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## On Lyapunov-type inequalities for $(n + 1)$ st order nonlinear differential equations with the antiperiodic boundary conditions

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**Abstract:** In this paper, we establish new Lyapunov-type inequalities for  $(n + 1)$ st order nonlinear differential equation including  $p$ -relativistic operator and  $q$ -prescribed curvature operator under the antiperiodic boundary conditions.

**Key words:** Lyapunov-type inequality,  $p$ -relativistic operator,  $q$ -prescribed curvature operator, the antiperiodic boundary condition

### 1. Introduction

In [12], Lyapunov obtained the following remarkable result: If  $q \in C([0, \infty), \mathbb{R})$  and  $x(t)$  is a solution of second order linear problem

$$x'' + q(t)x = 0 \quad (1.1)$$

with Dirichlet boundary condition

$$x(a) = x(b) = 0, \quad (1.2)$$

where  $a, b \in \mathbb{R}$  with  $b > a$  and  $x(t) \neq 0$  for  $t \in (a, b)$ , then the Lyapunov inequality

$$\int_a^b |q(s)| ds \geq \frac{4}{b-a} \quad (1.3)$$

holds. Thus, the Lyapunov inequality (1.3) provides a lower bound for the distance between two consecutive zeros of  $x$ . The inequality (1.3) is best possible in the sense that the constant 4 in the left hand side of (1.3) cannot be replaced by a larger number ([11, p. 267]). This result has found many applications in areas like eigenvalue problems, stability, oscillation theory, disconjugacy, etc. Since then, there have been several results to improve and generalize the linear equation (1.1) in many directions [1–4, 13–16].

More recently, by replacing  $x''$  and a relativistic acceleration  $\left(\frac{x'}{\sqrt{1-x'^2}}\right)'$  in Eq. (1.1), Yang et al. [16] have proved that if we assume that

$$q \in \mathcal{A} = \left\{ q \in L_{loc}^1((a, b), [0, \infty)) : \infty > \int_a^b (s-a)(b-s)q(s) ds \right\} \quad (1.4)$$

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and second order nonlinear problem

$$\left(\frac{x'}{\sqrt{1-x'^2}}\right)' + q(t)x = 0, \quad 1 > |x'|, \tag{1.5}$$

with Dirichlet boundary condition (1.2) has a positive solution, then the Lyapunov-type inequality

$$\int_a^b (s-a)(b-s)q(s)ds > b-a \tag{1.6}$$

holds.

In 2012, Wang [14] considered the following  $(n + 1)$ st order nonlinear differential equation

$$\left(|x^{(n)}|^{p-2}x^{(n)}\right)' + q(t)|x|^{p-2}x = 0, \tag{1.7}$$

where  $p > 1$ ,  $q(t) \in C([0, \infty), \mathbb{R})$ , and  $x(t)$  is a solution of (1.7) satisfying the antiperiodic boundary conditions

$$x^{(k)}(a) + x^{(k)}(b) = 0, \quad k = 0, 1, \dots, n, \tag{1.8}$$

where  $a, b \in \mathbb{R}$  with  $b > a$  and  $x(t) \not\equiv 0$  for  $t \in (a, b)$  and so it is obtained the following inequality

$$\int_a^b |q(s)|ds > 2\left(\frac{2}{b-a}\right)^{n(p-1)}. \tag{1.9}$$

In 2015, Wang et al. [13] considered the following  $(n + 1)$ st order nonlinear differential equation

$$\left(|x^{(n)}|^{p-2}x^{(n)}\right)' + \sum_{i=0}^n q_i(t)|x^{(i)}|^{p-2}x^{(i)} = 0, \tag{1.10}$$

where  $p > 1$ ,  $q_k(t) \in C([0, \infty), \mathbb{R})$ ,  $k = 0, 1, \dots, n$ ,  $n \in \mathbb{N}$ , and  $x(t)$  is a solution of (1.10) satisfying the antiperiodic boundary conditions (1.8) and so they obtained the following inequality

