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Quasinormal modes of charged fermions in linear dilaton black hole spacetime: exact frequencies

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Abstract: We study charged massless fermionic perturbations in the background of 4-dimensional linear dilaton black holes in Einstein–Maxwell–dilaton theory with double Liouville-type potentials. We present the analytical fermionic quasinormal modes, whose Dirac equations are solved in terms of hypergeometric functions. We also discuss the stability of these black holes under the charged fermionic perturbations.

Key words: Quasinormal modes, fermionic perturbations, dilaton, Dirac equation, Newman–Penrose, Liouville-type potential

1. Introduction

At high enough energies, it is quite possible that gravity is not described by the action of Einstein's general relativity theory. Today, the literature has a compelling evidence that in string theory (ST), gravity becomes a scalar-tensor [1]. The low-energy limit of the ST corresponds to Einstein's gravitational attraction, which is nonminimally coupled to a scalar dilaton field [2]. When the Einstein–Maxwell theory is combined with a dilaton field, the resulting spacetime solutions have important consequences. There have been significant studies in the literature to find exact solutions of the Einstein–Maxwell-dilaton (EMD) theory. For example, in the absence of a dilaton potential, EMD gravity's charged dilaton black hole (BH) solutions were found by many researchers [3–11]. Essentially, the presence of the dilaton alters the arbitrary structure of spacetime and causes to curvature singularities with finite radii. The obtained dilaton BHs in general have nonasymptotically flat (NAF) structure. In recent years, even in the various extensions of general relativity, there has been an active interest in the NAF spacetimes. Among them, probably the most important ones are the asymptotic anti-de Sitter (AdS) BHs [12], which play a vital role in string and quantum gravity theories as well as for the AdS/CFT (conformal field theory) correspondence, which is sometimes called Maldacena duality or gauge/gravity duality [13].

The BH solutions of EMD theory were studied by many researchers: the uncharged EMD BH solutions were found in [14–16], while the charged EMD BH solutions were considered in [17, 18]. With the inclusion of the Liouville-type potentials, the static charged BH solutions were found by [19–21]. The generalization to dyonic (having both electric and magnetic charges) BH solutions in 4-dimensional and higher-dimensional EMD gravity with single and double Liouville-type potentials were also found in [22]. In fact, similar to the

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Higgs potential, the double Liouville-type potentials have the ability to admit local extremes and critical points. However, a single Liouville potential lacks from these features. Besides, the double Liouville-type potentials also appear when the higher-dimensional theories are compressed into a 4-dimensional spacetime. Many of the BH solutions with the double Liouville-type potentials to date can be seen from [23] and references therein. Among them Mazharimousavi et al.'s BH [24] in the EMD theory with the double Liouville-type potentials has some unique features. First of all, those BHs cover Reissner–Nordström (RN) type BHs and Bertotti–Robinson (BR) spacetimes [25] interpolated within the same metric. Remarkably, in between the two spacetimes, there exists a linear dilaton BH (LDBH) [26] solution for the specific values of the dilaton, double Liouville-type potentials, and EMD parameters. The particular motivation of this work is to compute the quasinormal modes (QNMs) [27, 28] for the charged fermionic field perturbations [29–32] in the spacetime of those LDBHs. Since the QNMs can signal information about the stability of BHs, we shall also analyze the stability of the LDBHs under the charged fermionic perturbations.

Meanwhile, we would like to remind the reader for information purposes that QNMs are nothing but the energy dissipation of a perturbed BH. Similar to an ordinary object, when a BH is perturbed it begins to ring with its natural frequencies, which are the modes of its energy dissipation. These characteristic modes are called QNMs. The amplitudes of their oscillations decay in time. The amplitude of the oscillation can be approximated by

$$\Psi \approx e^{i\omega t} \; = e^{-\omega_I t}\cos(\omega_R t), \tag{1}$$

where $\omega = \omega_R + i\omega_I$ is the frequency of the QNM [28]. Here, ω_R and ω_I are the frequencies of oscillatory and exponential modes, respectively. After LIGO's great successes about the gravitational wave measurements [33], the subject of QNMs has gained considerable importance in the direct identification of BHs. Today, there are numerous well-known studies which show that the surrounding geometry of a BH experiences QNMs under perturbations (see for example [34–41] and references therein). One of the most important features to know about QNMs is that they allow us to analyze the quantum entropy/area spectrum of the BHs. In this regard, the reader is referred to [42–48] and references therein.

The organization of the present paper is as follows: In Section 2, we review the LDBH solution, which is obtained from the 4-dimensional action of the EMD theory having the double Liouville-type potentials. In Section 3, we first derive the spin-$\frac{1}{2}$ field equations by using 4-dimensional charged massless Dirac equations within the framework of Newman–Penrose (NP) formalism. Solution procedure to the obtained Dirac equations is given in Section 4. Section 5 is devoted to the computations of the QNM frequencies of the charged fermions. We summarize our results in Section 6. (Throughout the paper, we use the geometrized units of $G = c = \hbar = 1$.)

