3.2384E-04	2.50	6.5853E-04	2.01	6.5856E-04	2.01
512	8.0534E-05	2.42	2.0863E-04	2.00	2.0864E-04	2.00
$k=2$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	1.5164E-02	-	1.5483E-02	-	1.5483E-02	-
16	3.9880E-03	3.29	4.8323E-03	2.87	4.8326E-03	2.87
32	3.4084E-04	5.23	9.8501E-04	3.38	9.8507E-04	3.38
64	2.0500E-05	5.50	1.3843E-04	3.84	1.3843E-04	3.84
128	1.2958E-06	5.12	1.5583E-05	4.05	1.5584E-05	4.05
256	8.0291E-08	4.97	1.6713E-06	3.99	1.6713E-06	3.99
512	4.9838E-09	4.83	1.6633E-07	4.01	1.6622E-07	4.01
$k=3$	$\varepsilon=10^{-3}$		$\varepsilon=10^{-8}$		$\varepsilon=10^{-9}$	
8	2.5147E-03	-	3.1939E-03	-	3.1941E-03	-
16	9.3320E-05	8.12	5.4141E-04	4.38	5.4144E-04	4.38
32	1.7061E-06	8.51	4.7543E-05	5.18	4.7546E-05	5.18
64	2.2091E-08	8.51	2.4394E-06	5.81	2.4395E-06	5.81
128	3.2253E-10	7.84	8.6319E-08	6.20	8.6267E-08	6.20
256	4.8953E-12	7.48	2.7200E-09	6.18	2.7177E-09	6.19
512	4.1711E-13	4.28	5.9821E-11	6.63	5.9845E-11	6.63

As mentioned in the introduction, the standard numerical schemes in general do not give an acceptable accuracy unless some layer-adapted meshes are used in the discretization while numerical method using trigonometric quintic B-splines [14] and an implicit-explicit local differential method [31] have been proposed for solving SPPs. For comparison purpose, the problem (6.4) is also solved using the existing numerical approaches developed in [31] and [14]. As seen from Table 9, our method has uniform error estimates while the results in [31] and [14] suffer from the deterioration of numerical accuracy due to boundary layers when the perturbation parameter gets smaller $\varepsilon \ll 1$. In applications, the perturbation parameter ε should be chosen a sufficiently small number to reflect the nature of the singularly perturbed problem.

Table 9. Comparison of the numerical errors in the $||\cdot||_\infty$ norm for Example 6.3.

ε	[31]($\theta=0, k=6$)	[31]($\theta=0.5, k=6$)	[14]	The present method ($k=4$)
10^{-2}	2.08E-09	2.16E-12	2.32E-09	1.02E-13
10^{-3}	7.67E-07	2.15E-09	7.89E-08	5.46E-13
10^{-4}	4.12E-04	2.10E-08	8.12E-06	2.15E-09
10^{-5}	4.09E-01	0.90E-02	1.23E-02	2.16E-09

7. Conclusion

In this paper, the MWG-FEM is proposed and analyzed for solving one dimensional singularly perturbed convection-diffusion-reaction problems. A uniform error estimate is obtained by using a special type of interpolation operator and a special stabilization parameter in the proposed method. We theoretically confirm that the present method on the piecewise uniform Shishkin mesh has optimal and parameter-free convergent error estimate for higher order element in the energy norm. The numerical examples verify that the MWG-FEM solves efficiently and successfully linear SPPs as well as nonlinear SPPs. Similar error analysis can be carried out in two dimensional singularly perturbed convection-diffusion-reaction problems. An interpolation operator which is uniformly convergent on the tensor product of one dimensional uniform Shishkin mesh can be constructed for establishing the optimal and parameter-free convergent error estimates. This will be explored in the future work.

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