$$\int_a^b |q_n(s)|ds + \sum_{i=0}^{n-1} \left((b-a)S_{n-i}\right)^{(p-1)/2} \int_a^b |q_i(s)|ds > 2, \tag{1.11}$$

where

$$S_n = \frac{(2^{2n} - 1)(b-a)^{2n-1}}{2^{2n-1}\pi^{2n}} \zeta(2n), \quad n = 1, 2, \dots, \tag{1.12}$$

and  $\zeta(z) = \sum_{i=1}^{+\infty} \frac{1}{i^z}$ ,  $Re(z) > 1$  is the Riemann zeta function, and the constants  $\{S_n\}$  are sharp.

**Remark 1.1** If we take  $q_0(t) = q(t)$  and  $q_k(t) = 0$ ,  $k = 1, 2, \dots, n$  in (1.10), then, from

$$1 > \frac{2^2(2^{2n} - 1)}{\pi^{2n}} > \frac{2(2^{2n} - 1)\zeta(2n)}{\pi^{2n}}, \quad n > 1, \tag{1.13}$$

the inequality (1.11) is sharper than (1.9) when  $n > 1$  [13].

We now consider the  $(n + 1)$ st order nonlinear differential equation of the form

$$\left( \frac{|x^{(n)}|^{p-2} x^{(n)}}{(1 - |x^{(n)}|^p)^{(p-1)/p}} \right)' + \sum_{i=0}^{n-1} q_i(t) \frac{|x^{(i)}|^{p-2} x^{(i)}}{(1 + |x^{(i)}|^p)^{(p-1)/p}} = 0, \tag{1.14}$$

where  $p > 1$ ,  $q_k(t) \in C([0, \infty), \mathbb{R})$ ,  $k = 0, 1, \dots, n - 1$ ,  $n \in \mathbb{N}$ , and  $x(t)$  is a solution of (1.14) satisfying the antiperiodic boundary conditions (1.8).

In this paper, we prove new Lyapunov-type inequalities for the problem (1.14) with (1.8) including  $p$ -relativistic operator

$$\varphi_p(u) = \frac{|u|^{p-2} u}{(1 - |u|^p)^{(p-1)/p}}, \quad 1 > |u|, \tag{1.15}$$

and  $q$ -prescribed curvature operator

$$\varphi_q(u) = \frac{|u|^{q-2} u}{(1 + |u|^q)^{(q-1)/q}}, \quad u \in \mathbb{R}, \tag{1.16}$$

[5, 8–10]. Note that if we take  $p = q = 2$ , then they turn out to be the relativistic and the prescribed curvature operator  $\varphi_2$ , which have been attracting much attention in differential geometry and partial differential equation. It has many important applications in physics, biology and other inter disciplines [6, 7]. To the best of our knowledge, this is the first paper using the  $p$ -relativistic operator  $\varphi_p$  and  $q$ -prescribed curvature operator  $\varphi_q$  to study the above problem.

## 2. Main results

Firstly, we state some lemmas which we will use in the proof of our main results.

**Lemma 2.1** [13, Lemma 3.2] *If  $x(t)$  is a nontrivial solution of the problem (1.14) with (1.8), then we have*

$$\sqrt{(b - a) S_{n-k}} M_n \geq M_k, \quad k = 0, 1, \dots, n - 1, \tag{2.1}$$

where  $S_n$  is given in (1.12) and

$$M_n = \sup_{a \leq t \leq b} |x^{(n)}(t)|. \tag{2.2}$$

**Lemma 2.2** [14, Lemma 3.2] *If  $x(t)$  is a nontrivial solution of the problem (1.14) with (1.8), then we have*

$$\frac{1}{2} \left( \frac{b - a}{2} \right)^{(n-k)(p-1)} \int_a^b |x^{(n)}(s)|^p ds \geq |x^{(n-1)}(t)| |x^{(k)}(t)|^{p-1} \tag{2.3}$$

for  $k = 0, 1, \dots, n - 1$ .