2. LDBH spacetime in EMD theory with double Liouville-type potential

4-dimensional action of the EMD theory having the double Liouville-type potentials is given by [24]

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - \frac{1}{2}W(\phi)\left(F_{\lambda\sigma}F^{\lambda\sigma}\right)\right), \tag{2}$$

where

$$V(\phi) = V_1\mathrm{e}^{\beta_1\phi} + V_2\mathrm{e}^{\beta_2\phi}, \qquad W(\phi) = \lambda_1\mathrm{e}^{-2\gamma_1\phi} + \lambda_2\mathrm{e}^{-2\gamma_2\phi}, \tag{3}$$

where ϕ is the dilaton, $V_{1,2}$ are the double Liouville-type potentials, $\gamma_{1,2}$ denote the dilaton parameter, and $\lambda_{1,2}$, $\beta_{1,2}$ are constants. Moreover, R is the Ricci scalar and the Maxwell 2-form is given by

$$F = d\mathcal{A}. \tag{4}$$

For the choice of pure magnetic potential

$$\mathcal{A} = -Q\cos\theta d\varphi, \tag{5}$$

where Q denotes the charge, we have

$$F = Q\sin\theta d\theta \wedge d\varphi. \tag{6}$$

It is worth noting that with the current choice of $W(\phi)$, the magnetic-electric symmetry, which exists in the standard dilatonic coupling, namely $\lambda_1\,(\lambda_2) = 0$, is not valid anymore. Thus, as highlighted in [24], the charged LDBH spacetime is pure magnetic. Variations of the action (2) with respect to the gravitational field $g_{\mu\nu}$ and the dilaton ϕ yield the following EMD field equations

$$R_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + Vg_{\mu\nu} + W\left(2F_{\mu\lambda}F_\nu^\lambda - \frac{1}{2}F_{\lambda\sigma}F^{\lambda\sigma}g_{\mu\nu}\right), \tag{7}$$

$$\nabla^2\phi - V' - \frac{1}{2}W'\left(F_{\lambda\sigma}F^{\lambda\sigma}\right) = 0, \tag{8}$$

where $R_{\mu\nu}$ is the Ricci tensor and the prime symbol $(')$ denotes the derivative with respect to ϕ. Furthermore, the Maxwell equation is obtained with the variation with respect to $\mathcal{A}$ as

$$d\left(W^\star F\right) = 0 \tag{9}$$

in which the Hodge star $(^\star)$ refers to duality. After substituting the following metric (ansatz):

$$ds^2 = B(r)dt^2 - \frac{dr^2}{B(r)} - R(r)d\Omega_2^2, \tag{10}$$

into Eqs. (7) and (8) and in the sequel of compelling calculations as made in [24], the field equations result in the metric functions of the LDBH as follows:

$$B(r) = b\frac{(r-r_2)(r-r_1)}{r} \quad \text{and} \quad R(r) = A^2 r, \tag{11}$$

where

$$b = \frac{1}{A^2} - 2(V_1+V_2) > 0, \quad \text{and} \quad A^2 = 2\lambda_2 Q^2, \quad (A: \textit{Real constant}). \tag{12}$$

The inner and outer horizons of the LDBH are given by [24]

$$r_1 = \frac{1}{2b}\left(c - \sqrt{c^2-4ab}\right) \quad \text{and} \quad r_2 = \frac{1}{2b}\left(c + \sqrt{c^2-4ab}\right), \tag{13}$$

in which the physical parameters read

$$c = 4M \quad \text{and} \quad a = \frac{\lambda_1}{\lambda_2 A^2}, \tag{14}$$

where M denotes the quasilocal mass [49] and

$$\phi(r) = -\frac{1}{\sqrt{2}} \ln(r), \;\; \beta_1 = \beta_2 = \sqrt{2}, \;\; \gamma_1 = -\frac{1}{\sqrt{2}}, \;\; \gamma_2 = \frac{1}{\sqrt{2}}, \tag{15}$$

$$V = \frac{V_1 + V_2}{r}, \quad \text{and} \quad W = \frac{\lambda_1}{r} + \lambda_2 r. \tag{16}$$

The above spacetime corresponds to a phase transition geometry, which changes the structure of spacetime from RN to BR [24]. The case of $c^2 = 4ab$ gives us the extremal $(r_2 = r_1)$ LDBHs whose congenerics can be seen in [26, 50].

3. Charged Dirac equation in LDBH geometry

In the NP formalism [51], massless Dirac equations with charge coupling are given as follows [29, 52]:

$$\begin{aligned}
\left[D + iql^j\mathcal{A}_j + \varepsilon - \rho\right] F_1 + \left[\overline{\delta} + iq\overline{m}^j\mathcal{A}_j + \pi - \alpha\right] F_2 = 0,\\
\left[\delta + iqm^j\mathcal{A}_j \;\; + \beta - \tau\right] F_1 + \left[\Delta + iqn^j\mathcal{A}_j \;\; + \mu - \gamma\right] F_2 = 0,\\
\left[D + iql^j\mathcal{A}_j + \overline{\varepsilon} - \overline{\rho}\right] \widetilde{G}_2 - \left[\delta + iqm^j\mathcal{A}_j + \overline{\pi} - \overline{\alpha}\right] \widetilde{G}_1 = 0,\\
\left[\Delta + iqn^j\mathcal{A}_j \;\; + \overline{\mu} - \overline{\gamma}\right] \widetilde{G}_1 - \left[\overline{\delta} + iq\overline{m}^j\mathcal{A}_j + \overline{\beta} - \overline{\tau}\right] \widetilde{G}_2 = 0,
\end{aligned} \tag{17}$$

where q is the charge of the fermion and A_j represents the j^{th} component of the vector potential of the background electromagnetic field (5). The wave functions $F_1, F_2, \widetilde{G}_1, \widetilde{G}_2$ represent the Dirac spinors while $\alpha, \beta, \gamma, \varepsilon, \mu, \pi, \rho, \tau$ are the spin (Ricci rotation) coefficients. The directional derivatives for NP tetrads are defined as

$$D = l^j \nabla_j, \; \Delta = n^j \nabla_j, \; \delta = m^j \nabla_j, \; \bar{\delta} = \bar{m}^j \nabla_j. \tag{18}$$