The proof of Lemma 2.2 is similar to that of Lemma 3.2 given by Wang [14], and hence is omitted.

Now, we give main results obtained which are inspired by the results obtained by Wang et al. [13] and Wang [14].

**Theorem 2.3** *If  $x(t)$  is a nontrivial solution of the problem (1.14) with (1.8), then the following inequality*

$$\sum_{i=0}^{n-1} \left( (b-a) S_{n-i} \right)^{(p-1)/2} \int_a^b |q_i(s)| ds > 2 \tag{2.4}$$

holds, where  $S_n$  is given in (1.12).

**Proof** Assume that  $x(t)$  is a solution of Eq. (1.14) satisfying the antiperiodic boundary conditions  $x^{(k)}(a) + x^{(k)}(b) = 0$ ,  $k = 0, 1, \dots, n$ , where  $a, b \in \mathbb{R}$  with  $b > a$  and  $x(t) \neq 0$  for  $t \in (a, b)$ . Now, we define

$$H(t, s) = \begin{cases} 1/2 & , a \leq s \leq t, \\ -1/2 & , t \leq s \leq b. \end{cases} \tag{2.5}$$

Multiplying both sides of (1.14) by the function  $H$ , integrating from  $a$  to  $b$ , and using the antiperiodic boundary conditions (1.8) with  $i = n$ , we have

$$\begin{aligned} \frac{|x^{(n)}(t)|^{p-2} x^{(n)}(t)}{\left(1 - |x^{(n)}(t)|^p\right)^{(p-1)/p}} &= \int_a^b H(t, s) \left( \frac{|x^{(n)}(s)|^{p-2} x^{(n)}(s)}{\left(1 - |x^{(n)}(s)|^p\right)^{(p-1)/p}} \right)' ds \\ &= - \sum_{i=0}^{n-1} \int_a^b H(t, s) q_i(s) \frac{|x^{(i)}(s)|^{p-2} x^{(i)}(s)}{\left(1 + |x^{(i)}(s)|^p\right)^{(p-1)/p}} ds. \end{aligned} \tag{2.6}$$

Taking absolute value of (2.6) and using Lemma 2.1, we get

$$\begin{aligned} |x^{(n)}(t)|^{p-1} &\leq \frac{|x^{(n)}(t)|^{p-1}}{\left(1 - |x^{(n)}(t)|^p\right)^{(p-1)/p}} \\ &\leq \sum_{i=0}^{n-1} \int_a^b |H(t, s)| |q_i(s)| \frac{|x^{(i)}(s)|^{p-1}}{\left(1 + |x^{(i)}(s)|^p\right)^{(p-1)/p}} ds \\ &= \frac{1}{2} \sum_{i=0}^{n-1} \int_a^b |q_i(s)| \frac{|x^{(i)}(s)|^{p-1}}{\left(1 + |x^{(i)}(s)|^p\right)^{(p-1)/p}} ds \\ &\leq \frac{1}{2} \sum_{i=0}^{n-1} \int_a^b |q_i(s)| |x^{(i)}(s)|^{p-1} ds \\ &< \frac{1}{2} \sum_{i=0}^{n-1} M_i^{p-1} \int_a^b |q_i(s)| ds \\ &\leq \frac{M_n^{p-1}}{2} \sum_{i=0}^{n-1} \left( (b-a) S_{n-i} \right)^{(p-1)/2} \int_a^b |q_i(s)| ds \end{aligned} \tag{2.7}$$

and so

$$\frac{M_n^{p-1}}{2} \sum_{i=0}^{n-1} \left( (b-a) S_{n-i} \right)^{(p-1)/2} \int_a^b |q_i(s)| ds > M_n^{p-1}, \tag{2.8}$$

where  $S_n$  and  $M_n$  are given in (1.12) and (2.2), respectively. Now, we claim that  $M_n > 0$ . In fact, if it is not true, then we have  $M_n = 0$  or  $x^{(n)}(t) = 0$  for  $t \in [a, b]$ . By the antiperiodic boundary conditions (1.8), we obtain  $x(t) = 0$  for  $t \in [a, b]$ , which contradicts to the assumption that  $x(t)$  is a nontrivial solution of (1.14) with (1.8). Therefore, we obtain  $M_n > 0$ . Dividing both sides of the inequality (2.8) by  $M_n^{p-1}$ , we obtain

$$\sum_{i=0}^{n-1} \left( (b-a) S_{n-i} \right)^{(p-1)/2} \int_a^b |q_i(s)| ds > 2. \tag{2.9}$$

□

**Theorem 2.4** *If  $x(t)$  is a nontrivial solution of the problem (1.14) with (1.8), then the following inequality*

$$\sum_{i=0}^{n-1} \left( \frac{b-a}{2} \right)^{(n-i)(p-1)} \int_a^b |q_i(s)| ds > 2 \tag{2.10}$$

holds.

**Proof** Assume that  $x(t)$  is a solution of Eq. (1.14) satisfying the antiperiodic boundary conditions  $x^{(k)}(a) + x^{(k)}(b) = 0$ ,  $k = 0, 1, \dots, n$ , where  $a, b \in \mathbb{R}$  with  $b > a$  and  $x(t) \not\equiv 0$  for  $t \in (a, b)$ . Multiplying both sides of (1.14) by  $x^{(n-1)}(t)$  and integrating from  $a$  to  $b$ , we have

$$\int_a^b x^{(n-1)}(s) \left( \frac{|x^{(n)}(s)|^{p-2} x^{(n)}(s)}{(1 - |x^{(n)}(s)|^p)^{(p-1)/p}} \right)' ds + \sum_{i=0}^{n-1} \int_a^b q_i(s) x^{(n-1)}(s) \frac{|x^{(i)}(s)|^{p-2} x^{(i)}(s)}{(1 + |x^{(i)}(s)|^p)^{(p-1)/p}} ds = 0. \tag{2.11}$$

Integrating by parts the first integral of equation (2.11) and using the antiperiodic boundary conditions (1.8), we get

$$\int_a^b \frac{|x^{(n)}(s)|^p}{(1 - |x^{(n)}(s)|^p)^{(p-1)/p}} ds = \sum_{i=0}^{n-1} \int_a^b q_i(s) x^{(n-1)}(s) \frac{|x^{(i)}(s)|^{p-2} x^{(i)}(s)}{(1 + |x^{(i)}(s)|^p)^{(p-1)/p}} ds. \tag{2.12}$$

By Lemma 2.2, we have

$$\begin{aligned}
 \int_a^b |x^{(n)}(s)|^p ds &\leq \int_a^b \frac{|x^{(n)}(s)|^p}{\left(1 - |x^{(n)}(s)|^p\right)^{(p-1)/p}} ds \\
 &= \sum_{i=0}^{n-1} \int_a^b q_i(s) x^{(n-1)}(s) \frac{|x^{(i)}(s)|^{p-2} x^{(i)}(s)}{\left(1 + |x^{(i)}(s)|^p\right)^{(p-1)/p}} ds \\
 &\leq \sum_{i=0}^{n-1} \int_a^b |q_i(s)| \frac{|x^{(n-1)}(s)| |x^{(i)}(s)|^{p-1}}{\left(1 + |x^{(i)}(s)|^p\right)^{(p-1)/p}} ds \\
 &\leq \sum_{i=0}^{n-1} \int_a^b |q_i(s)| |x^{(n-1)}(s)| |x^{(i)}(s)|^{p-1} ds \\
 &< \sum_{i=0}^{n-1} \max_{a \leq t \leq b} |x^{(n-1)}(t)| |x^{(i)}(t)|^{p-1} \int_a^b |q_i(s)| ds \\
 &\leq \frac{1}{2} \int_a^b |x^{(n)}(s)|^p ds \sum_{i=0}^{n-1} \left(\frac{b-a}{2}\right)^{(n-i)(p-1)} \int_a^b |q_i(s)| ds. \tag{2.13}
 \end{aligned}$$