In the meantime, a bar over a quantity stands for the complex conjugation. We choose a complex null tetrad $\{l, n, m, \overline{m}\}$ (the covariant one-forms) for the LDBH geometry as

$$\begin{aligned}
l_j &= \frac{1}{\sqrt{2}} \left[\sqrt{B(r)}, -\frac{1}{\sqrt{B(r)}}, 0, 0\right],\\
n_j &= \frac{1}{\sqrt{2}} \left[\sqrt{B(r)}, \frac{1}{\sqrt{B(r)}}, 0, 0\right],\\
m_j &= A\sqrt{\frac{r}{2}} \left[0, 0, 1, i\sin\theta\right],\\
\overline{m}_j &= A\sqrt{\frac{r}{2}} \left[0, 0, 1, -i\sin\theta\right],
\end{aligned} \tag{19}$$

which all together satisfy the orthogonality conditions: $l.n = -m.\overline{m} = 1$. The nonzero spin coefficients are obtained as

$$\rho = \mu = -\frac{-1}{2\sqrt{2}}\frac{\sqrt{B(r)}}{r},$$
$$\epsilon = \gamma = \frac{b}{4\sqrt{2B(r)}}\left(1 - \frac{r_2 r_1}{r^2}\right),$$
$$\alpha = -\beta = \frac{\cot\theta}{2A\sqrt{2r}}. \tag{20}$$

The form of the Dirac equations (17) suggests that the spinors can be chosen as follows:

$$F_1 = f_1(r)\, A_1(\theta)\, e^{i(kt+m\varphi)},$$
$$\widetilde{G}_1 = g_1(r)\, A_2(\theta)\, e^{i(kt+m\varphi)},$$
$$F_2 = f_2(r)\, A_3(\theta)\, e^{i(kt+m\varphi)},$$
$$\widetilde{G}_2 = g_2(r)\, A_4(\theta)\, e^{i(kt+m\varphi)}, \tag{21}$$

where k is the frequency of the wave corresponding to the Dirac particle and m is the azimuthal quantum number.

4. Solution of charged Dirac equation in LDBH geometry

After substituting the spin coefficients (20) and the spinors (21) into the Dirac equations (17), one gets

$$\frac{1}{f_2}\widetilde{Z} f_1 - \frac{(LA_3)}{A_1} = 0,$$
$$\frac{1}{f_1}\overline{\widetilde{Z}} f_2 + \frac{\left(L^{\dagger} A_1\right)}{A_3} = 0,$$
$$\frac{1}{g_1}\widetilde{Z} g_2 - \frac{\left(L^{\dagger} A_2\right)}{A_4} = 0,$$
$$\frac{1}{g_2}\overline{\widetilde{Z}} g_1 + \frac{(LA_4)}{A_2} = 0. \tag{22}$$

The radial operators appear in the above simplified Dirac equations are

$$\widetilde{Z} = A\sqrt{\Lambda}\partial_r + H + \frac{iAkr}{\sqrt{\Lambda}},$$
$$\overline{\widetilde{Z}} = A\sqrt{\Lambda}\partial_r + H - \frac{iAkr}{\sqrt{\Lambda}}, \tag{23}$$

in which

$$\Lambda = b(r - r_2)(r - r_1), \tag{24}$$

$$H = \frac{A}{2r}\left[\frac{b(r^2 - r_2 r_1)}{2\sqrt{\Lambda}} + \sqrt{\Lambda}\right]. \tag{25}$$

The angular operators are

$$L = \partial_\theta + \frac{m}{\sin\theta} + \left(\frac{1}{2} - p\right)\cot\theta,$$
$$L^\dagger = \partial_\theta - \frac{m}{\sin\theta} + \left(\frac{1}{2} + p\right)\cot\theta, \tag{26}$$

where $p = qQ$. Further, choosing

$$f_1 = g_2,\ f_2 = g_1,\ A_1 = A_2,\ A_3 = A_4, \tag{27}$$

and introducing a real eigenvalue λ which is the separation constant of the complete Dirac equations, one can separate the Dirac equations into two sets of master equations. The radial master equations read

$$\overline{\widetilde{Z}} g_1 = -\lambda g_2,$$
$$\widetilde{Z} g_2 = -\lambda g_1, \tag{28}$$

and the angular master equations become

$$L^\dagger A_2 = \lambda A_4,$$
$$L A_4 = -\lambda A_2. \tag{29}$$

4.1. Angular equation

The laddering operators L and $L^\dagger$ [53] govern the spin-weighted spheroidal harmonics as

$$\left(\partial_\theta - \frac{m}{\sin\theta} - s\cot\theta\right)\left({}_sY_l^m(\theta)\right) = -\sqrt{(l-s)(l+s+1)}\,{}_{s+1}Y_l^m(\theta),$$
$$\left(\partial_\theta + \frac{m}{\sin\theta} + s\cot\theta\right)\left({}_sY_l^m(\theta)\right) = \sqrt{(l+s)(l-s+1)}\,{}_{s-1}Y_l^m(\theta). \tag{30}$$

Eigenfunctions ${}_sY_l^m(\theta)$, called the spin-weighted spheroidal harmonics, are complete and orthogonal, and have an explicit form [54]:

$$ {}_sY_l^m(\theta) = \sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!\,(l-m)!}{(l+s)!\,(l-s)!}}\left(\sin\frac{\theta}{2}\right)^{2l}$$
$$\times \sum_{r=-l}^{l}(-1)^{l+m-r}\binom{l-s}{r-s}\binom{l+s}{r-m}\left(\cot\frac{\theta}{2}\right)^{2r-m-s}, \tag{31}$$

where l and s are the angular quantum number and the spin-weight, respectively. They satisfy the following expressions:

$$l = |s|, |s|+1, |s|+2, ... \text{ and } -l < m < +l. \tag{32}$$

Comparison between the angular master equations (29) and (30) leads us to identify

$$A_2 = {}_{-\left(\frac{1}{2}+p\right)}Y_l^m,$$
$$A_4 = {}_{\left(\frac{1}{2}-p\right)}Y_l^m. \tag{33}$$

Since l and s both must be integers or half-integers, we impose the "Dirac quantization condition" [55]:

$$2qQ = 2p = n, \qquad n = 0, \pm 1, \pm 2...., \tag{34}$$

and obtain the eigenvalue of the spin-weighted spheroidal harmonic equation as

$$\lambda = -\sqrt{\left(l+\frac{1}{2}\right)^2 - p^2}, \quad \left(\text{Real: } \left(l+\frac{1}{2}\right) \geq p\right),$$
$$\therefore \ \lambda^2 = \left(l+\frac{1}{2}\right)^2 - p^2. \tag{35}$$

4.2. Radial equation and Zerilli potential

Letting

$$g_1 = \frac{G_1}{A(r\Lambda)^{\frac{1}{4}}},$$
$$g_2 = \frac{G_2}{A(r\Lambda)^{\frac{1}{4}}}, \tag{36}$$

and substituting them into the radial master equations (28), we obtain

$$G_1' - \frac{ik}{B} = -\frac{\lambda G_2}{A\sqrt{\Lambda}},$$
$$G_2' + \frac{ik}{B} = -\frac{\lambda G_1}{A\sqrt{\Lambda}}. \tag{37}$$

Introducing the tortoise coordinate [26]

$$dr_* = \frac{dr}{B} \quad \longrightarrow \quad r_* = \frac{1}{b(r_2 - r_1)} \ln\left[\frac{(r-r_2)^{r_2}}{(r-r_1)^{r_1}}\right], \tag{38}$$

one can rewrite Eq. (37) as

$$G_{1,r_*} - ik = -\frac{\lambda\sqrt{\Lambda}G_2}{Ar},$$
$$G_{2,r*} + ik = -\frac{\lambda\sqrt{\Lambda}G_1}{Ar}. \tag{39}$$

Setting the solutions of the above equations into the following forms

$$G_1 = P_1 + P_2,$$
$$G_2 = P_1 - P_2, \tag{40}$$

we get two decoupled radial equations, which correspond to 1-dimensional Schrödinger equation or the so-called Zerilli equation [26]:

$$P_{j,r*r*} + k^2 P_j = V_j P_j, \qquad j = 1, 2 \tag{41}$$

where the effective potentials are given by

$$V_j = \lambda^2 \frac{B}{R} + (-1)^j \frac{\lambda b}{2r^2}\sqrt{\frac{B}{R}}\left[r(r_1+r_2) - 2r_2r_1\right]. \tag{42}$$

The near horizon and asymptotic limits of the potentials are as follows:

$$\lim_{r\to r_2} V_j \equiv \lim_{r^*\to -\infty} V_j = 0,$$

$$\lim_{r\to \infty} V_j \equiv \lim_{r^*\to \infty} V_j = \frac{\lambda^2 b}{A^2}. \tag{43}$$

For the outer region of LDBH, the behaviors of potentials $V_{j=1,2}$ according to the various angular quantum numbers are depicted in Figures 1 and 2, respectively.

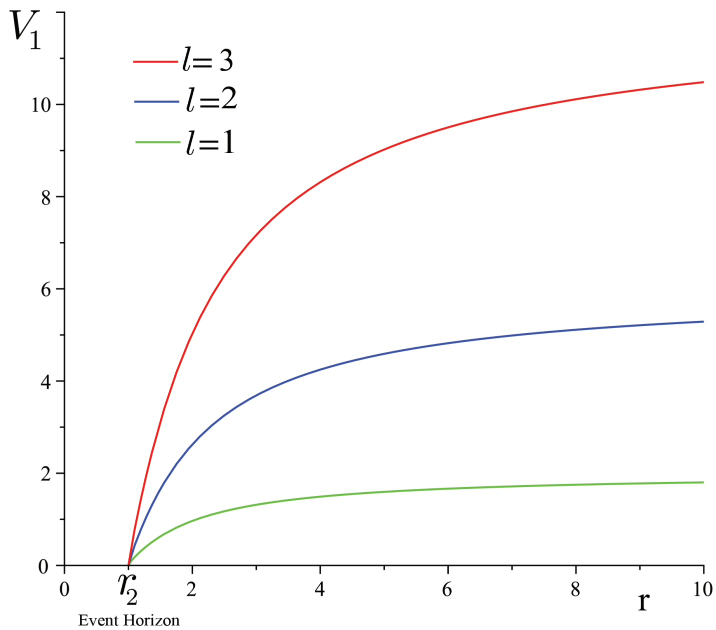

Figure 1. V_1 versus r graph. The physical parameters are chosen as $A = b = r_2 = 2r_1 = 2p = 1$. Plot is governed by Eq. (42).