Now, we claim that  $\int_a^b |x^{(n)}(s)|^p ds > 0$ . In fact, if it is not true, then we have  $\int_a^b |x^{(n)}(s)|^p ds = 0$  or  $x^{(n)}(t) = 0$  for  $t \in [a, b]$ . By the antiperiodic boundary conditions (1.8), we obtain  $x(t) = 0$  for  $t \in [a, b]$ , which contradicts to the assumption that  $x(t)$  is a nontrivial solution of (1.14) with (1.8). Therefore, we have  $\int_a^b |x^{(n)}(s)|^p ds > 0$ . Dividing both sides of the inequality (2.13) by  $\int_a^b |x^{(n)}(s)|^p ds$ , we obtain

$$\sum_{i=0}^{n-1} \left(\frac{b-a}{2}\right)^{(n-i)(p-1)} \int_a^b |q_i(s)| ds > 2. \tag{2.14}$$

□

**Remark 2.5** If we take  $n = 1$  in Eq. (1.14), then the inequality (2.4) is equal to (2.10). Note that as  $m$  increases, the ratio  $R = \left(\frac{2(2^{2^m}-1)\zeta(2m)}{\pi^{2m}}\right)^{1/2}$  decreases rapidly (especially when  $m \geq 2$ ,  $R$  is smaller than 1) [15]. Therefore, the inequality (2.4) is sharper than (2.10) when  $n > 1$ .

As immediate consequences of Theorems 2.3 and 2.4, the following result gives sufficient conditions for the nonexistence of nontrivial solutions of the boundary value problem (1.14) with (1.8).

**Corollary 2.6** (a) Assume

$$2 \geq \sum_{i=0}^{n-1} \left((b-a) S_{n-i}\right)^{(p-1)/2} \int_a^b |q_i(s)| ds, \tag{2.15}$$

where  $S_n$  is given in (1.12). The problem (1.14) with (1.8) has no nontrivial solution.

(b) Assume

$$2 \geq \sum_{i=0}^{n-1} \left(\frac{b-a}{2}\right)^{(n-i)(p-1)} \int_a^b |q_i(s)| ds. \tag{2.16}$$

The problem (1.14) with (1.8) has no nontrivial solution.

Now we give a sufficient condition for the uniqueness of the solution of the following nonhomogenous boundary value problem

$$\begin{cases} \left( \frac{|x^{(n)}|^{p-2} x^{(n)}}{(1-|x^{(n)}|^p)^{(p-1)/p}} \right)' + \sum_{i=0}^{n-1} q_i(t) \frac{|x^{(i)}|^{p-2} x^{(i)}}{(1+|x^{(i)}|^p)^{(p-1)/p}} = f(t), & t \in (a, b), \\ x^{(k)}(a) + x^{(k)}(b) = 0, & k = 0, 1, \dots, n, \end{cases} \tag{2.17}$$

where  $p > 1$  and  $f, q_k \in C([a, b], \mathbb{R})$ ,  $k = 0, 1, \dots, n-1$ ,  $n \in \mathbb{N}$ , as an application of Lyapunov-type inequalities (2.4) and (2.10).