Near horizon ($r \to r_2$) solutions of the radial equations (41) read

$$P_j = C_{1j}e^{ikr_*} + C_{2j}e^{-ikr_*}, \tag{44}$$

and thus we have

$$G_1 = P_1 + P_2 \quad \longrightarrow \quad G_1 = D_1e^{-ikr*} + D_3e^{ikr_*}, \tag{45}$$

$$G_2 = P_1 - P_2 \quad \longrightarrow \quad G_2 = D_2e^{-ikr*} + D_4e^{ikr_*}.$$

We impose one of the QNM conditions that the wave should be purely ingoing at the horizon by letting $D_3 = D_4 = 0$; thence

$$G_j = D_je^{-ikr*}. \tag{46}$$

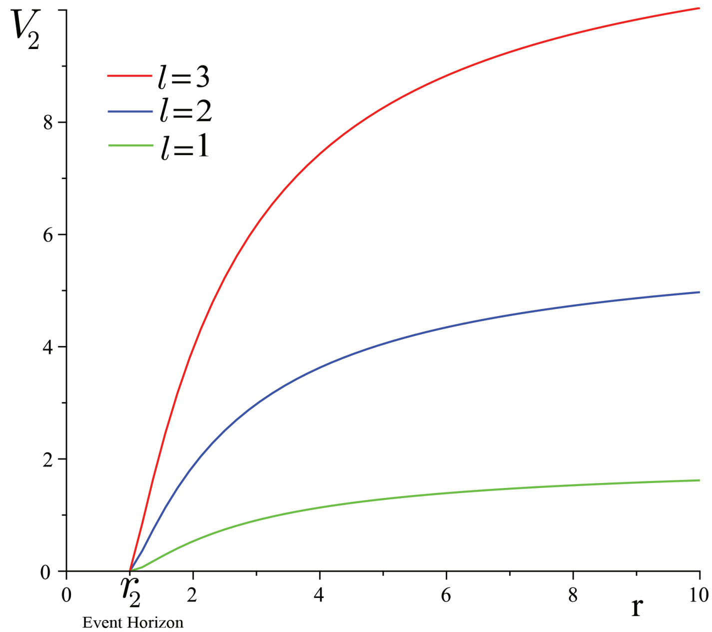

Figure 2. V_2 versus r graph. The physical parameters are chosen as $A = b = r_2 = 2r_1 = 2p = 1$. Plot is governed by Eq. (42).

Asymptotic ($r \to \infty$) solutions of the radial equations (41) are found as

$$P_j = \widetilde{C}_{1j} e^{i\eta r_*} + \widetilde{C}_{2j} e^{-i\eta r*}, \tag{47}$$

where

$$\eta = \frac{1}{A}\sqrt{k^2 A^2 - \lambda^2 b}. \tag{48}$$

For $k^2 > \frac{\lambda^2 b}{A^2}$, the Dirac waves can propagate to spatial infinity without fading. The asymptotic solutions to the radial equations are then found to be

$$\begin{aligned} G_1 = P_1 + P_2 \quad &\longrightarrow \quad G_1 = \widetilde{D}_1 e^{i\eta r_*} + \widetilde{D}_3 e^{-i\eta r*}, \\ G_2 = P_1 - P_2 \quad &\longrightarrow \quad G_2 = \widetilde{D}_2 e^{i\eta r_*} + \widetilde{D}_4 e^{-i\eta r*}. \end{aligned} \tag{49}$$

We impose the other QNM condition which requires the propagating waves to be purely outgoing at spatial infinity. To this end, we simply set $\widetilde{D}_3 = \widetilde{D}_4 = 0$ in Eq. (49) and obtain

$$G_j = \widetilde{D}_j e^{i\eta r_*}. \tag{50}$$

At the asymptotic region, the tortoise coordinate becomes

$$r_* = \frac{1}{b(r_2 - r_1)} \ln\left[\frac{(r - r_2)^{r_2}}{(r - r_1)^{r_1}}\right] \approx \frac{1}{b(r_2 - r_1)} \ln r^{r_2 - r_1}. \tag{51}$$

Therefore, one can see that

$$\eta r_* \approx \frac{\widetilde{\alpha}}{r_2 - r_1} \ln r^{r_2 - r_1}, \tag{52}$$

where $\widetilde{\alpha} = \frac{\eta}{b}$ and $e^{i\eta r_*} = r^{i\widetilde{\alpha}}$. By this way, the asymptotic solutions (50) can be expressed in terms of the radial coordinate as follows:

$$G_j = \widetilde{D}_j r^{i\widetilde{\alpha}}. \tag{53}$$

An alternative way of getting the latter result is the decoupling of G_j from Eq. (37):

$$\Lambda G_j'' - \frac{b(r_2 + r_1 - 2r)}{2} G_j' + \left\{(-1)^j ik \left[\frac{b(r_2 r_1 - r^2)}{\Lambda} - \frac{b(r_2 + r_1 - 2r)}{2\Lambda}\right] + \frac{k^2 r^2}{\Lambda} - \frac{\lambda^2}{A^2}\right\} G_j = 0. \tag{54}$$

As $r \to \infty$, Eq. (54) takes the following form

$$r^2 G_j'' + r G_j' + (\frac{k^2 A^2 - \lambda^2 b}{A^2 b^2} = \widetilde{\alpha}^2) G_j = 0, \tag{55}$$

whose solutions are found to be

$$G_j = \widetilde{D}_j r^{i\widetilde{\alpha}} + \widehat{C}_j r^{-i\widetilde{\alpha}}. \tag{56}$$

Since $e^{i\eta r_*} = r^{i\widetilde{\alpha}}$ and the wave should be purely outgoing at the spatial infinity, we set $\widehat{C}_j = 0$ and thus find $G_j = \widetilde{D}_j r^{i\widetilde{\alpha}}$ which is nothing but the same result that was obtained in Eq. (53).

5. Exact QNM frequencies

To find the analytical solutions of G_j, we first introduce

$$G_j = H_j e^{(-1)^{j+1} ik r_*}, \tag{57}$$

and change the variable to

$$z = -x = -\frac{r - r_2}{r_2 - r_1}. \tag{58}$$

After inserting Eq. (58) in Eq. (54), we obtain

$$z(1-z) H_j'' + \left[\boldsymbol{c} - (1 + \boldsymbol{a} + \boldsymbol{b}) z\right] H_j' - \boldsymbol{ab} H_j = 0, \tag{59}$$

which is the Euler's hypergeometric differential equation [56] with

$$\boldsymbol{a} = i\left[(-1)^{j+1}\frac{k}{b} + \widetilde{\alpha}\right],$$
$$\boldsymbol{b} = i\left[(-1)^{j+1}\frac{k}{b} - \widetilde{\alpha}\right],$$
$$\boldsymbol{c} = \frac{1}{2} + 2ik(-1)^{j+1}\frac{r_2}{b(r_2 - r_1)}. \tag{60}$$

The general solution of Eq. (59) is given by [56]

$$H_j = C_2 F(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c}; z) + C_1 z^{1-\boldsymbol{c}} F(\widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}}; \widetilde{\boldsymbol{c}}; z), \tag{61}$$

where C_1 and C_2 are the integration constants and $F(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c};z)$ and $F(\widetilde{\boldsymbol{a}},\widetilde{\boldsymbol{b}};\widetilde{\boldsymbol{c}};z)$ are the hypergeometric (or Gaussian) functions with

$$\begin{aligned}
\widetilde{\boldsymbol{a}} &= \boldsymbol{a}-\boldsymbol{c}+1 = \frac{1}{2} - i\left[k(-1)^{j+1}\frac{r_2+r_1}{b(r_2-r_1)}+\widetilde{\alpha}\right],\\
\widetilde{\boldsymbol{b}} &= \boldsymbol{b}-\boldsymbol{c}+1 = \frac{1}{2} - i\left[k(-1)^{j+1}\frac{r_2+r_1}{b(r_2-r_1)}-\widetilde{\alpha}\right],\\
\widetilde{\boldsymbol{c}} &= 2-\boldsymbol{c} = \frac{3}{2} - 2ik(-1)^{j+1}\frac{r_2}{b(r_2-r_1)}.
\end{aligned} \tag{62}$$

In the near horizon region: $r \to r_2,\ r_* \to -\infty \quad \Rightarrow \quad z \approx e^{r_*} \to 0$. By recalling that $F(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c};0) = 1$ [56], the physical solutions (57), which fulfill the QNM condition that only ingoing waves can propagate near the horizon, are found to be as follows:

$$\begin{aligned}
G_2 &= H_2 e^{-ikr_*} = C_2 e^{-ikr_*} F(\boldsymbol{a},\boldsymbol{b};\boldsymbol{c};z),\\
G_1 &= H_1 e^{ikr_*} = C_1 z^{1-\boldsymbol{c}} e^{+ikr_*} F(\widetilde{\boldsymbol{a}},\widetilde{\boldsymbol{b}};\widetilde{\boldsymbol{c}};z).
\end{aligned} \tag{63}$$

By using one of the special features of the hypergeometric functions [56]:

$$\begin{aligned}
F(\boldsymbol{a},\mathbf{b};\boldsymbol{c};y) &= \frac{\Gamma(\boldsymbol{c})\Gamma(\boldsymbol{b}-\boldsymbol{a})}{\Gamma(\boldsymbol{b})\Gamma(\boldsymbol{c}-\boldsymbol{a})}(-y)^{-\boldsymbol{a}} F(\boldsymbol{a},\boldsymbol{a}+1-\boldsymbol{c};\boldsymbol{a}+1-\boldsymbol{b};1/y)\\
&+ \frac{\Gamma(\boldsymbol{c})\Gamma(\boldsymbol{a}-\boldsymbol{b})}{\Gamma(\boldsymbol{a})\Gamma(\boldsymbol{c}-\boldsymbol{b})}(-y)^{-\boldsymbol{b}} F(\boldsymbol{b},\boldsymbol{b}+1-\boldsymbol{c};\boldsymbol{b}+1-\boldsymbol{a};1/y),
\end{aligned} \tag{64}$$

one can extend the solutions (63) to spatial infinity ($r \to \infty,\ r_* \to \infty \ \Rightarrow \ \frac{1}{z} \to 0$) as follows:

$$\begin{aligned}
G_1 &= C_1 z^{1-\boldsymbol{c}} e^{ikr_*}\left[\frac{\Gamma(\widetilde{\boldsymbol{c}})\Gamma(\widetilde{\boldsymbol{b}}-\widetilde{\boldsymbol{a}})}{\Gamma(\widetilde{\boldsymbol{b}})\Gamma(\boldsymbol{c}-\widetilde{\boldsymbol{a}})}(x)^{-\widetilde{\boldsymbol{a}}} F(\widetilde{\boldsymbol{a}},\widetilde{\boldsymbol{a}}+1-\widetilde{\boldsymbol{c}};\widetilde{\boldsymbol{a}}+1-\widetilde{\boldsymbol{b}};1/z)\right.\\
&\left.+ \frac{\Gamma(\widetilde{\boldsymbol{c}})\Gamma(\widetilde{\boldsymbol{a}}-\widetilde{\boldsymbol{b}})}{\Gamma(\widetilde{\boldsymbol{a}})\Gamma(\widetilde{\boldsymbol{c}}-\widetilde{\boldsymbol{b}})}(x)^{-\widetilde{\boldsymbol{b}}} F(\widetilde{\boldsymbol{b}},\widetilde{\boldsymbol{b}}+1-\widetilde{\boldsymbol{c}};\widetilde{\boldsymbol{b}}+1-\widetilde{\boldsymbol{a}};1/z)\right]\\
G_2 &= C_2 e^{-ikr_*}\left[\frac{\Gamma(\boldsymbol{c})\Gamma(\boldsymbol{b}-\boldsymbol{a})}{\Gamma(\boldsymbol{b})\Gamma(\boldsymbol{c}-\boldsymbol{a})}(x)^{-\boldsymbol{a}} F(\boldsymbol{a},\boldsymbol{a}+1-\boldsymbol{c};\boldsymbol{a}+1-\boldsymbol{b};1/z)\right.\\
&\left.+ \frac{\Gamma(\boldsymbol{c})\Gamma(\boldsymbol{a}-\boldsymbol{b})}{\Gamma\boldsymbol{a})\Gamma(\boldsymbol{c}-\boldsymbol{b})}(x)^{-\boldsymbol{b}} F(\boldsymbol{b},\boldsymbol{b}+1-\boldsymbol{c};\boldsymbol{b}+1-\boldsymbol{a};1/z)\right].
\end{aligned} \tag{65}$$

Since $x = \frac{r-r_2}{r_2-r_1}$, and thus $x \simeq r \simeq e^{br_*}$ at the infinity, we have

$$G_1 \approx C_1 \frac{\Gamma(\widetilde{\boldsymbol{c}})\Gamma(\widetilde{\boldsymbol{b}}-\widetilde{\boldsymbol{a}})}{\Gamma(\widetilde{\boldsymbol{b}})\Gamma(\boldsymbol{c}-\widetilde{\boldsymbol{a}})} r^{i\widetilde{\alpha}} + C_1 \frac{\Gamma(\widetilde{\boldsymbol{c}})\Gamma(\widetilde{\boldsymbol{a}}-\widetilde{\boldsymbol{b}})}{\Gamma(\widetilde{\boldsymbol{a}})\Gamma(\widetilde{\boldsymbol{c}}-\widetilde{\boldsymbol{b}})} r^{-i\widetilde{\alpha}}. \tag{66}$$

$$G_2 \approx C_2 \frac{\Gamma(\boldsymbol{c})\Gamma(\boldsymbol{b}-\boldsymbol{a})}{\Gamma(\boldsymbol{b})\Gamma(\boldsymbol{c}-\boldsymbol{a})} r^{i\widetilde{\alpha}} + C_2 \frac{\Gamma(\boldsymbol{c})\Gamma(\boldsymbol{a}-\boldsymbol{b})}{\Gamma(\boldsymbol{a})\Gamma(\boldsymbol{c}-\boldsymbol{b})} r^{-i\widetilde{\alpha}}. \tag{67}$$

The correspondence between the asymptotic solutions (49) and the above solutions (67) yields

$$\begin{aligned}\widetilde{D}_1 &= C_1 \frac{\Gamma(\widetilde{\boldsymbol{c}})\Gamma(\widetilde{\boldsymbol{b}}-\widetilde{\boldsymbol{a}})}{\Gamma(\widetilde{\boldsymbol{b}})\Gamma(\boldsymbol{c}-\widetilde{\boldsymbol{a}})},\\ \widetilde{D}_2 &= C_2 \frac{\Gamma(\boldsymbol{c})\Gamma(\boldsymbol{b}-\boldsymbol{a})}{\Gamma(\boldsymbol{b})\Gamma(\boldsymbol{c}-\boldsymbol{a})},\\ \widetilde{D}_3 &= C_1 \frac{\Gamma(\widetilde{\boldsymbol{c}})\Gamma(\widetilde{\boldsymbol{a}}-\widetilde{\boldsymbol{b}})}{\Gamma(\widetilde{\boldsymbol{a}})\Gamma(\widetilde{\boldsymbol{c}}-\widetilde{\boldsymbol{b}})},\\ \widetilde{D}_4 &= C_2 \frac{\Gamma(\boldsymbol{c})\Gamma(\boldsymbol{a}-\boldsymbol{b})}{\Gamma(\boldsymbol{a})\Gamma(\boldsymbol{c}-\boldsymbol{b})}.\end{aligned} \tag{68}$$