**Theorem 2.7** *If (2.15) or (2.16) holds, then nonhomogenous boundary problem (2.17) has a unique solution.*

**Proof** To prove the uniqueness, it is sufficient to show that the homogeneous boundary value problem (1.14) with (1.8) has only trivial solution. Assume on the contrary that  $x(t) \not\equiv 0$  is a solution of the homogeneous boundary value problem (1.14) with (1.8). Then by using Lyapunov-type inequality (2.4) or (2.10), we have

$$\sum_{i=0}^{n-1} ((b-a) S_{n-i})^{(p-1)/2} \int_a^b |q_i(s)| ds > 2 \tag{2.18}$$

or

$$\sum_{i=0}^{n-1} \left(\frac{b-a}{2}\right)^{(n-i)(p-1)} \int_a^b |q_i(s)| ds > 2 \tag{2.19}$$

which gives contradiction to (2.15) or (2.16), respectively. Therefore the homogeneous boundary value problem (1.14) with (1.8) has only trivial solution. Because of the theory of boundary value problems, the nonhomogenous boundary problem (2.17) has a unique solution.  $\square$

As an example of Theorem 2.7, we consider the following nonhomogenous boundary value problem

$$\begin{cases} \left( \frac{x'}{\sqrt{1-|x'|^2}} \right)' + \frac{x}{4\sqrt{1+|x|^2}} = -\frac{1}{\sin^2 t} + \frac{\sin t}{4\sqrt{1+\sin^2 t}}, & t \in (0, \pi), \\ x(0) + x(\pi) = 0, \\ x'(0) + x'(\pi) = 0. \end{cases} \tag{2.20}$$

Since the condition (2.16) is satisfied, the problem (2.20) has a unique solution. One such solution of the problem (2.20) is  $x(t) = \sin t$ . Note that  $|x'|$  is bounded with 1 and the homogeneous boundary value problem corresponding to the problem (2.20) has only trivial solution.



In the following part, we apply the obtained Lyapunov-type inequalities (2.4) and (2.10) to the eigenvalue problems associated with the nonlinear boundary value problem

$$\begin{cases} \left( \frac{|x^{(n)}|^{p-2} x^{(n)}}{(1-|x^{(n)}|^p)^{(p-1)/p}} \right)' + \lambda \sum_{i=0}^{n-1} q_i(t) \frac{|x^{(i)}|^{p-2} x^{(i)}}{(1-|x^{(i)}|^p)^{(p-1)/p}} = 0, \\ x^{(k)}(a) + x^{(k)}(b) = 0, \quad k = 0, 1, \dots, n, \end{cases} \tag{2.21}$$

where  $p > 1$ ,  $q_k \in C([a, b], \mathbb{R})$ ,  $k = 0, 1, \dots, n - 1$ ,  $n \in \mathbb{N}$ , and  $\lambda \in \mathbb{R}$  is an eigenvalue parameter. As direct consequences of Theorems 2.3 and 2.4, we obtain the following result.

**Theorem 2.8** (a) Assume  $\lambda$  is an eigenvalue of the boundary value problem (2.21). Then

$$|\lambda| > \frac{2}{\sum_{i=0}^{n-1} ((b-a) S_{n-i})^{(p-1)/2} \int_a^b |q_i(s)| ds}, \tag{2.22}$$

where  $S_n$  is given in (1.12).

(b) Assume  $\lambda$  is an eigenvalue of the boundary value problem (2.21). Then

$$|\lambda| > \frac{2}{\sum_{i=0}^{n-1} \left(\frac{b-a}{2}\right)^{(n-i)(p-1)} \int_a^b |q_i(s)| ds}. \tag{2.23}$$

**Open Problem.** Note that we have only considered the case  $p = q$  in the following  $(n + 1)$ st order nonlinear differential equation

$$\left( \varphi_p \left( x^{(n)}(t) \right) \right)' + \sum_{i=0}^{n-1} q_i(t) \varphi_q \left( x^{(i)}(t) \right) = 0, \tag{2.24}$$

where  $p, q > 1$ ,  $q_k \in C([0, \infty), \mathbb{R})$ , and the operators  $\varphi_p$  and  $\varphi_q$  are given in (1.15) and (1.16), respectively. It is an open problem to obtain similar results for Eq. (2.24) without any restrictions on  $p$  and  $q$ .