Therefore, to have only pure outgoing waves at the spatial infinity (i.e. $\widetilde{D}_3, \widetilde{D}_4 = 0$), we appeal to the pole structure of the Gamma functions. The Gamma functions $\Gamma(\widetilde{x})$ have poles at $\widetilde{x} = -n$ for $n = 0, 1, 2 ....$ [56]. Hence, we obtain the QNMs by using the following relations:

$$\begin{aligned}\boldsymbol{a} &= i\left[-\frac{k}{b} + \widetilde{\alpha}\right] = -n,\\ \boldsymbol{c}-\boldsymbol{b} &= \frac{1}{2} + i\left[-k\frac{r_2+r_1}{b(r_2-r_1)} - \widetilde{\alpha}\right] = -n,\\ \widetilde{\boldsymbol{a}} &= \frac{1}{2} - i\left[k\frac{r_2+r_1}{b(r_2-r_1)} + \widetilde{\alpha}\right] = -n,\\ \widetilde{\boldsymbol{c}}-\widetilde{\boldsymbol{b}} &= 1 + i\left[k\frac{r_2+r_1}{b(r_2-r_1)} - \widetilde{\alpha}\right] = -n.\end{aligned} \tag{69}$$

Recall that the time dependence of the QNMs is governed by e^{ikt} [see Eq. (21)]. Therefore, having QNMs is conditional on $Im(k) > 0$, which guaranties the stability [40]. The conditions under which frequencies obtained from G_1 and G_2 spinor solutions produce stable modes (i.e. QNMs) and which ones generate the unstable modes are summarized in Tables 1 and 2:

Table 1. QNM frequencies for G_1. The results are obtained from Eqs. (66) and (69).

Frequencies	Stable modes (QNMs)	Unstable modes
$k = -\frac{I}{2A(\beta^2-1)}\left[A\beta b(2n+1) + 2\sqrt{b}\sqrt{A^2 b(n+\frac{1}{2})^2 + \lambda^2(\beta^2-1)}\right)$	if $A < 0$ and $A < \frac{\lambda}{\sqrt{b}(n+\frac{1}{2})}$	if $A > 0$ or if $A < 0$ and $A > \frac{\lambda}{\sqrt{b}(n+\frac{1}{2})}$
$k = \frac{I}{2A(\beta^2-1)}\left[A\beta b(n+1) + \sqrt{b}\sqrt{A^2 b(n+1)^2 + \lambda^2(\beta^2-1)}\right)$	if $A > 0$ if $A < 0$ and $A < \frac{\lambda}{\sqrt{b}(n+1)}$	if $A < 0$ and $A > \frac{\lambda}{\sqrt{b}(n+1)}$

In Tables 1 and 2, the parameter of β is given by

$$1 < \beta = \frac{r_2+r_1}{r_2-r_1} < \infty. \tag{70}$$

As can be seen from above, depending on the values of A, λ, b, n some QNM frequencies can have positive/negative imaginary part, and therefore the LDBHs can be stable/unstable under charged fermionic perturbations.

Table 2. QNM frequencies for G_2. The results are obtained from Eqs. (67) and (69).

Frequencies	Stable modes (QNMs)	Unstable modes
$k = \frac{I}{2A^2 n}(\lambda^2 - n^2 A^2 b)$	if $A > 0$ and $A < \frac{-\lambda}{\sqrt{b}n}$ if $A < 0$ and $A > \frac{\lambda}{\sqrt{b}n}$	if $A > 0$ and $A > \frac{-\lambda}{\sqrt{b}n}$ if $A < 0$ and $A < \frac{\lambda}{\sqrt{b}(n}$
$k = -\frac{I}{2A(\beta^2-1)}\,[A\beta b(2n+1)+ 2\sqrt{b}\sqrt{A^2 b(n+\frac{1}{2})^2 + \lambda^2(\beta^2-1))}]$	if $A < 0$ and $A < \frac{\lambda}{\sqrt{b}(n+\frac{1}{2})}$	if $A > 0$ or if $A < 0$ and $A > \frac{\lambda}{\sqrt{b}(n+\frac{1}{2})}$

6. Conclusion

In this paper, we have analytically computed the QNMs of charged fermionic perturbations for the 4-dimensional LDBHs which are the solutions to the EMD theory with the double Lioville-type potentials. The fermionic fields are the solutions of the Dirac equation. For the LDBH geometry, we have found that the massless but charged fermionic fields are governed by the Gaussian hypergeometric functions. We matched the exact solutions obtained with the solutions found for near horizon and asymptotic regions. Thus, we have shown that there are two sets of frequencies for each component (G_1 and G_2) of the fermionic fields. As being summarized in Tables 1 and 2, in each set the frequencies obtained can belong to either the stable modes (QNMs) or the unstable modes depending on the relations between the parameters of A, λ, b, and n. Meanwhile, the reader may question what the advantage/disadvantage of perturbing the LDBH with fermionic fields instead of the bosonic fields is. If precise measurements of QNM frequencies become possible in the future and if the fermionic QNM data obtained for spin-up and spin-down fields (i.e. G_1 and G_2) coincide with (a kind of double check) the analytical expressions that we have derived, those results will likely reveal the presence of LDBHs more accurately than the bosonic QNM ones.

As is known, rotating BH solutions are more considerable for testing relevant theories with the astrophysical observations. For this reason, in the near future, we plan to extend our study to the rotating LDBHs. To this end, we shall apply the Newman–Janis algorithm [57] to the metric (10) and perturb the obtained rotating LDBH to reveal the effect of rotation on their QNMs.

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