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### References

- [1] Aktas MF, Cakmak D, Tiryaki A. Lyapunov-type inequality for quasilinear systems with anti-periodic boundary conditions. *Journal of Mathematical Inequalities* 2014; 8 (2): 313-320. doi:10.7153/jmi-08-22
- [2] Cakmak D. Lyapunov-type integral inequalities for certain higher order differential equations. *Applied Mathematics and Computation* 2010; 216 (2): 368-373. doi:10.1016/j.amc.2010.01.010
- [3] Cakmak D. Lyapunov-type inequalities for two classes of nonlinear systems with anti-periodic boundary conditions. *Applied Mathematics and Computation* 2013; 223: 237-242. doi:10.1016/j.amc.2013.07.099
- [4] Cakmak D. On Lyapunov-type inequality for a class of quasilinear systems. *Electronic Journal of Qualitative Theory of Differential Equations* 2014; (9): 1-10. doi:10.14232/ejqtde.2014.1.9
- [5] Carletti T, Villari G. Existence of limit cycles for some generalization of the Lienard equations: the relativistic and the prescribed curvature cases. *Electronic Journal of Qualitative Theory of Differential Equations* 2020; (2): 1-15. doi:10.14232/ejqtde.2020.1.2

- [6] Dosly O, Rehak P. Half-linear Differential Equations. North-Holland Mathematics Studies, 202. Amsterdam: Elsevier Science BV, 2005.
- [7] Gilbarg D, Trudinger NS. Elliptic Partial Differential Equations of Second Order. Berlin, New York: Springer Verlag, 1983.
- [8] Jebelean P, Mawhin J, Serban C. Multiple critical orbits to partial periodic perturbations of the  $p$ -relativistic operator. Applied Mathematics Letters 2020; (104): 106220. doi:10.1016/j.aml.2020.106220
- [9] Jebelean P, Mawhin J, Serban C. A vector  $p$ -Laplacian type approach to multiple periodic solutions for the  $p$ -relativistic operator. Communications in Contemporary Mathematics 2017; 19 (3): 1-16. doi:10.1142/S0219199716500292
- [10] Jebelean P, Serban C. Boundary value problems for discontinuous perturbations of singular  $\varphi$ -Laplacian operator. Journal of Mathematical Analysis and Applications 2015; 431 (1): 662-681. doi:10.1016/j.jmaa.2015.06.004
- [11] Kelley WG, Peterson AC. The Theory of Differential Equations. Classical and Qualitative, New York: Springer, 2010.
- [12] Liapunov AM. Probleme general de la stabilite du mouvement. Annales de la Faculte Des Sciences de Toulouse 1907; 2 (9): 203-407.
- [13] Wang Y, Cui Y, Li Y. Lyapunov-type inequalities for  $(m + 1)$ th order half-linear differential equations with anti-periodic boundary conditions. Electronic Journal of Qualitative Theory of Differential Equations 2015; (14): 1-7. doi:10.14232/ejqtde.2015.1.14
- [14] Wang Y. Lyapunov-type inequalities for certain higher order differential equations with anti-periodic boundary conditions. Applied Mathematics Letters 2012; 25 (12): 2375-2380. doi:10.1016/j.aml.2012.07.006
- [15] Watanabe K, Yamagishi H, Kametaka Y. Riemann zeta function and Lyapunov-type inequalities for certain higher order differential equations. Applied Mathematics and Computation 2011; 218 (7): 3950-3953. doi:10.1016/j.amc.2011.08.105
- [16] Yang R, Sim I, Lee YH. Lyapunov-type inequalities for one-dimensional Minkowski-curvature problems. Applied Mathematics Letters 2019; 91: 188-193. doi:10.1016/j.aml.2018.11.006